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EXPLICIT HOMOGENIZED EQUATIONS OF THE PIEZOELECTRICITY THEORY IN A TWO-DIMENSIONAL DOMAIN WITH A VERY ROUGH INTERFACE OF COMB-TYPE

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Abstract. The main purpose of this paper is to derive explicit homogenized equations of the linear piezoelectricity in two-dimensional domains separated by a very rough interface of comb-type. In order to do that, first, the basic equations of the theory of piezoelectricity are written down in matrix form. Then, following the techniques presented recently by these authors, the explicit homogenized equation in matrix form and the associate continuity condition, for generally anisotropic piezoelectric materials, are derived. They are then written down in component form for a special case when the solids are made of tetragonal crystals of class $\bar{4}2m$. Since the obtained equations are totally explicit, they are significant in use.

Keywords: Homogenization, very rough interfaces, homogenized equations, piezoelectricity.

1. INTRODUCTION

Boundary-value problems in domains with rough boundaries or interfaces appear in many fields of natural sciences and technology such as: scattering of waves on rough boundaries [1, 4], transmission and reflection of waves on rough interfaces [5]-[8], mechanical problems concerning the plates with densely spaced stiffeners [9], flows over rough walls [10], vibrations of strongly inhomogeneous elastic bodies [11], propagations of surface waves in half-spaces with cracked surfaces [12]-[13], nearly circular holes and inclusions in plane elasticity and thermoelasticity [14]-[15], and so on. When the amplitude (height) of the roughness is much small in comparison with its period, the problems are usually analyzed by perturbation methods [16]. When the amplitude is much larger than its period, i.e. the boundaries and interfaces are very rough, the homogenization method is required, see for instance: [17]-[19]. The mentioned above boundary-value problems are originated from various physical theories.

For the elasticity theory, Nevard and Keller [20] examined the homogenization of a very rough three-dimensional interface that oscillates between two parallel planes and separates two linear anisotropic solids. By applying the homogenization method, the authors have derived the homogenized equations, but these equations are still implicit. They are therefore not convenient in use. In some recent papers [21, 22, 23], the explicit homogenized equations of the linear elasticity in two-dimensional domains with interfaces rapidly oscillating between two parallel straight lines and between two concentric circles have been obtained.

Because piezoelectric materials exhibit electromechanical coupling phenomenon, they have been widely used in various fields of the modern engineering, such as the field of electroacoustics, transducers and control of structure vibration, etc (see [24]). The consideration the boundary-value problems of the piezoelectricity theory in domains with very rough boundaries or interfaces is therefore significant and of great theoretical and practical as well interest.

The main aim of this paper is to find explicit homogenized equations of the linear theory of piezoelectricity in two-dimensional domains including a very rough interface which is assumed to be of the comb-type (see Fig. 1). Note that interfaces and boundaries

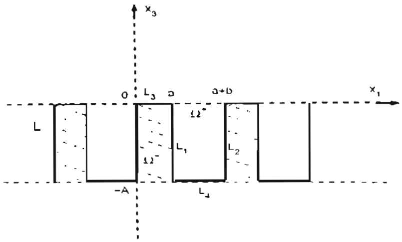

Fig. 1. Two-dimensional domains Ω^+ and Ω^- are separated by a very rough interface L of comb-type $(0 < a + b << A)$.

of the comb-type have been the subject of many investigations of wave scattering and wave reflection/transmission. see, for examples, studies [25, 26, 27] and references therein. To derive the explicit homogenized equation, first, the basic equations of the linear theory of piezoelectricity are written down in matrix form. Then, following the techniques presented recently by these authors [21, 22, 23], the homogenized equation and the associate continuity condition in explicit form, for generally anisotropic piezoelectric materials, are derived. They are written down in component form, as an example, for a special case when the solids are made of tetragonal crystals of class $\bar{4}2m$.

