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SPECTRAL ANALYSIS OF MULTIPLE CRACKED BEAM SUBJECTED TO MOVING LOAD

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Abstract. In present paper, the spectral approach is proposed for analysis of multiple cracked beam subjected to general moving load that allows us to obtain explicitly dynamic response of the beam in frequency domain. The obtained frequency response is traightforward to calculate time history response by using the FFT algorithm and provides a novel tool to investigate effect of position and depth of multiple cracks on the dynamic response. The analysis is important to develop the spectral method for identification of multiple cracked beam by using its response to moving load. The theoretical development is illustrated and validated by numerical case study.

Keywords: Multiple cracked beam, moving load problem, frequency domain solution, modal method, spectral analysis.

1. INTRODUCTION

The moving load problem has attracted attention of researchers and engineers in the field of structural engineering and it is so far an actual topic in dynamics of structures. The mathematical fundamentals of the problem were formulated in [1–3]. The mathematical representation of the problem is strictly associated with the model adopted for moving load and structure subjected to the load. The models adopted for moving load are constant or harmonic force [4]; moving mass [5, 6] and more complicated vehicle system [7, 8]. The structure taken into this issue is firstly the simple and intact beam like structures and, recently, more complicated structures [9–14]. Most of the aforementioned studies have investigated the moving load problem in time domain by using either the mode superposition (modal) method or the finite element one. The modal method [15] relies on the eigenvalue problem that is not easily for damaged structures. On the other hand, the FEM [16, 17] requires a time consumable task to identify position of moving load for computing nodal load. Moreover, both the methods are poorly applicable for evaluating the shear force [18] and high frequency components [19]. Jiang et al have demonstrated in [20, 21] that the moving load problem can be investigated straightforwardly in the frequency domain. Khiem et al. [22] have proposed a spectral approach to the moving load problem that is solved completely in frequency domain.

The present paper aims to use the spectral approach for analysis of frequency response of beam with arbitrary number of cracks. The equivalent spring model [23] is adopted to represent open cracks in a beam element. The conventional time history response can be easily calculated from the frequency response in arbitrary frequency range. The theoretical development is illustrated and validated by numerical examples.

2. THE GOVERNING EQUATIONS OF DYNAMIC SYSTEM

Let's consider a dynamic system that comprises a simply supported Euler-Bernoulli beam and a vehicle moving on the beam, see Fig. 1. Suppose that E, ρ, A, I, ℓ are parameters of the beam and m, c, k are respectively the mass, damping coefficient and stiffness of vehicle. Moreover, the beam is assumed to be cracked at the position $e_1, \ldots, e_n$ with the depth $a_1, \ldots, a_n$ respectively.

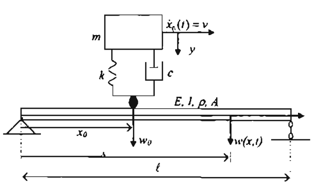

Fig. 1. Dynamic model of beam subjected to moving vehicle

By introducing the notations $w(x,t)$, $z(t)$, $x_0(t)$ respectively for transverse deflection of the beam at section x, vertical displacement of the vehicle and distance of the vehicle from the left end ($x = 0$) of beam, the governing equations for the system can be derived as follow

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + \rho A\eta\frac{\partial w(x,t)}{\partial t} + \rho A\frac{\partial^2 w(x,t)}{\partial t^2} = P(t)\delta[x - x_0(t)], \tag{1}$$

$$P(t) = mg + c\dot{y}(t) + ky(t) = m[g - \ddot{z}(t)], \tag{2}$$

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = -m\ddot{w}_0(t), y(t) = [z(t) - w_0(t)], w_0(t) = w[x_0(t), t] + \zeta[x_0(t)]. \tag{3}$$

