Vietnam Journal of Mechanics, VAST, Vol. 36, No. 3 (2014), pp. 173 – 183

FUZZY ANALYSIS OF LATERALLY-LOADED PILE IN LAYERED SOIL

Pham Hoang Anh
National University of Civil Engineering, Hanoi, Vietnam
E-mail: anhpham.nuce@gmail.com
Received March 09, 2014

Abstract. A fuzzy finite-element method for analysis of laterally loaded pile in multi-layer soil with uncertain properties is presented. The finite-element formulation is established using a beam-on-two-parameter foundation model. Uncertainty propagation of the soil parameters to the pile response is evaluated by a perturbation technique. This approach requires a few number of classical finite-element equations to be solved and provides reasonable results A comparison with vertex method is made in a numerical example.

Keywords: Fuzzy finite element analysis, laterally-loaded pile, multi-layered soil.

1. INTRODUCTION

Piles subjected to lateral loading can be found in many civil engineering structures such as offshore platforms, bridge piers and high-rise buildings. For the design of pile foundations of such structures, special attention needs to be concentrated not only on the bearing capacity but also on the horizontal displacements of the piles under lateral loading conditions. The deterministic analysis of lateral load-displacement behavior of piles is complicated and in general requires a numerical solution procedure (e.g., the finite difference method, finite element method). On the other hand, uncertainty is often present in the input data, especially in geotechnical engineering data. These uncertainties can be accounted for by using probabilistic methods, e.g., methods proposed in [1–6]. However, very often the input data fall in the category of non-statistical uncertainty. The reasons for this uncertainty may be because the observations made can best be categorized with linguistic variables (e.g., the soil may be described with linguistic variables such as "very soft", "soft", or "stiff"; "loose", "dense", or "very dense"), or because only a limited number of samples are available and a particular soil property are unknown or vary from location to other location. These types of uncertainties can be appropriately represented in the mathematical model as fuzziness [7].

In this paper, a laterally loaded pile in multi-layer soil with uncertain parameters is considered. It is assumed that only rough estimates of the soil parameters are available and these are modeled as fuzzy values. The analysis of the pile-soil interaction is based on a

"*Beam-on-two-parameter-linear-elastic-foundation*" model. A finite element of the pile-soil system is formulated and the fuzzy pile deflection is developed by a perturbation-based technique. The fuzzy behavior of the pile is illustrated and compared with results obtained by vertex method via a numerical example.

2. MODEL OF ANALYSIS

Consider a vertical pile embed in a soil deposit containing $n$layers, with the thickness of layer i given by H_i (Fig. 1(a)). The top of the pile is at the ground surface and the bottom end of the pile is considered embedded in the n-th layer. Each soil layer is assumed to behave as a linear, elastic material with the compressive resistance parameter k_i and shear resistance parameter t_i. The pile is subjected to a lateral force F_0 and a moment M_0 at the pile top. The pile behaves as an Euler-Bernoulli beam with length L_p and a constant flexural rigidity EI. The governing differential equation for pile deflection w_i within any layer i is given in [8]

$$EI\frac{d^4w_i}{dz^4} + k_i w_i - 2t_i\frac{d^2w_i}{dz^2} = 0. \tag{1}$$

The Eq. (1) is exactly the same as the equation for the "*Beam-on-two-parameter-linear-elastic-foundation*" model introduced by Vlasov and Leont'ev [9]. The use of linear elastic analysis in the laterally loaded pile problem, especially in the prediction of deformations at working stress levels, has become a widely accepted model in geotechnical engineering. Also in the real problem where nonlinear stress-strain relationships for the soil must be used, linear elastic solution provides the framework for the analysis, in which the elastic properties of the soil will be changed with the changing deformation of the soil mass (e.g., the "p-y" method [10]).