2. BASIC EQUATIONS

Consider a linear piezoelectric body occupied two-dimensional domains Ω^+ and Ω^- of the plane x_1x_3 whose interface is the curve L of comb-type as illustrated in Fig. 1. We consider the generalized plane strain (see [28]) for which the displacement components u_1, u_2, u_3 and the electric potential ϕ are of the form

$$u_1 = u_1(x_1, x_3, t), \quad u_2 = u_2(x_1, x_3, t), \quad u_3 = u_3(x_1, x_3, t), \quad \phi = \phi(x_1, x_3, t) \tag{1}$$

The strain ε_{ij} and the components of the electric field vector E_i are expressed as [29, 30, 31]

$$\varepsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \qquad E_i = -\frac{\partial \phi}{\partial x_i} \tag{2}$$

The stress σ_{ij} and the components of the electric displacement vector D_i are related to the strains ε_{ij} and the components E_i of the electric field vector by the following relations [29, 31 32]

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} - e_{lij}E_l, \qquad D_i = e_{ikl}\varepsilon_{kl} + \epsilon_{il}E_l \tag{3}$$

where commas indicate differentiation with respect to x_i, c_{ijkl}, e_{ijk} and ϵ_{ij} are respectively the elastic (measured in a constant electric field), piezoelectric (measured at a constant strain or electric field) and the dielectric(measured at a constant strain) moduli which have the following classical properties of symmetry

$$c_{jikl} = c_{ijkl} = c_{klij}, \quad e_{kij} = e_{kji}, \quad \epsilon_{ij} = \epsilon_{ji} \tag{4}$$

and they are defined as

$$c_{ijkl}, \; e_{ijk}, \; \epsilon_{ij} = \begin{cases} c_{ijkl+}, \; e_{ijk+}, \; \epsilon_{ij+} \text{ for } (x_1, x_3) \in \Omega_+ \\ c_{ij-}, \; e_{ijk-}, \; \epsilon_{ij-} \text{ for } (x_1, x_3) \in \Omega_- \end{cases} \tag{5}$$

c_{ijkl+}, e_{ijk+}, ϵ_{ij+}, c_{ijkl-}, e_{ijk-}, ϵ_{ij-} are constant.

Using the Voigt contracted notations, the relations (3) are written in matrix form as (see [29])

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \\ D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & e_{11} & e_{21} & e_{31} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} & e_{12} & e_{22} & e_{33} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} & e_{13} & e_{23} & e_{33} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} & e_{14} & e_{24} & e_{34} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} & e_{15} & e_{25} & e_{35} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} & e_{16} & e_{26} & e_{36} \\ e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} & -\epsilon_{11} & -\epsilon_{12} & -\epsilon_{13} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} & -\epsilon_{12} & -\epsilon_{22} & -\epsilon_{23} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} & -\epsilon_{13} & -\epsilon_{23} & -\epsilon_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \\ \phi_{,1} \\ \phi_{,2} \\ \phi_{,3} \end{bmatrix} \tag{6}$$

Equations of motion and Gauss's law are [29, 31]

$$
\begin{aligned}
\sigma_{11,1} + \sigma_{13,3} + f_1 &= \rho \ddot{u}_1 \\
\sigma_{12,1} + \sigma_{23,3} + f_2 &= \rho \ddot{u}_2 \\
\sigma_{13,1} + \sigma_{33,3} + f_3 &= \rho \ddot{u}_3 \\
D_{1,1} + D_{3,3} - q &= 0
\end{aligned}
\tag{7}
$$

for $(x_1, x_3) \notin L$, where ρ is the mass density (taking different constants ρ^+, ρ^- in Ω^+ and Ω^-, respectively), f_1, f_2 and f_3 are the components of body forces, q is the electric charge density, a dot indicates differentiation with respect to the time t. In addition to Eqs. (7) is required the continuity condition on the interface L, namely

$$[u_k]_L = 0 \, (k = 1, 2, 3), [\phi]_L = 0, \ [\Sigma_{nk}]_L = 0 \ (k = 1, 2, 3), \ [D_n]_L = 0 \tag{8}$$

where $[w]_L = w_+ - w_-$, and

$$\Sigma_{nk} = \sigma_{k1} n_1 + \sigma_{k3} n_3 \ (k = 1, 2, 3), \ D_n = D_1 n_1 + D_3 n_3 \tag{9}$$

n_k are the components of the unit normal to the curve L.

