ON EXISTENCE OF INFINITELY MANY WEEK SOLUTIONS TO A FRACTIONAL KIRCHHOFF PROBLEM

Pham Thi Thuy[1*], Do Thi Mai Huong[2]
[1]TNU - University of Education,
[2]Thai Nguyen Pedagogical College

ABSTRACT

In this paper, we consider the following nonlocal problem:

$$\begin{cases} -(a+b||u||^2_{X_0})\mathcal{L}_K u &= f(x,u)+\lambda u \text{ in } \Omega, \\ u &= 0 \qquad\qquad \text{in } \mathbb{R}^3 \setminus \Omega, \end{cases},$$

where λ is a real parameter and Ω is an open bounded subset of R3 with Lipschitz boundary $\partial\Omega$, $s \in (3/4, 1)$, and the term f is a continuous function satisfying some suitable conditions. Using Fountain Theorem and variational method in fractional Sobolev space, we prove that there exist infinitely many weak solutions with unbounded energy to above problem.

Keywords: *Fractional Laplace equation; Fountain Theorem; Kirchhoff type problem; Cerami condition.*

Received: 02/10/2020; Revised: 30/11/2020; Published: 30/11/2020

VỀ SỰ TỒN TẠI VÔ HẠN NGHIỆM YẾU CỦA BÀI TOÁN KIRCHHOFF THỨ

Phạm Thị Thủy[1*], Đỗ Thị Mai Hương[2]
[1]Trường Đại học Sư phạm - ĐH Thái Nguyên
[2]Trường Cao đẳng Sư phạm Thái Nguyên

TÓM TẮT

Trong bài báo bày, chúng tôi nghiên cứu sự tồn tại vô hạn nghiệm yếu của bài toán Kirchhoff chứa toán tử vi tích phân:

$$\begin{cases} -(a+b||u||^2_{X_0})\mathcal{L}_K u &= f(x,u)+\lambda u \text{ in } \Omega, \\ u &= 0 \qquad\qquad \text{in } \mathbb{R}^3 \setminus \Omega, \end{cases}$$

trong đó λ là tham số thực và Ω là một miền mở bị chặn trong R3 với biên $\partial\Omega$ Lipschitz, $s \in (3/4, 1)$, f là hàm liên tục thỏa mãn một số điều kiện thích hợp. Sử dụng Định lý Fountain và phương pháp biến phân trong không gian Sobolev thứ, chúng tôi chứng minh sự tồn tại vô hạn nghiệm yếu với năng lượng không bị chặn của bài toán trên.

Từ khóa: *Toán tử Laplace thứ; định lý Fountain; bài toán kiểu Kirchhoff; điều kiện Cerami.*

Ngày nhận bài: 02/10/2020; Ngày hoàn thiện: 30/11/2020; Ngày đăng: 30/11/2020

* Corresponding author. *Email: p.thuysptn@gmail.com*
https://doi.org/10.34238/tnu-jst.3670

1 Introduction and main result

In this paper, we consider the following nonlocal problem:

$$\begin{cases} -(a+b\|u\|_{X_0}^2) & \mathcal{L}_K u = f(x,u) + \lambda u \text{ in } \Omega, \\ u & = 0 \qquad \text{in } \mathbb{R}^3 \setminus \Omega, \end{cases}$$

where λ is a real parameter, and Ω is an open bounded subset of $\mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$, $s \in (3/4, 1)$, and the term f is a continuous function verifying the conditions stated in the sequel. Moreover, a, b denote two positive real constants and

$$\|u\|_{X_0}^2 := \int_{\mathbb{R}^6} |u(x)-u(y)|^2 K(x-y)dxdy.$$

The $\mathcal{L}_K$ is the integrodifferential operator which is defined as following:

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^3} (u(x+y)+u(x-y) - 2u(x))K(y)dy, \quad x \in \mathbb{R}^3, \tag{1.1}$$

where the kernel $K : \mathbb{R}^3 \setminus \{0\} \to (0,+\infty)$ is such that

$$mK \in L^1(\mathbb{R}^3), \quad \text{where } m(x) = \min\{|x|^2, 1\}, \tag{1.2}$$

and there exists $\theta > 0$ such that

$$K(x) \geq \theta |x|^{-(3+2s)} \tag{1.3}$$

for any $x \in \mathbb{R}^3 \setminus \{0\}$. A model for K is given by the singular kernel $K(x) = |x|^{-(3+2s)}$ which gives rise to the fractional Laplace operator $-(-\Delta)^s$, that may defined (up to a normalizing constant) by the Riesz potential as follows:

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^3} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}} dy$$

for any $x \in \mathbb{R}^3$.