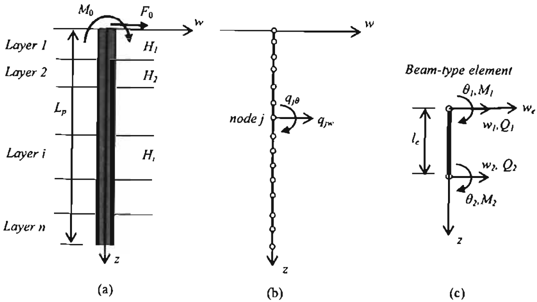


Fig. 1. (a) A laterally-loaded pile in a layered soil; (b) FE discretization; (c) Beam-type element

In this paper, this *Beam-on-linear-elastic-foundation* model is the basis for the finite element formulation of the laterally loaded pile problem which will be presented in the next section.

3. FINITE ELEMENT FORMULATION

While the finite-difference method has sometimes been the preferred numerical solution technique for Eq. (1), this paper uses the finite-element approach, which offers a convenient vehicle for dealing with boundary conditions and variable material properties, especially the fuzzy soil properties described later in the paper.

The pile is divided into m finite elements and to each j-th node of their interconnection, two degrees of freedom are allowed: q_{jw} - the deflection and $q_{j\theta}$ - the rotation of cross section with positive direction as in Fig. 1(b). Element of EB-beam type is chosen for each pile element with length l_e and two nodes, one at each end. The element is connected to other elements only at the nodes. To each element, two degrees of freedom are allowed at both ends: deflection, w_1 and rotation, θ_1, and w_2, θ_2 respectively, positives in the system of local axes from Fig. 1(c). With these displacements, the element nodal displacement vector $\{q\}_e$ and the element nodal force vector $\{r\}_e$ of respect to the system of local axes, are defined:

$$\{q\}_e = \{w_1\ \theta_1\ w_2\ \theta_2\}^T, \quad \{r\}_e = \{Q_1\ M_1\ Q_2\ M_2\}^T \tag{2}$$

It is noted that Q_1 and Q_2 from (2) include shear force in the pile section and also shear force in the soil.

We assume the displacement function within an element in the form of cubic polynomial

$$w_e = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3. \tag{3}$$

Applying the boundary conditions

$$\begin{cases} w_e(0) = w_1, -\dfrac{dw_e}{dz}(0) = \theta_1 \\ w_e(l_e) = w_2, -\dfrac{dw_e}{dz}(l_e) = \theta_2 \end{cases} \tag{4}$$

will give the coefficients of displacement function in terms of element nodal displacements, which are substitute in (3) to obtain the expression of the deflection as

$$w_e = N_1(z)\,w_1 + N_2(z)\,\theta_1 + N_3(z)\,w_2 + N_4(z)\,\theta_2 = [N]\{q\}_e, \tag{5}$$

where $N_i(z), i = 1, \dots, 4$ are the shape functions (interpolation functions)

$$\begin{cases} N_1(z) = 1 - \dfrac{3z^2}{l_e^2} + \dfrac{2z^3}{l_e^3}, N_2(z) = -z + \dfrac{2z^2}{l_e} - \dfrac{z^3}{l_e^2} \\ N_3(z) = \dfrac{3z^2}{l_e^2} - \dfrac{2z^3}{l_e^3}, N_4(z) = \dfrac{z^2}{l_e} - \dfrac{z^3}{l_e^2} \end{cases} \tag{6}$$

The strain energy in the beam element is

$$U_b = \frac{1}{2}\int_V \sigma_z\varepsilon_z dV = \frac{1}{2}\int_0^{l_e} EI\left(\frac{d^2w_e}{dz^2}\right)^2 dz$$
$$= \frac{1}{2}EI\int_0^{l_e} \{q\}_e^T\left(\frac{d^2}{dz^2}[N]\right)^T\left(\frac{d^2}{dz^2}[N]\right)\{q\}_e dz, \tag{7}$$

or

$$U_b = \frac{1}{2}\{q\}_e^T[k]_b\{q\}_e, \quad \text{with} \quad [k]_b = EI\int_0^{l_e}\left(\frac{d^2}{dz^2}[N]\right)^T\left(\frac{d^2}{dz^2}[N]\right)dz. \tag{8}$$