3. EXPLICIT HOMOGENIZED EQUATION IN MATRIX FORM

Using (6) in (7) and taking into account (1) and (2) yield a system of equations for the displacement components and the electric potential whose matrix form is

$$(\mathbf{A}_{hk} \mathbf{u}_{,k})_{,h} + \mathbf{F} = \rho \mathbf{I} \ddot{\mathbf{u}} \tag{10}$$

where $\mathbf{u} = [u_1, u_2, u_3, \phi]^T$, $\mathbf{F} = [f_1, f_2, f_3, -q]^T$ and

$$
\mathbf{A}_{11} = \begin{bmatrix} c_{11} & c_{16} & c_{15} & e_{11} \\ c_{16} & c_{66} & c_{56} & e_{16} \\ c_{15} & c_{56} & c_{55} & e_{15} \\ e_{11} & e_{16} & e_{15} & -\epsilon_{11} \end{bmatrix}, \quad
\mathbf{A}_{13} = \begin{bmatrix} c_{15} & c_{14} & c_{13} & e_{31} \\ c_{56} & c_{46} & c_{36} & e_{36} \\ c_{55} & c_{45} & c_{35} & e_{35} \\ e_{15} & e_{14} & e_{13} & -\epsilon_{13} \end{bmatrix}
$$

$$
\mathbf{A}_{31} = \begin{bmatrix} c_{15} & c_{56} & c_{55} & e_{15} \\ c_{14} & c_{46} & c_{45} & e_{14} \\ c_{13} & c_{36} & c_{35} & e_{13} \\ e_{31} & e_{36} & e_{35} & -\epsilon_{13} \end{bmatrix}, \quad
\mathbf{A}_{33} = \begin{bmatrix} c_{55} & c_{45} & c_{35} & e_{35} \\ c_{45} & c_{44} & c_{34} & e_{34} \\ c_{35} & c_{34} & c_{33} & e_{33} \\ e_{35} & e_{34} & e_{33} & -\epsilon_{33} \end{bmatrix}
\tag{11}
$$

$$
\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

here, symbol "T" indicates the transpose of a matrix. In addition to Eq. (10) is required the continuity condition on L, namely

$$[\mathbf{u}]_L = 0 \tag{12}$$

$$\Big[(\mathbf{A}_{11} \mathbf{u}_{,1} + \mathbf{A}_{13} \mathbf{u}_{,3}) n_1 + (\mathbf{A}_{31} \mathbf{u}_{,1} + \mathbf{A}_{33} \mathbf{u}_{,3}) n_3 \Big]_L = 0 \tag{13}$$

which originate from (8).

Following Bensoussan et al. [18], Bakhvalov & Panasenko [19], Kohler et al. [17] we suppose that $\mathbf{u}(x_1, x_3, t, \epsilon)=\mathbf{U}(x_1, y, x_3, t, \varepsilon)$, and we express $\mathbf{U}$ as follows (see [21, 22, 23])

$$\begin{aligned}\mathbf{U} &= \mathbf{V} + \varepsilon(\mathbf{N}^1\mathbf{V} + \mathbf{N}^{11}\mathbf{V}_{,1} + \mathbf{N}^{13}\mathbf{V}_{,3}) + \varepsilon^2(\mathbf{N}^2\mathbf{V} + \mathbf{N}^{21}\mathbf{V}_{,1} + \mathbf{N}^{23}\mathbf{V}_{,3} \\ &+ \mathbf{N}^{211}\mathbf{V}_{,11} + \mathbf{N}^{213}\mathbf{V}_{,13} + \mathbf{N}^{233}\mathbf{V}_{,33}) + O(\varepsilon^3)\end{aligned} \tag{14}$$

where $\mathbf{V}=\mathbf{V}(x_1, x_3, t)$ (being independent of y), $\mathbf{N}^1$, $\mathbf{N}^{11}$, $\mathbf{N}^{13}$, $\mathbf{N}^2$, $\mathbf{N}^{21}$, $\mathbf{N}^{23}$, $\mathbf{N}^{211}$, $\mathbf{N}^{213}$, $\mathbf{N}^{233}$ are 4×4-matrix functions of y and x_3 (not depending on x_1, t), and they are y-periodic with the period 1. The matrix functions $\mathbf{N}^1$, ..., $\mathbf{N}^{233}$ are determined so that equation (10) and boundary conditions (12) and (13) are satisfied.