In the latter equation function $\zeta(x)$ represents rough surface of the beam on which the vehicle is traveling. Furthermore, solution of Eq. (1) is subject to boundary conditions

$$w(0,t) = w''(0,t) = w(\ell,t) = w''(\ell,t) = 0, \tag{4}$$

and compatibility conditions at the crack positions

$$\begin{gathered} w(e_k + 0, t) = w(e_k - 0, t); w''(e_k + 0, t) = w''(e_k - 0, t); w'''(e_k + 0, t) = w'''(e_k - 0, t), \\ [w'(e_k + 0, t) - w'(e_k - 0, t)] = \gamma_k w''(e_k, t), \gamma_k = EI\theta(a_k). \end{gathered} \tag{5}$$

Function $\theta(a)$ in Eq. (5) is defined in the theory of cracked beam [23]

Note that moving load (2) expressed in the form

$$P(t) = P_0[a + b\xi(t)], \tag{6}$$

represents a number of earlier models of the moving load. Namely, for the case when relative vertical displacement of vehicle and acceleration of beam are negligible one has $a = 1$, $b = -1/g$, $P_0 = mg, \xi(t) = d^2\zeta[x_0(t)]/dt^2$. The conventional case of constant force moving on smooth surface of beam corresponds to $a = 1, b = 0$. In the case of a concentrated harmonic load, $a = 0$, $b = P_0$, $\xi(t) = \sin(\omega_e t + \varphi)$ that gives rise $P(t) = P_0 \sin(\omega_e t + \varphi)$ and in the moving mass case, $a = 1$, $b = -1/g$, $P_0 = mg$, i.e. $P(t) = m[g - \ddot{w}_0(t)]$. In this study we investigate the problem with moving load given generally in a discrete form $\{P(t_1), \dots, P(t_M)\}$.

3. FREQUENCY RESPONSE OF CRACKED BEAM TO GENERAL MOVING LOAD

Supposing that the force $P(t)$ is travelling on the beam with constant speed, i.e. $x_0(t) = vt$, the Fourrier transform leads Eq. (1) to

$$\frac{d^4\phi(x,\omega)}{dx^4} - \lambda^4\phi(x,\omega) = Q(x,\omega),\ \lambda^4 = (\omega^2 - i\eta\omega)/a^2,\ a = \sqrt{EI/\rho A}, \tag{7}$$

$$\phi(x,\omega) = \int_{-\infty}^{\infty} w(x,t)e^{-i\omega t}dt,\quad Q(x,\omega) = P(x/v)e^{-i\omega x/v}/EIv. \tag{8}$$

It is well known that general solution of Eq. (7) is

$$\phi(x,\omega) = \phi_0(x,\omega) + \int_0^x h(x-s)Q(s,\omega)ds, \tag{9}$$

with $\phi_0(x,\omega)$ being general solution of homogeneous equation

$$\frac{d^4\phi(x,\omega)}{dx^4} - \lambda^4\phi(x,\omega) = 0, \tag{10}$$

and

$$h(x) = (1/2\lambda^3)[\sinh \lambda x - \sin \lambda x]. \tag{11}$$

Since $h(0) = h''(0) = h''(0) = 0$ function

$$\phi_1(x,\omega) = \int_0^x h(x-s)Q(s,\omega)ds \tag{12}$$

satisfies the conditions $\phi_1(0,\omega) = \phi_1''(0,\omega) = 0$ so that solution (9) will satisfy the boundary conditions at the left end of beam together with function $\phi_0(x,\omega)$. It is easily to verify that solution $\phi_0(x,\omega)$ of Eq. (10) satisfying conditions

$$[\phi'(e_k+0) - \phi'(e_k-0)] = \gamma_k\phi''(e_k), \tag{13}$$

$$\phi(e_k+0) = \phi(e_k-0),\ \phi''(e_k+0) = \phi''(e_k-0), \phi'''(e_k+0) = \phi'''(e_k-0),$$

can be expressed in the form

$$\phi_0(x,\lambda) = L_0(x,\lambda) + \sum_{k=1}^{n} \mu_k K(x - e_k), \tag{14}$$