Strain energy in the two-parameter elastic foundation corresponding to the beam element is given by

$$U_f = \frac{1}{2}\int_0^{l_e} kw_e^2 dz + \frac{1}{2}\int_0^{l_e} 2t\left(\frac{dw_e}{dz}\right)^2 dz$$
$$= \frac{1}{2}\{q_e\}^T\left[k\int_0^{l_e}[N]^T[N]dz + 2t\int_0^{l_e}\left(\frac{d}{dz}[N]\right)^T\left(\frac{d}{dz}[N]\right)dz\right]\{q_e\}, \tag{9}$$

or

$$U_f = \frac{1}{2}\{q\}_e^T\left([k]_w + [k]_t\right)\{q\}_e,$$
$$\text{with} \quad [k]_w = k\int_0^{l_e}[N]^T[N]dz, \quad [k]_t = 2t\int_0^{l_e}\left(\frac{d}{dz}[N]\right)^T\left(\frac{d}{dz}[N]\right)dz. \tag{10}$$

The total strain energy of the coupled element is

$$U_e = U_b + U_f = \frac{1}{2}\{q\}_e^T\left([k]_b + [k]_w + [k]_t\right)\{q\}_e = \frac{1}{2}\{q\}_e^T[k]_e\{q\}_e. \tag{11}$$

In Eq. (11), $[k]_e = [k]_b + [k]_w + [k]_t$ represents the stiffness matrix of one-dimension finite element of pile on two-parameter elastic foundations. The terms of $[k]_b$, $[k]_w$, $[k]_t$ matrices are calculated using the relation (8) and (10). We obtain

$$[k]_b = \frac{EI}{l_e^3}\begin{bmatrix} 12 & -6l_e & -12 & -6l_e \\ -6l_e & 4l_e^2 & 6l_e & 2l_e^2 \\ -12 & 6l_e & 12 & 6l_e \\ -6l_e & 2l_e^2 & 6l_e & 4l_e^2 \end{bmatrix}, \tag{12}$$

$$[k]_w = \frac{kl_e}{420}\begin{bmatrix} 156 & -22l_e & 54 & 13l_e \\ -22l_e & 4l_e^2 & -13l_e & -3l_e^2 \\ 54 & -13l_e & 156 & 22l_e \\ 13l_e & -3l_e^2 & -3l_e^2 & 4l_e^2 \end{bmatrix}, \tag{13}$$

$$[k]_t = \frac{2t}{30l_e}\begin{bmatrix} 36 & -3l_e & -36 & -3l_e \\ -3l_e & 4l_e^2 & 3l_e & -l_e^2 \\ -36 & 3l_e & 36 & 3l_e \\ -3l_e & -l_e^2 & 3l_e & 4l_e^2 \end{bmatrix}. \tag{14}$$

The potential of element nodal loads is

$$W_e = \{q\}_e^T \{r\}_e. \tag{15}$$

The total potential energy functional of the element is

$$\Pi_e = U_e - W_e = \frac{1}{2}\{q\}_e^T [k]_e \{q\}_e - \{q\}_e^T \{r\}_e. \tag{16}$$

The equilibrium condition of the element is the first variation of (16) equals to zero, with arbitrary variation of the displacement $\delta\{q\}_e \neq 0$

$$\delta\Pi_e = \frac{\partial\Pi_e}{\partial\{q\}_e}\delta\{q\}_e = ([k]_e\{q\}_e - \{r\}_e)\,\delta\{q\}_e = 0, \tag{17}$$

or

$$[k]_e\{q\}_e = \{r\}_e. \tag{18}$$

Eq. (18) is the equilibrium equation of element. This is followed by assembly, implementation of boundary conditions, introduction of loads and equation solution. To review the finite element solution, two examples of laterally-loaded pile with deterministic inputs are analyzed and compared with analytical solution (exact solution). Later in this paper, the soil parameters k and t in Eqs. (12), (13), (14) will be treated as fuzzy variable.

The first example is taken from [11]. A pile of length $L_p = 20\,\text{m}$, and flexural rigidity $EI = 50{,}000\,\text{kNm}^2$ is driven into one-layer clay soil and subjected to a horizontal force $F_0 = 300\,\text{kN}$ and moment $M_0 = 100\,\text{kNm}$ at pile top. The lateral soil stiffness k is constant, and given by $k = 4{,}000\,\text{kPa}$. The analytical solution of the deflection at the top for this case is 63.4802 mm [11], which is compared with finite-element analysis using four, eight and twenty equal-length elements in Tab. 1. Gocd agreement is obtained using even coarse finite-element mesh.