Our main purpose is to find the explicit homogenized equation of the value-boundary problem (10), (12) and (13), i. e. the equation for the leading term $\mathbf{V} = [V_1, V_2, V_3, \Phi]^T$ in the asymptotic expansion (14), and the associate continuity conditions. Following the same procedure as was carried out in [21, 22, 23], the explicit homogenized equation of the problem (10), (12) and (13) in matrix form is

$$\begin{gathered}(\mathbf{A}^+_{hk}\mathbf{V}_{,k})_{,h} + \mathbf{F}^+ = \rho^+\mathbf{I}\ddot{\mathbf{V}}, \ x_3 > 0 \\ \langle\mathbf{A}_{11}^{-1}\rangle^{-1}\mathbf{V}_{,11} + \left[\langle\mathbf{A}_{11}^{-1}\rangle^{-1}\langle\mathbf{A}_{11}^{-1}\mathbf{A}_{13}\rangle + \langle\mathbf{A}_{31}\mathbf{A}_{11}^{-1}\rangle\langle\mathbf{A}_{11}^{-1}\rangle^{-1}\right]\mathbf{V}_{,13} \\ +\left[\langle\mathbf{A}_{33}\rangle + \langle\mathbf{A}_{31}\mathbf{A}_{11}^{-1}\rangle\langle\mathbf{A}_{11}^{-1}\rangle^{-1}\langle\mathbf{A}_{11}^{-1}\mathbf{A}_{13}\rangle - \langle\mathbf{A}_{31}\mathbf{A}_{11}^{-1}\mathbf{A}_{13}\rangle\right]\mathbf{V}_{,33} \\ +\langle\mathbf{F}\rangle = \langle\rho\rangle\mathbf{I}\ddot{\mathbf{V}}, \ -A < x_3 < 0 \\ (\mathbf{A}^-_{hk}\mathbf{V}_{,k})_{,h} + \mathbf{F}^- = \rho^-\mathbf{I}\ddot{\mathbf{V}}, \ x_3 < -A\end{gathered} \tag{15}$$

and the continuity conditions on the straight lines $x_3 = 0$ and $x_3 = -A$ have the form

$$\begin{aligned}&\left[\langle\mathbf{A}_{31}\mathbf{A}_{11}^{-1}\rangle\langle\mathbf{A}_{11}^{-1}\rangle^{-1}\mathbf{V}_{,1} + \left(\langle\mathbf{A}_{33}\rangle + \langle\mathbf{A}_{31}\mathbf{A}_{11}^{-1}\rangle\langle\mathbf{A}_{11}^{-1}\rangle^{-1}\langle\mathbf{A}_{11}^{-1}\mathbf{A}_{13}\rangle\right.\right. \\ &\left.\left. - \langle\mathbf{A}_{31}\mathbf{A}_{11}^{-1}\mathbf{A}_{13}\rangle\right)\mathbf{V}_{,3}\right]_{L^*} = 0, \ [\mathbf{V}|_{L^*} = 0\end{aligned} \tag{16}$$

where L^* is lines: $x_3 = 0, x_3 = -A$.
Here, matrices A_{hk} are given by (11), and

$$\langle g\rangle = \frac{1}{a+b}(ag_- + bg_+) \tag{17}$$

where g_+ and g_- are the values of g in Ω^+ and Ω^-, respectively (see Fig. 1). Note that due to the positive definiteness of the strain energy, $\det\mathbf{A}_{11} \neq 0$, the matrix $\mathbf{A}_{11}^{-1}$ therefore exists.

Equations of motion and Gauss's law are [29, 31]

$$\begin{aligned} \sigma_{11,1} + \sigma_{13,3} + f_1 &= \rho\ddot{u}_1 \\ \sigma_{12,1} + \sigma_{23,3} + f_2 &= \rho\ddot{u}_2 \\ \sigma_{13,1} + \sigma_{33,3} + f_3 &= \rho\ddot{u}_3 \\ D_{1,1} + D_{3,3} - q &= 0 \end{aligned} \tag{7}$$

for $(x_1, x_3) \notin L$, where ρ is the mass density (taking different constants ρ^+, ρ^- in Ω^+ and Ω^-, respectively), f_1, f_2 and f_3 are the components of body forces, q is the electric charge density, a dot indicates differentiation with respect to the time t. In addition to Eqs. (7) is required the continuity condition on the interface L, namely

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where $[w]_L = w_+ - w_-$, and

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n_k are the components of the unit normal to the curve L.