where $L_0(x,\lambda)$ is a particular continuous solution of Eq. (10) satisfying the condition $L_0(0,\lambda) = L_0''(0,\lambda) = 0$ and

$$K(x) = \begin{cases} 0 & \text{for } x \le 0 \\ S(x) & \text{for } x \succ 0 \end{cases} \quad K''(x) = \begin{cases} 0 & \text{for } x \le 0 \\ S''(x) & \text{for } x \succ 0 \end{cases},$$

$$S(x) = (\sinh \lambda x + \sin \lambda x)/2\lambda, \quad S''(x) = \lambda(\sinh \lambda x - \sin \lambda x)/2, \tag{15}$$

$$\mu_j = \gamma_j [L_0''(e_j,\lambda) + \sum_{k=1}^{j-1} \mu_k S''(e_j - e_k)]. \tag{16}$$

Representing the solution $L_0(x,\lambda)$ as

$$L_0(x) = CL_1(x,\lambda) + DL_2(x,\lambda), \tag{17}$$

and substituting it together with expression (14) into Eq. (9) one obtains

$$\phi(x,\omega) = CL_1(x,\lambda) + DL_2(x,\lambda) + \sum_{k=1}^{n} \mu_k K(x - e_k) + \phi_1(x,\omega). \tag{18}$$

Obviously, the latter function (18) satisfies boundary conditions at the left end of beam and the unknown constants C, D can be determined from the boundary conditions

$$\phi(\ell,\omega) = \phi''(\ell,\omega) = 0,$$

that is rewritten in more detail as

$$CL_1(\ell,\lambda) + DL_2(\ell,\lambda) = -\sum_{k=1}^{n} \mu_k S(1 - e_k) - \phi_1(\ell,\omega),$$
$$CL_1''(\ell,\lambda) + DL_2''(\ell,\lambda) = -\sum_{k=1}^{n} \mu_k S''(1 - e_k) - \phi_1''(\ell,\omega).$$

Solution of the latter equations is easily obtained in the form

$$C = C_0 + \sum_{k=1}^{n} C_k \mu_k, \; D = D_0 + \sum_{k=1}^{n} D_k \mu_k, \tag{19}$$

where

$$C_0 = \frac{[L_2(\ell,\lambda)\phi_1''(\ell,\omega) - L_2''(\ell,\lambda)\phi_1(\ell,\omega)]}{d_0(\lambda)}, \; D_0 = \frac{[L_1''(\ell,\lambda)\phi_1(\ell,\omega) - L_1(\ell,\lambda)\phi_1''(\ell,\omega)]}{d_0(\lambda)}, \tag{20}$$

$$C_k = \frac{[L_2(\ell,\lambda)S''(\ell - e_k) - L_2''(\ell,\lambda)S(\ell - e_k)]}{d_0(\lambda)}, \; D_k = \frac{[L_1''(\ell,\lambda)S(\ell - e_k) - L_1(\ell,\lambda)S''(\ell - e_k)]}{d_0(\lambda)},$$

$$d_0(\lambda) = L_1(\ell,\lambda)L_2''(\ell,\lambda) - L_1''(\ell,\lambda)L_2(\ell,\lambda). \tag{21}$$

Now substituting expression (17) with coefficient (19) into (16) yields

$$[\mathbf{I} - \mathbf{\Gamma}(\gamma)\mathbf{B}(\boldsymbol{\lambda},\mathbf{e})]\boldsymbol{\mu} = \mathbf{\Gamma}(\gamma)\mathbf{b}(\boldsymbol{\lambda},\mathbf{e}), \tag{22}$$

where the following matrices and vectors are introduced

$$\mathbf{B}(\lambda, \mathbf{e}) = [b_{jk}, j, \ k = 1, \dots, n],\ b_{jk} = C_k L_1''(e_j, \lambda) + D_k L_2''(e_j, \lambda) + K''(e_j - e_k),$$
$$\mathbf{\Gamma}(\boldsymbol{\gamma}) = diag\{\gamma_1, \dots, \gamma_n\}, \boldsymbol{\mu} = (\mu_{11}, \dots, \mu_n)^T, \mathbf{e} = (e_1, \dots, e_n)^T, \boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)^T, \tag{23}$$
$$\mathbf{b} = (b_1, \dots, b_n)^T,\ b_j = C_0 L_1''(e_j, \lambda) + D_0 L_2''(e_j, \lambda), j = 1, \dots, n.$$