Table 1. Pile top deflection by finite-element and analytical solutions (mm)

Analytical	FE solution: number of elements		
	4	8	20
63.4802	62.2033	63.3163	63.4753

Table 2. Pile top deflection in the second example (mm)

Analytical	FE solution: number of elements		
	8	20	40
5.8428	5.8080	5.8414	5.8427

The second example is adapted from [12]. A pile of length $L_p = 20\,\mathrm{m}$, radius $r_p = 0.3\,\mathrm{m}$ and modulus $E_p = 25 \times 106\,\mathrm{kNm}^2$ is subjected to a lateral force $F_0 = 300\,\mathrm{kN}$ and a moment $M_0 = 100\,\mathrm{kNm}$ at the pile head. The soil deposit has four layers with $H_1 = H_2 = H_3 = 5\,\mathrm{m}$. A two-parameter foundation model with $k_1 = 56.0\,\mathrm{MPa}$, $k_2 = 140.0\,\mathrm{MPa}$, $k_3 = 155.0\,\mathrm{MPa}$ and $k_4 = 200.0\,\mathrm{MPa}$, and $t_1 = 11.0\,\mathrm{MN}$, $t_2 = 28.0\,\mathrm{MN}$, $t_3 = 40.0\,\mathrm{MN}$ and $t_4 = 60.0\,\mathrm{MN}$ is assumed. The analytical solution for this case is obtained using the method proposed by Pham [13]. The top deflection is 5.8428 mm, which is shown in the analytical column of Tab. 2. The finite-element solutions are obtained using eight, twenty and forty equal-element length elements and also shown in Tab. 2. It is shown clearly that the finite-element results will converge to the exact solution when the finite-element mesh is refined.

4. FUZZY ANALYSIS METHOD FOR LATERALLY-LOADED PILE

In practical engineering problems, there are randomness and fuzziness with mechanical parameter values of soil. It follows that the stiffness matrix and the pile response will be fuzzy. According to the finite element method, we have

$$[\tilde{K}]\{\tilde{q}\} = \{f\}. \tag{19}$$

In which, $[\tilde{K}]$ is the fuzzy system stiffness matrix, $\{f\}$ is the external force vector and $\{\tilde{q}\}$ is the fuzzy displacement vector (consisting of nodal deflections and nodal rotations).

Basically, to evaluate fuzzy outputs through a finite-element model the concept of α-level discretization is adopted. All fuzzy input parameters are discretized using the same number of α-levels (often 5 to 10). The core procedure is an α-level optimization and can be operated according to any optimization algorithm. For each same α-level of the input parameters, the largest and the smallest output values can be determined, thus two points of the membership function of the output are known. By this procedure the fuzzy results are yield α-level by α-level.

Although the optimization strategy is acknowledged as the standard procedure for fuzzy finite element analysis, it is often a time consuming process because finite element analysis has to be carried out for every evaluation in the input spaces. On the other hand, for the case of laterally-loaded pile, only some output quantities are of interest (e.g., pile top deflection, maximum bending moment). Therefore, methods which can yield faster results are desirable. The present paper introduces a perturbation-based approach for estimation of fuzzy deflection of laterally-loaded pile and adopts the vertex method [14] for comparison.

4.1. Perturbation-based approach

For simplicity, we assume that soil parameter a_i (here a_i can be compressive parameters or shear parameters) are modeled as triangular fuzzy numbers. The fuzzy soil parameter denoted as $\tilde{a}_i$ is then given by $\tilde{a}_i = (a_i^L, a_i^M, a_i^R)$, where $a_i^L \leq a_i^M \leq a_i^R$, and a_i^M is the main value of $\tilde{a}_i$, which is the value of a_i with membership level 1. The fuzzy number $\tilde{a}_i$ can be determined as a sum of a distinct value a_i^M and a deviation δa_i, so that for membership level α

$$\tilde{a}_{i\alpha} = a_i^M + \delta a_{i\alpha}, \tag{20}$$

where δa_i is a triangular fuzzy number given by

$$\delta a_i = \left(\delta a_i^L, 0, \delta a_i^R\right) = \left(a_i^L - a_i^M, 0, a_i^R - a_i^M\right). \tag{21}$$