3. EXPLICIT HOMOGENIZED EQUATION IN MATRIX FORM

Using (6) in (7) and taking into account (1) and (2) yield a system of equations for the displacement components and the electric potential whose matrix form is

$$(\mathbf{A}_{hk}\mathbf{u}_{,k})_{,h} + \mathbf{F} = \rho\mathbf{I}\ddot{\mathbf{u}} \tag{10}$$

where $\mathbf{u} = [u_1, u_2, u_3, \phi]^T$, $\mathbf{F} = [f_1, f_2, f_3, -q]^T$ and

$$\mathbf{A}_{11} = \begin{bmatrix} c_{11} & c_{16} & c_{15} & e_{11} \\ c_{16} & c_{66} & c_{56} & e_{16} \\ c_{15} & c_{56} & c_{55} & e_{15} \\ e_{11} & e_{16} & e_{15} & -\epsilon_{11} \end{bmatrix}, \quad \mathbf{A}_{13} = \begin{bmatrix} c_{15} & c_{14} & c_{13} & e_{31} \\ c_{56} & c_{46} & c_{36} & e_{36} \\ c_{55} & c_{45} & c_{35} & e_{35} \\ e_{15} & e_{14} & e_{13} & -\epsilon_{13} \end{bmatrix}$$

$$\mathbf{A}_{31} = \begin{bmatrix} c_{15} & c_{56} & c_{55} & e_{15} \\ c_{14} & c_{46} & c_{45} & e_{14} \\ c_{13} & c_{36} & c_{35} & e_{13} \\ e_{31} & e_{36} & e_{35} & -\epsilon_{13} \end{bmatrix}, \quad \mathbf{A}_{33} = \begin{bmatrix} c_{55} & c_{45} & c_{35} & e_{35} \\ c_{45} & c_{44} & c_{34} & e_{34} \\ c_{35} & c_{34} & c_{33} & e_{33} \\ e_{35} & e_{34} & e_{33} & -\epsilon_{33} \end{bmatrix} \tag{11}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

here, symbol "T" indicates the transpose of a matrix. In addition to Eq. (10) is required the continuity condition on L, namely

$$[\mathbf{u}]_L = 0 \tag{12}$$

$$\Big[(\mathbf{A}_{11}\mathbf{u}_{,1} + \mathbf{A}_{13}\mathbf{u}_{,3}) n_1 + (\mathbf{A}_{31}\mathbf{u}_{,1} + \mathbf{A}_{33}\mathbf{u}_{,3}) n_3 \Big]_L = 0 \tag{13}$$

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$$\begin{aligned}\mathbf{U} &= \mathbf{V} + \varepsilon(\mathbf{N}^1\mathbf{V} + \mathbf{N}^{11}\mathbf{V}_{,1} + \mathbf{N}^{13}\mathbf{V}_{,3}) + \varepsilon^2(\mathbf{N}^2\mathbf{V} + \mathbf{N}^{21}\mathbf{V}_{,1} + \mathbf{N}^{23}\mathbf{V}_{,3} \\ &+ \mathbf{N}^{211}\mathbf{V}_{,11} + \mathbf{N}^{213}\mathbf{V}_{,13} + \mathbf{N}^{233}\mathbf{V}_{,33}) + O(\varepsilon^3)\end{aligned} \tag{14}$$

where $\mathbf{V} = \mathbf{V}(x_1, x_3, t)$ (being independent of y), $\mathbf{N}^1$, $\mathbf{N}^{11}$, $\mathbf{N}^{13}$, $\mathbf{N}^2$, $\mathbf{N}^{21}$, $\mathbf{N}^{23}$, $\mathbf{N}^{211}$, $\mathbf{N}^{213}$, $\mathbf{N}^{233}$ are 4×4-matrix functions of y and x_3 (not depending on x_1, t). and they are y-periodic with the period 1. The matrix functions $\mathbf{N}^1$, ..., $\mathbf{N}^{233}$ are determined so that equation (10) and boundary conditions (12) and (13) are satisfied.