Definition 1. We say that $u \in X_0$ is a weak solution of problem (1.1) if

$$(a+b\|u\|_{X_0}^2) \int_{\mathbb{R}^3\times\mathbb{R}^3} (u(x)-u(y)) \times (\varphi(x)-\varphi(y))K(x-y)dxdy = \int_\Omega f(x,u(x))\varphi(x)dx + \lambda \int_\Omega u(x)\varphi(x)dx$$

for any $\varphi \in X_0$. Here, the space X_0 is defined by

$$X_0 := \{g \in X : g = 0 \text{ in } x \in \mathbb{R}^3 \setminus \Omega\},$$

where the functional space X denotes by the linear space of Lebesgue measurable functions from $\mathbb{R}^3$ to $\mathbb{R}$ such that the restriction of any function g in X to Ω belong to $L^2(\Omega)$ and the map

$$(x,y) \to (g(x)-g(y))\sqrt{K(x-y)}$$

is in $L^2((\mathbb{R}^3 \times \mathbb{R}^3) \setminus (C\Omega \times C\Omega), dxdy)$, $C\Omega := \mathbb{R}^3 \setminus \Omega$.

We denote $F(x,t) := \int_0^t f(x,\tau)d\tau$ and $G(x,t) = f(x,t)t - 4F(x,t)$, for all $(x,t) \in \Omega \times \mathbb{R}$.

We assume that $f \in C(\Omega \times \mathbb{R})$ satisfies following conditions hold:

(f_0) There exists a positive constant C such that

$$|f(x,t)| \leq C(1+|t|^{q-1}), \forall (x,t) \in \Omega \times \mathbb{R}$$

for some $q \in (4, \dfrac{6}{3-2s})$;

(f_1) $tf(x,t) \geq 0$ in $\Omega \times \mathbb{R}$;

(f_2) $\lim_{|t|\to+\infty} \dfrac{f(x,t)}{t^3} = +\infty$, uniformly in $x \in \Omega$.

(f_3) There exists $\gamma_* \geq 1$ and $W \in L^1(\Omega)$ satisfying $W(x) \geq 0$ for all $x \in \Omega$, such that

$$G(x,s) \leq \gamma_* G(x,t) + W(x) \tag{1.4}$$

for all $x \in \Omega$ and $0 \leq |s| \leq |t|$.

(f_4) There is $\delta > 0$ such that

$$F(x,t) \leq ht^2,$$

for every $x \in \Omega$ and $t \in (-\delta, \delta)$, where $h \neq 0$ is a real number.

Our result is given as follows:

Theorem 2. *Let Ω be a bounded domain in $\mathbb{R}^3$ with continuous boundary $\partial\Omega$ and $s \in (\frac{3}{4}, 1)$. Further, let $K : \mathbb{R}^3 \setminus \{0\} \to (0, +\infty)$ be a function satisfying assumptions (1.2) and (1.3). Let $f \in C(\overline{\Omega} \times \mathbb{R})$ satisfies the conditions $(f_0) - (f_4)$ and $f(x, -t) = -f(x, t)$ for all $(x, t) \in \overline{\Omega} \times \mathbb{R}$. Then, for any $\lambda \in \mathbb{R}$, the problem (1.1) has infinitely many solutions $u_j \in X_0, j \in \mathbb{N}$, whose energy $\mathcal{J}_{K,\lambda}(u_j) \to +\infty$ as $j \to +\infty$.*

In order to study problem (1.1), we consider the Euler-Lagrange equation of energy functional $\mathcal{J}_{K,\lambda} : X_0 \to \mathbb{R}$ defined as

$$\mathcal{J}_{K,\lambda}(u) := \frac{a}{2}||u||^2_{X_0} + \frac{b}{4}||u||^4_{X_0} - \frac{\lambda}{2}\int_\Omega |u(x)|^2 dx - \int_\Omega F(x, u(x))dx. \quad (1.5)$$

2 Some preliminary results

Now, we recall some basic results on the spaces X and X_0. In the sequel we set $Q = \mathbb{R}^6 \setminus \mathcal{O}$, where $\mathcal{O} = C\Omega \times C\Omega \subset \mathbb{R}^6$.