According to Eq. (12), it can be seen that the stiffness matrix is linear with respect to the soil parameters. Therefore, the fuzzy stiffness matrix can be expanded as

$$[\tilde{K}] = [K^0] + \sum_i [\dot{K}]_i \delta a_i, \tag{22}$$

where $[\dot{K}]_i$ is the partial derivative of the stiffness matrix with respect to parameter, a_i taken at main values of all parameters. In the same manner, the displacement response is expanded as

$$\{\tilde{q}\} = \{q^0\} + \sum_i \{\dot{q}\}_i \delta a_i. \tag{23}$$

Note that, the relation (23) is only an approximation of the actual displacement response. In the above formula $[K^0]$, $\{q^0\}$ are the stiffness matrix and the corresponding displacement vector, respectively, taken at a_i^M. Substitute Eqs. (22) and (23) into (19), comparing similar items on δ, followings can be obtained

$$[K^0]\{q^0\} = \{F\}, \tag{24}$$

$$[K^0]\{\dot{q}\}_i = -[\dot{K}]_i\{q^0\}. \tag{25}$$

The above equations are deterministic equations, from which $\{q^0\}$, $\{\dot{q}\}_i$ can be calculated. The fuzzy sets $\{\tilde{q}\}$ can then be approximated from fuzzy sets δa_i based on the principle of expansion given by (23). At α membership level, the relationship between the two is

$$\{\tilde{q}\}_\alpha = \{q^0\} + \sum_i \{\dot{q}\}_i \delta a_{i\alpha}. \tag{26}$$

According to the decomposition theorem, the membership function of a fuzzy set can be determined by its membership in each level $\alpha \in [0,1]$. We can see in each membership level $\alpha \in [0,1]$ on $\tilde{a}_i$, $\delta a_{i\alpha}$ are defined by interval, i.e. $\delta a_{i\alpha} = [\delta a_{i\alpha}^L, \delta a_{i\alpha}^R]$. The fuzzy nodal displacement, $\tilde{q}_j$ at the membership level α defined by $q_{j\alpha} = [q_{j\alpha}^L, q_{j\alpha}^R]$ can be easily obtained by the following formula,

$$q_{j\alpha}^L = q_j^0 + \sum_i \min(\dot{q}_{ji}\delta a_{i\alpha}^L, \dot{q}_{ji}\delta a_{i\alpha}^R), \tag{27}$$

$$q_{j\alpha}^R = q_j^0 + \sum_i \max(\dot{q}_{ji}\delta a_{i\alpha}^L, \dot{q}_{ji}\delta a_{i\alpha}^R). \tag{28}$$

Eqs. (27) and (28) determine the lower and upper bounds of a fuzzy nodal displacement corresponding to membership level α.

It can be seen that, this method requires solving $N + 1$ finite-element equations, with N is the number of fuzzy soil parameters.

4.2. Vertex method for pile top deflection

In practice, often only the pile top deflection is of interest. Moreover, it can be shown that the pile top deflection is monotonic in each soil parameters k_i and t_i. Therefore, the membership of the deflection can be evaluated by determining the membership at the endpoints of the level cuts of membership of each k_i, t_i. This method, which is the well known "Vertex method" introduced by Dong and Shah [14], will be adopted to evaluate the fuzzy deflection at pile top and compared with the above perturbation-based method in a numerical example.

It is noted that, the number of finite-element solutions will increase (total 2^N deterministic finite element analyses for each membership level).

5. NUMERICAL EXAMPLE

Consider the same pile as in the second example in section 3. However, the soil properties are uncertain and given by triangular fuzzy numbers. Three cases are examined:

Case 1. Only soil parameters of layer 1 are fuzzy, while other layers have non-fuzzy properties: $k_1 = (33.6, 56.0, 78.4)\,\text{MPa}$, $t_1 = (6.6, 11.0, 15.4)\,\text{MN}$, other soil parameters are the same as the deterministic example.