Our main purpose is to find the explicit homogenized equation of the value-boundary problem (10), (12) and (13), i. e. the equation for the leading term $\mathbf{V} = [V_1, V_2, V_3, \Phi]^T$ in the asymptotic expansion (14), and the associate continuity conditions. Following the same procedure as was carried out in [21, 22, 23], the explicit homogenized equation of the problem (10), (12) and (13) in matrix form is

$$\begin{gathered}(\mathbf{A}^+_{hk}\mathbf{V}_{,k})_{,h} + \mathbf{F}^+ = \rho^+\mathbf{I}\ddot{\mathbf{V}}, \ x_3 > 0 \\ \langle\mathbf{A}_{11}^{-1}\rangle^{-1}\mathbf{V}_{,11} + \left[\langle\mathbf{A}_{11}^{-1}\rangle^{-1}\langle\mathbf{A}_{11}^{-1}\mathbf{A}_{13}\rangle + \langle\mathbf{A}_{31}\mathbf{A}_{11}^{-1}\rangle\langle\mathbf{A}_{11}^{-1}\rangle^{-1}\right]\mathbf{V}_{,13} \\ + \left[\langle\mathbf{A}_{33}\rangle + \langle\mathbf{A}_{31}\mathbf{A}_{11}^{-1}\rangle\langle\mathbf{A}_{11}^{-1}\rangle^{-1}\langle\mathbf{A}_{11}^{-1}\mathbf{A}_{13}\rangle - \langle\mathbf{A}_{31}\mathbf{A}_{11}^{-1}\mathbf{A}_{13}\rangle\right]\mathbf{V}_{,33} \\ + \langle\mathbf{F}\rangle = \langle\rho\rangle\mathbf{I}\ddot{\mathbf{V}}, \ -A < x_3 < 0 \\ (\mathbf{A}^-_{hk}\mathbf{V}_{,k})_{,h} + \mathbf{F}^- = \rho^-\mathbf{I}\ddot{\mathbf{V}}, \ x_3 < -A\end{gathered} \tag{15}$$

and the continuity conditions on the straight lines $x_3 = 0$ and $x_3 = -A$ have the form

$$\begin{aligned}&\Bigg[\langle\mathbf{A}_{31}\mathbf{A}_{11}^{-1}\rangle\langle\mathbf{A}_{11}^{-1}\rangle^{-1}\mathbf{V}_{,1} + \Big(\langle\mathbf{A}_{33}\rangle + \langle\mathbf{A}_{31}\mathbf{A}_{11}^{-1}\rangle\langle\mathbf{A}_{11}^{-1}\rangle^{-1}\langle\mathbf{A}_{11}^{-1}\mathbf{A}_{13}\rangle \\ &- \langle\mathbf{A}_{31}\mathbf{A}_{11}^{-1}\mathbf{A}_{13}\rangle\Big)\mathbf{V}_{,3}\Bigg]_{L^*} = 0, \ |\mathbf{V}|_{L^*} = 0\end{aligned} \tag{16}$$

where L^* is lines: $x_3 = 0, x_3 = -A$.
Here, matrices A_{hk} are given by (11), and

$$\langle g\rangle = \frac{1}{a+b}(ag_- + bg_+) \tag{17}$$

where g_+ and g_- are the values of g in Ω^+ and Ω^-, respectively (see Fig. 1). Note that due to the positive definiteness of the strain energy, $\det\mathbf{A}_{11} \neq 0$, the matrix $\mathbf{A}_{11}^{-1}$ therefore exists.