The space X is endowed with the norm defined as

$$||g||_X = ||g||_{L^2(\Omega)} + \Big(\int_Q |g(x) - g(y)|^2 K(x - y)dxdy\Big)^{1/2}. \quad (2.1)$$

It is easily seen that $||.||_X$ is a norm on X (see, for instance, [3] for a proof). Futhermore, X_0 is endowed with norm

$$||g||_{X_0} = \Big(\int_{\mathbb{R}^3\times\mathbb{R}^3} |g(x) - g(y)|^2 K(x - y)dxdy\Big)^{1/2}, \quad (2.2)$$

and $(X_0, ||.||_{X_0})$ is a Hilbert space (see [3], Lemma 7), with scalar product

$$< u, v >_{X_0} = \int_{\mathbb{R}^3\times\mathbb{R}^3} (u(x) - u(y)) \times (v(x) - v(y))K(x - y)dxdy. \quad (2.3)$$

In the following we denote $H^s(\Omega)$ the usual fractional Sobolev space endowed with norm (the so-call *Gagliardo norm*)

$$||g||_{H^s(\Omega)} = ||g||_{L^2(\Omega)} + \Big(\int_{\Omega\times\Omega} \frac{|g(x) - g(y)|^2}{|x - y|^{3+2s}}dxdy\Big)^{1/2}. \quad (2.4)$$

We recall that the space X_0 is nonempty (see Lemma 5.2 [1]). Finally, we recall that the eigenvalue problem driven by $-\mathcal{L}_K$, namely

$$\begin{cases} -\mathcal{L}_K u &= \lambda u \text{ in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^3 \setminus \Omega. \end{cases} \quad (2.5)$$

We know that (2.5) [2] possesses a divergent sequence of positive eigenvalues

$$\lambda_1 < \lambda_2 < \cdots \le \lambda_k \le \lambda_{k+1} \le \dots,$$

whose corresponding eigenfunctions will be denoted by e_k, each eigenvalue λ_k has finite multiplicity. By Proposition 9 in [2], we know that $\{e_k\}_{k\in\mathbb{N}}$ can be choosen in such a way that this sequence provides an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in X_0.

The following result due to Servadei-Valdioci which give the characteristic for embedding from X_0 into $L^\nu(\mathbb{R}^3), \nu \in [1, 2^*_s], 2^*_s = \dfrac{6}{3 - 2s}$:

Lemma 1. *[4] Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ be a function satisfying (1.2)- (1.3). Then, the following assertions hold true:*
a) if Ω is a bounded domain with continuous boundary, then embedding $X_0 \hookrightarrow L^\nu(\mathbb{R}^3)$ is compact for any $\nu \in [1, 2^*_s)$;
b) the embedding $X_0 \hookrightarrow L^\nu(\mathbb{R}^3)$ is continuous for all $\nu \in [1, 2^*_s]$.

From Lemma 1, we have embedding $X_0 \hookrightarrow L^\nu(\mathbb{R}^3)$ is continuous for all $\nu \in [1, 2^*_s]$. Then there exists the best constant

$$S_\nu = \inf_{v\in X_0, v\neq 0} \frac{\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|v(x) - v(y)|^2}{|x - y|^{3+2s}}dxdy}{\Big(\int_{\mathbb{R}^3} |v(x)|^\nu dx\Big)^{2/\nu}}. \quad (2.6)$$

We have

$$< \mathcal{J}'_{K,\lambda}(u), \varphi >= (a + b||u||^2_{X_0})\times$$
$$\int\limits_{\mathbb{R}^3\times\mathbb{R}^3} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y)dxdy$$
$$- \int\limits_{\Omega} f(x, u(x))\varphi(x)dx - \lambda \int\limits_{\Omega} u(x)\varphi(x)dx.$$

Certainly, solutions of problems (1.1) is critical point of the energy function $\mathcal{J}_{K,\lambda}$.