Case 2. Soil parameters of the two upper layers are fuzzy, while other layers have non-fuzzy properties: $k_1 = (33.6, 56.0, 78.4)\,\text{MPa}$, $k_2 = (84.0, 140.0, 196.0)\,\text{MPa}$, and $t_1 = (6.6, 11.0, 15.4)\,\text{MN}$, $t_2 = (16.8, 28.0, 39.2)\,\text{MN}$.

Case 3. All soil parameters are fuzzy: $k_1 = (33.6, 56.0, 78.4)\,\text{MPa}$, $k_2 = (84.0, 140.0, 196.0)\,\text{MPa}$, $k_3 = (93.0, 155.0, 217.0)\,\text{MPa}$ and $k_4 = (120.0, 200.0, 280.0)\,\text{MPa}$, and $t_1 = (6.6, 11.0, 15.4)\,\text{MN}$, $t_2 = (16.8, 28.0, 39.2)\,\text{MN}$, $t_3 = (24.0, 40.0, 56.0)\,\text{MN}$ and $t_4 = (36.0, 60.0, 84.0)\,\text{MN}$.

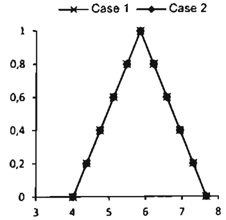
In all three cases, each fuzzy parameter has the relative variation at different levels of membership with respect to the clear value at the membership of 1 not exceed 40%.

A finite-element model of forty elements with equal length 0.5 m is used for the analysis. The results for membership function of pile top deflection in three cases are given in Tab. 3. In comparison with *case* 1, *case* 2 shows very small variation of the membership function, and case 3 gives almost the same results as case 2 (Fig. 2(a) and Tab. 3). It implies that the fuzziness of pile top deflection depends largely on the properties of the first soil layer and the variation of soil parameters of lower layers has insignificant influence on the pile behavior.

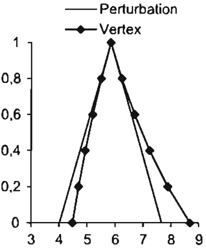
On the other hand, different results are obtained by the two methods, which can also be seen in Fig. 2(b). The vertex method gives exact bounds of the deflection in each membership level, while the perturbation method produces approximate results. At the membership level $\alpha = 0$, difference between the results of the perturbation analysis and those of vertex analysis is about 13% (comparison is made for the support width of membership functions). Nevertheless, for relatively small variation of the soil parameters, the perturbation results and vertex results are basically consistent. When membership $\alpha \geq 0.4$ (in this case, the relative change of fuzzy variables with respect to clear value at membership of 1 less than 25%), the support width of perturbation results and vertex results differ not more than 5%. With the increase in membership, the accuracy of perturbation results corresponding to the membership levels also increase, because with the increase in membership, the relative variation of fuzzy parameters is reduced, improving the accuracy of the calculation, which is the characteristics of perturbation method.

Table 3. Top deflection (10^{-3} m) in different membership levels

	α	Case 1	Case 2	Case 3
Perturbation-based analysis	1	5.8427	5.8427	5.8427
	0.8	[5.4778, 6.2077]	[5.4768, 6.2087]	[5.4768, 6.2087]
	0.6	[5.1128, 6.5726]	[5.1107, 6.5747]	[5.1107, 6.5747]
	0.4	[4.7479, 6.9376]	[4.7448, 6.9407]	[4.7448, 6.9407]
	0.2	[4.3829, 7.3026]	[4.3788, 7.3067]	[4.3788, 7.3067]
	0	[4.0180, 7.6675]	[4.0128, 7.6727]	[4.0128, 7.6727]
Vertex analysis	1	5.8427	5.8427	5.8427
	0.8	[5.5015, 6.2352]	[5.5006, 6.2364]	[5.5006, 6.2364]
	0.6	[5.2018, 6.6921]	[5.2003, 6.6951]	[5.2003, 6.6951]
	0.4	[4.9362, 7.2318]	[4.9343, 7.2373]	[4.9343, 7.2373]
	0.2	[4.6989, 7.8806]	[4.6968, 7.8896]	[4.6968, 7.8896]
	0	[4.4855, 8.6778]	[4.4833, 8.6917]	[4.4833, 8.6917]