4. EXPLICIT HOMOGENIZED EQUATIONS IN COMPONENT FORM FOR TETRAGONAL CRYSTALS OF CLASS $\bar{4}2m$

Consider tetragonal crystals of class $\bar{4}2m$ (see [29]), for which the matrices A_{hk} are of the form

$$\mathbf{A}_{11} = \begin{bmatrix} c_{11} & 0 & 0 & 0 \\ 0 & c_{66} & 0 & 0 \\ 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & -\epsilon_{11} \end{bmatrix}, \quad \mathbf{A}_{13} = \begin{bmatrix} 0 & 0 & c_{13} & 0 \\ 0 & 0 & 0 & e_{36} \\ c_{55} & 0 & 0 & 0 \\ 0 & e_{14} & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}_{31} = \begin{bmatrix} 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & e_{14} \\ c_{13} & 0 & 0 & 0 \\ 0 & e_{36} & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_{33} = \begin{bmatrix} c_{55} & 0 & 0 & 0 \\ 0 & c_{55} & 0 & 0 \\ 0 & 0 & c_{33} & 0 \\ 0 & 0 & 0 & -\epsilon_{33} \end{bmatrix} \tag{18}$$

From (6) one can see that

$$\begin{aligned} &\sigma_{13} = c_{55}(u_{1,3} + u_{3,1}), \ \sigma_{23} = e_{14}\phi_{,1} + c_{55}u_{2,3} \\ &\sigma_{33} = c_{13}u_{1,1} + c_{33}u_{3,3}, \ D_3 = e_{36}u_{2,1} - \epsilon_{33}\phi_{,3} \end{aligned} \tag{19}$$

Substituting (18) into (15), (16) and after some manipulations, we obtain the explicit homogenized equations in component form and the associate continuity conditions, namely
- For $x_3 > 0$

$$\begin{cases} c_{11+}V_{1,11} + c_{55+}V_{1,33} + (c_{13+} + c_{55+})V_{3,13} + f_{1+} = \rho^+\ddot{V}_1 \\ c_{66+}V_{2,11} + c_{55+}V_{2,33} + (e_{14+} + e_{36+})\Phi_{,13} + f_{2+} = \rho^+\ddot{V}_2 \\ c_{55+}V_{3,11} + c_{33+}V_{3,33} + (c_{13+} + c_{55+})V_{1,13} + f_{3+} = \rho^+\ddot{V}_3 \\ (e_{14+} + e_{36+})V_{2,13} - \epsilon_{11+}\Phi_{,11} - \epsilon_{33+}\Phi_{,33} - q_+ = 0 \end{cases} \tag{20}$$

- For $-A < x_3 < 0$

$$\begin{aligned}
&\langle c_{11}^{-1}\rangle^{-1}V_{1,11} + \Big[\langle c_{55}^{-1}\rangle^{-1} + \langle c_{13}c_{11}^{-1}\rangle\langle c_{11}^{-1}\rangle^{-1}\Big]V_{3,13} + \langle c_{55}^{-1}\rangle^{-1}V_{1,33} + \langle f_1\rangle = \langle\rho\rangle\ddot{V}_1 \\
&\langle c_{66}^{-1}\rangle^{-1}V_{2,11} + \Big[\langle e_{36}c_{66}^{-1}\rangle\langle c_{66}^{-1}\rangle^{-1} + \langle e_{14}\epsilon_{11}^{-1}\rangle\langle\epsilon_{11}^{-1}\rangle^{-1}\Big]\Phi_{,13} + \Big[\langle c_{55}\rangle + \langle e_{14}^2\epsilon_{11}^{-1}\rangle \\
&- \langle e_{14}\epsilon_{11}^{-1}\rangle^2\langle\epsilon_{11}^{-1}\rangle^{-1}\Big]V_{2,33} + \langle f_2\rangle = \langle\rho\rangle\ddot{V}_2 \\
&\langle c_{55}^{-1}\rangle^{-1}V_{3,11} + \Big[\langle c_{55}^{-1}\rangle^{-1} + \langle c_{13}c_{11}^{-1}\rangle\langle c_{11}^{-1}\rangle^{-1}\Big]V_{1,13} + \Big[\langle c_{13}c_{11}^{-1}\rangle^2\langle c_{11}^{-1}\rangle^{-1} \\
&- \langle c_{11}^{-1}c_{13}^2\rangle + \langle c_{33}\rangle\Big]V_{3,33} + \langle f_3\rangle = \langle\rho\rangle\ddot{V}_3 \\
&- \langle\epsilon_{11}^{-1}\rangle^{-1}\Phi_{,11} + \Big[\langle e_{14}\epsilon_{11}^{-1}\rangle\langle\epsilon_{11}^{-1}\rangle^{-1} + \langle e_{36}c_{66}^{-1}\rangle\langle c_{66}^{-1}\rangle^{-1}\Big]V_{2,13} \\
&+ \Big[\langle e_{36}c_{66}^{-1}\rangle^2\langle c_{66}^{-1}\rangle^{-1} - \langle e_{36}^2c_{66}^{-1}\rangle - \langle\epsilon_{33}\rangle\Big]\Phi_{,33} - \langle q\rangle = 0
\end{aligned} \tag{21}$$