Eq. (22) can be solved with respect to μ as

$$\boldsymbol{\mu} = [\mathbf{I} - \mathbf{\Gamma}(\boldsymbol{\gamma})\mathbf{B}(\lambda, \mathbf{e})]^{-1}\mathbf{\Gamma}(\boldsymbol{\gamma})\mathbf{b}(\lambda, \mathbf{e}). \tag{24}$$

Therefore, frequency response of multiple cracked beam can be represented as

$$\phi(x, \omega) = \alpha_0(x, \omega) + \sum_{k=1}^{n} \mu_k \alpha_k(x, \mathbf{e}, \boldsymbol{\gamma}, \omega), \tag{25}$$

where

$$\alpha_0(x, \omega) = C_0 L_1(x, \lambda) + D_0 L_2(x, \lambda) + \phi_1(x, \omega), \tag{26}$$
$$\alpha_k(x, \omega) = C_k L_1(x, \lambda) + D_k L_2(x, \lambda) + K(x - e_k),\ k = 1, \dots, n.$$

Since the static response is defined as the frequency response at $\omega = 0$, it can be conducted by solving the equation $d^4\phi(x, 0)/dx^4 = Q(x, 0)$. So that the static solution $\phi(x, 0)$ satisfying the given boundary conditions is

$$\phi(x, 0) = Q_4(x) - Q_4''(\ell)x^3/6\ell + [Q_4''(\ell)\ell \ / \ 6 - Q_4(\ell) \ / \ \ell]x, \tag{27}$$

$$Q_4(x) = \int_0^x ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \int_0^{s_3} Q(s, 0)ds. \tag{28}$$

If the moving load $P(t)$ has been given at the time mesh $(t_1, \dots, t_M)$ the function $Q(x, \omega)$ would be defined in the form

$$Q(x_j, \omega) = P(t_j)e^{-i\omega t_j}/EIv, x_j = vt_j j = 1, \dots, M. \tag{29}$$

Hence, the function defined in (12) can be calculated

$$\phi_1(x, \omega) = \int_0^x h(x - s)Q(s, \omega)ds = (1/EI)\sum_{r=1}^{M} H(x - vt_r)P(t_r))e^{-i\omega t_r}\Delta t_r, \tag{30}$$

$$\phi_1''(x, \omega) = \int_0^x h''(x - s)Q(s, \omega)ds = (1/EI)\sum_{r=1}^{M} H''(x - vt_r)P(t_r)e^{-i\omega t_r}\Delta t_r, \tag{31}$$

$$H(x) = \begin{cases} 0, & x \leq 0 \\ h(x), & x \geq 0 \end{cases}, H''(x) = \begin{cases} 0, & x \leq 0 \\ h''(x), & x \geq 0 \end{cases}, \Delta t_r = t_r - t_{r-1}, \tag{32}$$

that allow the coefficients C_0, D_0 to be completely calculated with expressions (20). Thus, the frequency response (25), (26) is fully determined for the given discretely moving load. Once the frequency response $\phi(x, \omega)$ has been known the time history response

$$w(x, t) = (1/2\pi) \int_{-\infty}^{\infty} \phi(x, \omega)e^{i\omega t}d\omega, \tag{33}$$

could be usually evaluated at the discrete time mesh $t_r = rT/N, r = 0, \ldots, N$ in a finite interval of time $[0, T]$ as

$$w(x, t_r) = (2/T) \sum_{k=0}^{N-1} \phi(x, \omega_k) e^{2i\pi kr/N)}, \; r = 0, \ldots, N, \tag{34}$$