3 Proof of Theorem 2

In [5, 6], Cerami introduced the so-called *Cerami condition*, as a weak version of the Palais-Smale assumption. With our notation, it can be written as follows:

Cerami condition. The function $\mathcal{J}_{K,\lambda}$ satisfies the *Cerami compactness condition* at level $c \in \mathbb{R}$ if any sequence $\{u_j\}_{j\in\mathbb{N}}$ in X_0 such that $\mathcal{J}_{K,\lambda}(u_j) \to c$ and $(1 + ||u_j||_{X_0}) \sup_{||\varphi||_{X_0}=1} | < \mathcal{J}'_{K,\lambda}(u_j), \varphi > | \to 0$, admits a strongly convergent subsequence in X_0.

We show that $\mathcal{J}_{K,\lambda}$ satisfies the Cerami condition.

Lemma 2. *Let $f : \Omega\times\mathbb{R} \to \mathbb{R}$ be a function verifying conditions $(f_0) - (f_4)$. Then $\mathcal{J}_{K,\lambda}$ satisfies the Cerami condition at level $c \in \mathbb{R}$.*

Proof. Let $c \in \mathbb{R}$ and $\{u_j\}_{j\in\mathbb{N}}$ be a Cerami sequence in X_0, that is $\{u_j\}_{j\in\mathbb{N}}$ satisfying

$$\mathcal{J}_{K,\lambda}(u_j) = \frac{a}{2}||u_j||^2_{X_0} + \frac{b}{4}||u_j||^4_{X_0}$$
$$- \frac{\lambda}{2}\int\limits_{\Omega} |u(x)|^2 dx - \int\limits_{\Omega} F(x, u_j(x))dx \to c, \tag{3.1}$$

$$(1 + ||u_j||_{X_0}) \sup_{||\varphi||_{X_0}=1} \{| < \mathcal{J}'_{K,\lambda,0}(u_j), \varphi > |\} \to 0 \tag{3.2}$$

as $j \to \infty$. Hence

$$c = \mathcal{J}_{K,\lambda}(u_j) + o(1). \tag{3.3}$$

From (3.2) and (3.3), we see

$$\sup_{||\varphi||_{X_0}=1} \{| < \mathcal{J}'_{K,\lambda}(u_j), \varphi > |\} \to 0, \tag{3.4}$$

and

$$||u_j||_{X_0} \sup_{||\varphi||_{X_0}=1} \{| < \mathcal{J}'_{K,\lambda}(u_j), \varphi > |\} \to 0 \tag{3.5}$$

as $j \to +\infty$. Since

$$| < \mathcal{J}'_{K,\lambda}(u_j), u_j > |$$
$$\le ||u_j||_{X_0} \sup_{||\varphi||_{X_0}=1} \{| < \mathcal{J}'_{K,\lambda}(u_j), \varphi > |\},$$

we also have that

$$< \mathcal{J}'_{K,\lambda}(u_j), u_j >= o(1), \tag{3.6}$$

where $o(1) \to 0$, as $j \to \infty$. We have

$$4\mathcal{J}_{K,\lambda}(u_j) - < \mathcal{J}'_{K,\lambda}(u_j), u_j > \tag{3.7}$$
$$= a||u_j||^2_{X_0} - \lambda \int\limits_{\Omega} |u_j(x)|^2 dx$$
$$+ \int\limits_{\Omega} \Big(f(x, u_j(x))u_j(x) - 4F(x, u_j(x))\Big)dx.$$

First, we show that the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0. Conversly, if $\{u_j\}_{j\in\mathbb{N}}$ is unbounded in X_0, that is, suppose that, up to a subsequence, still denoted by $\{u_j\}_{j\in\mathbb{N}}$,

$$||u_j||_{X_0} \to +\infty. \tag{3.8}$$

For any $j \in \mathbb{N}$, we take

$$v_j = \frac{u_j}{||u_j||_{X_0}}. \tag{3.9}$$

Therefore $||v_j|| = 1$, so $\{v_j\}_{j\in\mathbb{N}}$ is bounded. By Lemma 1, up to a subsequence, there exists v_∞ such that