- For $x_3 < -A$

$$\begin{cases} c_{11-}V_{1,11} + c_{55-}V_{1,33} + (c_{13-} + c_{55-})V_{3,13} + f_{1-} = \rho^- \ddot{V}_1 \\ c_{66-}V_{2,11} + c_{55-}V_{2,33} + (e_{14-} + e_{36-})\Phi_{,13} + f_{2-} = \rho^- \ddot{V}_2 \\ c_{55-}V_{3,11} + c_{33-}V_{3,33} + (c_{13-} + c_{55-})V_{1,13} + f_{3-} = \rho^- \ddot{V}_3 \\ (e_{14-} + e_{36-})V_{2,13} - \epsilon_{11-}\Phi_{,11} - \epsilon_{33-}\Phi_{,33} - q_- = 0 \end{cases} \tag{22}$$

and

$$V_1, V_2, V_3, \Phi, \sigma_{13}^0, \sigma_{23}^0, \sigma_{33}^0, D_3^0 \text{ are continuous on lines } x_3 = 0, x_3 = -A \tag{23}$$

where σ_{ij}^0, D_3^0 are the coefficients of ϵ^0 (i. e. they are leading terms) in their asymptotic expansions, and they are given by

$$\begin{aligned} \sigma_{13}^0 &= \langle c_{55}^{-1}\rangle^{-1}(V_{3,1} + V_{1,3}) \\ \sigma_{23}^0 &= \langle \epsilon_{11}^{-1}\rangle^{-1}\langle \epsilon_{11}^{-1}e_{14}\rangle\Phi_{,1} + \Big(\langle c_{55}\rangle + \langle \epsilon_{11}^{-1}e_{14}^2\rangle - \langle \epsilon_{11}^{-1}\rangle^{-1}\langle \epsilon_{11}^{-1}e_{14}\rangle^2\Big)V_{2,3} \\ \sigma_{33}^0 &= \langle c_{11}^{-1}c_{13}\rangle\langle c_{11}^{-1}\rangle^{-1}V_{1,1} + \Big(\langle c_{33}\rangle + \langle c_{11}^{-1}\rangle^{-1}\langle c_{11}^{-1}c_{13}\rangle^2 - \langle c_{11}^{-1}c_{13}^2\rangle\Big)V_{3,3} \\ D_3^0 &= \langle c_{66}^{-1}\rangle^{-1}\langle c_{66}^{-1}e_{36}\rangle V_{2,1} + \Big(\langle c_{66}^{-1}\rangle^{-1}\langle c_{66}^{-1}e_{36}\rangle^2 - \langle c_{66}^{-1}e_{36}^2\rangle - \langle \epsilon_{33}\rangle\Big)\Phi_{,3} \end{aligned} \tag{24}$$

It is readily to see that, when the materials of Ω^+ and Ω^- are the same, equations (20), (21) and (22) coincide with each other, and σ_{k3}^0 and D_3^0 become, respectively, σ_{k3} and D_3 given by (19). Also note that, for this case V_1 and V_3 are decoupled from V_2 and Φ.

5. CONCLUSIONS

In this paper, we consider the homogenization of a two-dimensional very rough interface of comb-type which separates two piezoelectric solids. Following the same procedure as was carried out in [21, 22, 23], the explicit homogenized equation and associate continuity conditions are derived. They are written down in component form for a special case when the solids are made of tetragonal crystals of class $\bar{4}2m$. Since obtained homogenized equations are totally explicit, they are very useful in use.

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