where $\omega_k = k\Delta\omega = k(2\pi/T)$ and N is chosen accordingly to the frequency range of interest. For instance, if Ω is Nyquist frequency of the response, then

$$N = \Omega/\Delta\omega = \Omega T/2\pi. \tag{35}$$

4. RESULTS AND DISCUSSION

An example of the beam with $E = 2.1 \times 10^{11}, \rho = 7860$ kg/m^3, $\ell = 50$ m, $h = 1.0$ m, $b = 0.5$ m subjected to moving constant force is examined by using the proposed spectral method. Deflection, slope, bending moment and shear force distributed along the beam length are computed with different speeds of moving load and various scenarios of multiple cracks. Namely, the quantities are computed at the frequencies $f = f_1$; $1.5f_1$; $2f_1$; $3f_1$, where f_1 is the fundamental frequency of uncracked beam with speed equal to a half of critical speed $v = 0.5v_c$. Results of computation are given in Fig. 1. Fig. 2 presents the deflection, slope, moment and shear response at fundamental frequency for various speed ratios, $v/v_c = 0.1 - 2.0$. The frequency response for beam with different scenarios of crack position and depth is presented in Figs. 3-4, where crack position is roving from 5 m to 45 m and crack depth is varying from 0% to 50%. Fig. 5 shows the response computed for different numbers of cracks appeared in the beam. In all the figures the deflection, slope, bending, moment and shear are plotted along the beam length, i.e. versus $x \in (0, \ell)$.

It can be noted from Fig. 2 that waveform of defection, slope, moment and shear response vary strongly with frequency and is much dissimilar to the vibration mode shape. The response at lower frequency may appear as higher frequency mode shape that is perhap caused by multi-resonance phenomenon for forced vibration under moving load.

Vibration amplitude increases with the speed growing up to the critical one except the speed $v = 0.5v_c$ that shows to be intiresonant speed (Fig. 3). Further increase of speed from the critical one leads to reduced vibration amplitude so that maximum effect is observed at critical speed.

Furthermore, any crack inside beam makes uniformly distributed change in frequency response so that crack position cannot be visible from the frequency response plotted along the beam length. However, the largest change is observed when crack occurred at position 20 m from the left end. It can be seen from Fig. 4 that symmetric (about the beam middle) cracks lead to not equal change in frequency response that is important to solve the nonunique solution problem in crack detection for beam with symmetric boundary conditions.

Figs. 5 and 6 show that while the frequency response monotony increases with crack depth, multiple cracks occurred additionally to the right of beam middle make reduction of the response. This implies that frequency response is monotony increasing with amount of cracks located on the left of beam midpoint and decreasing with growing number of cracks on the right of the midpoint.

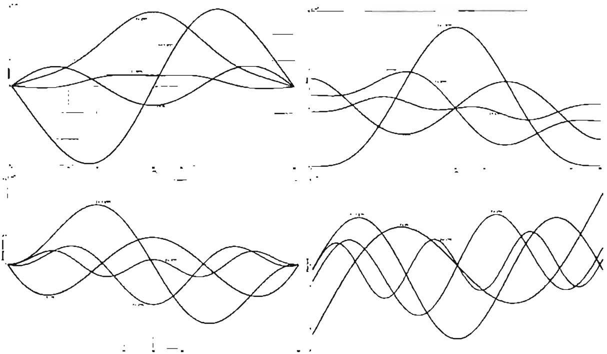

Fig. 2. Frequency response for deflection, slope, bending moment and shear of uncracked beam at natural frequencies $f = [1.0; 1.5; 2.0; 3.0] \times f_1$, speed $v = 0.5v_c$