$$v_j \to v_\infty \text{ in } L^2(\mathbb{R}^3), \tag{3.10}$$
$$v_j \to v_\infty \text{ in } L^q(\mathbb{R}^3), \tag{3.11}$$
$$v_j \to v_\infty \text{ in } \mathbb{R}^3 \tag{3.12}$$

as $j \to \infty$. Futhermore, by Lemma A.1 [7], there exists $l \in L^q(\mathbb{R}^3)$ such that

$$\{|v_\infty(x)|, |v_j(x)|\} \le l(x) \text{ in } \mathbb{R}^3 \tag{3.13}$$

for all $j \in \mathbb{N}$. Next, we consider two cases when $v_\infty \equiv 0$ and $v_\infty \not\equiv 0$.

Case 1. Suppose that

$$v_\infty \equiv 0. \tag{3.14}$$

For any $j \in \mathbb{N}$, there exists $t_j \in [0,1]$ such that

$$\mathcal{J}_{K,\lambda}(t_j u_j) = \max_{t\in[0,1]} \mathcal{J}_{K,\lambda}(tu_j). \tag{3.15}$$

From (3.8), for any $m \in \mathbb{N}$, we choose $r_m = \sqrt[4]{\frac{8m}{b}}$ such that

$$r_m ||u_j||^{-1}_{X_0} \in (0,1), \tag{3.16}$$

provided j is large enough, say $j > \bar{j}$, with $\bar{j} = \bar{j}(m)$. From (3.10) and (3.14), we have

$$\int_\Omega |r_m v_j(x)|^2 dx \to 0. \tag{3.17}$$

By the continuity of the function F, we get that

$$F(x, r_m v_j(x)) \to F(x, r_m v_\infty(x)) \text{ on } \Omega \tag{3.18}$$

as $j \to \infty$, for any $m \in \mathbb{N}$. By (f_0) and (3.13), using Hölder inequality, we have

$$\begin{aligned}|F(x, r_m v_j(x))| &\le C(|r_m v_j(x)| + |r_m v_j(x)|^q)\\ &\le C(|r_m l(x)| + |r_m l(x)|^q) \in L^1(\Omega),\end{aligned} \tag{3.19}$$

for any $m, j \in \mathbb{N}$. Therefore, from (3.18), (3.19) and the Dominated Convergence Theorem lead to that

$$F(., r_m v_j(.)) \to F(., r_m v_\infty(.)) \text{ in } L^1(\Omega) \tag{3.20}$$

as $j \to \infty$, for any $m \in \mathbb{N}$. Because $F(x,0) = 0$ for all $x \in \Omega$, from (3.14) and (3.20), we have

$$\int_\Omega F(x, r_m v_j(x)) dx \to 0 \tag{3.21}$$

as $j \to \infty$, for any $m \in \mathbb{N}$. Denote $v_{j,m} = r_m v_j$, we have $||v_{j,m}||^2_{X_0} = \left|\left|\sqrt[4]{\frac{8m}{b}} w_j\right|\right|^2_{X_0} = \sqrt{\frac{8m}{b}}$, as well as $||v_{j,m}||^4_{X_0} = \frac{8m}{b}$. Hence, from (3.16)-(3.18) and (3.21), we get

$$\begin{aligned}&\mathcal{J}_{K,\lambda}(t_j u_j) \ge \mathcal{J}_{K,\lambda}(r_m ||u_j||^{-1}_{X_0} u_j)\\ &= \frac{a}{2}||r_m v_j||^2_{X_0} + \frac{b}{4}||r_m v_j||^4_{X_0} - \frac{\lambda}{2}\int_\Omega |r_m v_j(x)|^2 dx\\ &- \int_\Omega F(x, r_m v_j(x)) dx \ge 2m - \frac{\lambda}{2}\int_\Omega |r_m v_j(x)|^2 dx\\ &- \int_\Omega F(x, r_m v_j(x)) dx \ge m,\end{aligned}$$

for all j large enough and for any $m \in \mathbb{N}$. Thus, we deduce that

$$\mathcal{J}_{K,\lambda}(t_j u_j) \to +\infty \tag{3.22}$$

as $j \to +\infty$. We note that $\mathcal{J}_{K,\lambda}(0) = 0$ and (3.1) holds, combining with (3.22), we see that there exists $t_j \in (0,1)$ and so by (3.15), we obtain

$$\frac{d}{dt}\Big|_{t=t_j} \mathcal{J}_{K,\lambda}(tu_j) = 0$$

for any $j \in \mathbb{N}$. We have

$$< \mathcal{J}'_{K,\lambda}(t_j u_j), t_j u_j >= t_j \frac{d}{dt}\Big|_{t=t_j} \mathcal{J}_{K,\lambda}(tu_j) = 0. \tag{3.23}$$