(a)



(b)

Fig. 2. Membership function of top deflection (10^{-3} m)

Using the proposed perturbation method, the envelope of the pile deflection, which is the possible minimum and maximum deflections along pile length, can also be easily

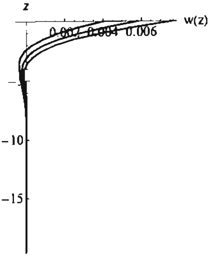


Fig. 3. Envelope of deflection along pile (mm)-Case 3

obtained as in Fig. 3. Pile deflection determined with main values of soil parameters is also plotted in the same figure. This gives the picture of the variation of pile behavior under uncertain soil conditions.

6. CONCLUSION

This paper has presented a fuzzy analysis method for laterally-loaded pile in multi-layered soil. The pile is idealized as one-dimensional beam and the soil as two-parameter elastic foundation model. A fast fuzzy finite element algorithm was developed using the perturbation technique. This solving procedure is similar with the conventional finite element method and in principle does not require solving a large number of finite-element equations as often found in the optimization strategy.

The method was established for the analysis of the pile behavior considering fuzziness in soil parameters. Numerical results show that the variation of the top soil layer properties has large influence on the pile deflection, while the fuzziness of lower layers has (almost) no impact. When the variation of soil parameters is small, the results are generally consistent with the results of vertex method. In this case, the fuzzy analysis method in this paper provides a feasible way for a reasonable solution to practical engineering analysis and design problems.

REFERENCES

[1] H. Fan and R. Liang. Application of Monte Carlo simulation to laterally loaded piles. *Geo-Congress 2012*, (2012), pp. 376–386.
[2] H. Fan and R. Liang. Performance-based reliability analysis of laterally loaded drilled shafts. *Journal of Geotechnical and Geoenvironmental Engineering*, **139**, (12), (2013), pp. 2020–2027.
[3] H. Fan and R. Liang. Reliability-based design of laterally loaded piles considering soil spatial variability In *Foundation Engineering in the Face of Uncertainty*. ASCE, (2013), pp. 475–486.

[4] C. L. Chan and B. K. Low. Reliability analysis of laterally loaded piles involving nonlinear soil and pile behavior. *Journal of Geotechnical and Geoenvironmental Engineering*, **135**, (3), (2009), pp. 431–443.
[5] V. Tandjiria, C. I. Teh, and B. K. Low. Reliability analysis of laterally loaded piles using response surface methods. *Structural Safety*, **22**, (4), (2000), pp. 335–355.
[6] C. L. Chan and B. K. Low. Probabilistic analysis of laterally loaded piles using response surface and neural network approaches. *Computers and Geotechnics*, **43**, (2012), pp. 101–110.
[7] L. A. Zadeh. Fuzzy set. *Information Control*, **8**, (1), (1965), pp. 338–353.
[8] D. Basu and R. Salgado. Elastic analysis of laterally loaded pile in multi-layered soil. *Geomechanics and Geoengineering: An International Journal*, **2**, (3), (2007), pp. 183–196.
[9] V. Z. Vlasov and N. N. Leont'ev. *Beams, plates and shells on elastic foundations*. Israel Program for Scientific Translations, Jerusalem, (1966).
[10] L. C. Reese and W. F. Van Impe. *Single piles and pile groups under lateral loading*. A.A. Balkema: Rotterdam, Netherlands, (2001).
[11] C. L. Chan. *Reliability assessment of laterally loaded single piles*. PhD thesis, Nanyang Technological University, Singapore, (2012).
[12] D. Basu, R. Salgado, and M. Prezzi. *Analysis of laterally loaded piles in multilayered soil deposits*. Publication FHWA/IN/JTRP-2007/23, Joint Transportation Research Program, Department of Transportation and Purdue University, West Lafayette, Indiana, (2008).
[13] P. H. Anh. Analytical solution for beams on elastic foundation with complex conditions. *J. Science and Technology in Civil Eng., NUCE*, **18**, (2014), pp. 59–65.
[14] W. Dong and H. C. Shah. Vertex method for computing functions of fuzzy variables. *Fuzzy sets and Systems*, **24**, (1), (1987), pp. 65–78.