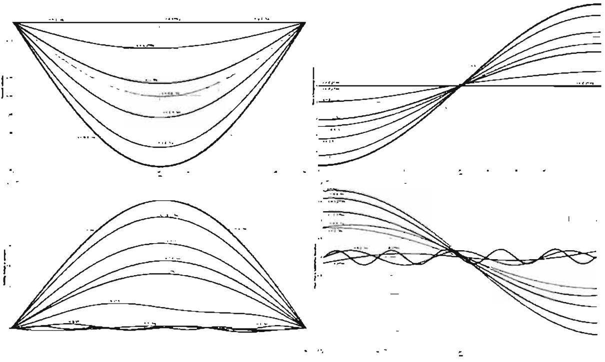

Fig. 3. Frequency response for deflection, slope, bending moment and shear at fundamental frequency in different speed ratios (0 1;0 2;0.3;0.5;0.6,0.8;1.0;1.2;1.5;2.0)

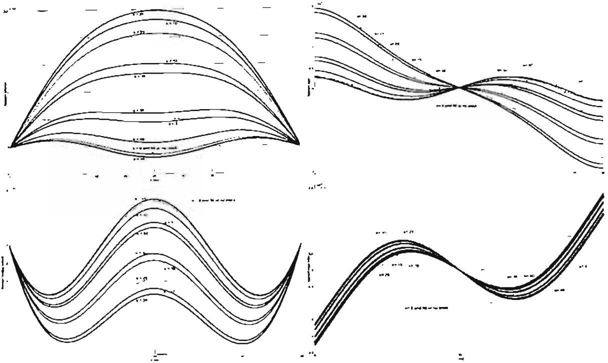

Fig. 4. Effect of crack position (5; 10; 15; 20; 25; 30; 35; 40; 45m) on the deflection, slope, bending moment and shear force response at fundamental resonant frequency

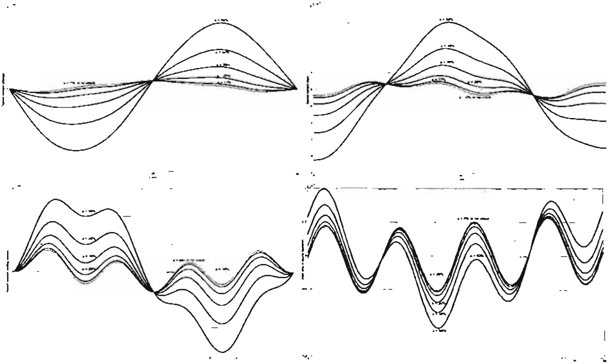

Fig. 5. Effect of crack depth (0; 10; 20; 30; 40; 50%) on the deflection, slope, bending moment and shear force response at fundamental resonant frequency

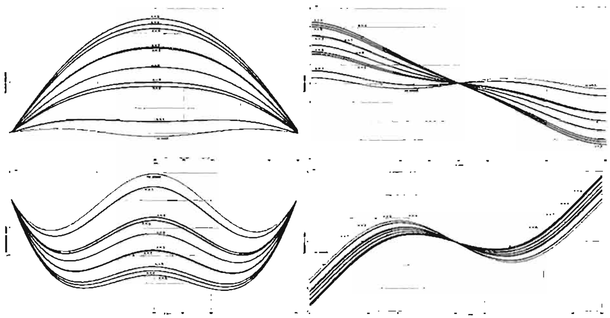

Fig. 6. Effect of number of cracks (1; 2; 3; 4; 5; 6, 7; 8; 9) on the deflection, slope, bending moment and shear force response at fundamental resonant frequency

5. CONCLUSION

In present paper the spectral method has been developed for dynamic analysis of multiple cracked beams subjected to general moving load in frequency domain. A closed form solution for frequency response to moving load was conducted for beam with arbitrary number of cracks. The obtained solution is straightforward to calculate time history response and provides a novel tool for dynamic analysis of response at arbitrary frequency. Numerical results have shown that a localized crack makes uniformly distributed change in waveform of the frequency response; due to moving load the cracks occurred to symmetric positions affect not symmetrically on the response; amplitude of forced vibration is not monotony increasing with growing number of cracks.

The proposed method can be used for dynamic analysis in the case of more complicated moving load and crack detection problem by measurement of dynamic response of beam-like structure subjected to moving load.

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