We show that

$$\limsup_{j\to\infty} \mathcal{J}_{K,\lambda}(t_j u_j) \le \kappa \tag{3.24}$$

for a suitable positive constant. From (3.23), we get

$$\begin{aligned}&\frac{4}{\gamma_*}\mathcal{J}_{K,\lambda}(t_j u_j)\\ &= \frac{1}{\gamma_*}(4\mathcal{J}_{K,\lambda}(t_j u_j) - < \mathcal{J}'_{K,\lambda}(t_j u_j), t_j u_j >)\\ &= \frac{1}{\gamma_*}(a||t_j u_j||^2_{X_0} - \lambda \int_\Omega |t_j u_j(x)|^2 dx)\\ &+ \frac{1}{\gamma_*}\Big(\int_\Omega (f(x, t_j u_j(x)) t_j u_j(x) - 4F(x, t_j u_j(x))) dx\Big)\\ &= \frac{1}{\gamma_*}(a||t_j u_j||^2_{X_0} - \lambda \int_\Omega |t_j u_j(x)|^2 dx)\\ &+ \frac{1}{\gamma_*}\int_\Omega G(x, t_j u_j(x)) dx.\end{aligned}$$

By (f_3), we have

$$\begin{aligned}
&\frac{4}{\gamma_*}\mathcal{J}_{K,\lambda}(t_ju_j)\\
&\leq \frac{1}{\gamma_*}(a||t_ju_j||^2_{X_0} - \lambda\int\limits_\Omega |t_ju_j(x)|^2dx)\\
&+\int\limits_\Omega G(x,u_j(x))dx + \int\limits_\Omega W(x)dx\\
&= \frac{1}{\gamma_*}(a||t_ju_j||^2_{X_0} - \lambda\int\limits_\Omega |t_ju_j(x)|^2dx)\\
&+\int\limits_\Omega \Big(u_j(x)f(x,u_j(x)) - 4F(x,u_j(x))\Big)dx\\
&+\int\limits_\Omega W(x)dx.
\end{aligned}$$

Using above inequality and (3.7), we get

$$\begin{aligned}
&\frac{4}{\gamma_*}\mathcal{J}_{K,\lambda}(t_ju_j)\\
&\leq \frac{1}{\gamma_*}(a||t_ju_j||^2_{X_0} - \lambda\int\limits_\Omega |t_ju_j(x)|^2dx)\\
&-(a||u_j||^2_{X_0} - \lambda\int\limits_\Omega |u_j(x)|^2dx)\\
&+4\mathcal{J}_{K,\lambda}(u_j) - <\mathcal{J}'_{K,\lambda}(u_j),u_j> + \int\limits_\Omega W(x)dx.
\end{aligned}$$

Note that

$$\begin{aligned}
&\frac{1}{\gamma_*}(a||t_ju_j||^2_{X_0} - \lambda\int\limits_\Omega |t_ju_j(x)|^2dx)\\
&-(a||u_j||^2_{X_0} - \lambda\int\limits_\Omega |u_j(x)|^2dx)\\
&= (\frac{t_j^2}{\gamma_*}-1)(a||u_j||^2_{X_0} - \lambda\int\limits_\Omega |u_j(x)|^2dx)\\
&\leq (\frac{t_j^2}{\gamma_*}-1)B_{K,\lambda,0}||u_j||^2_{X_0} \leq 0,
\end{aligned}$$

where (see [2], Lemma 16)

$$B_{K,\lambda} = \begin{cases} a & \text{if } \lambda \leq 0\\ a - \dfrac{\lambda}{\lambda_1} & \text{if } 0 < \dfrac{\lambda}{a} < \lambda_1\\ a - \dfrac{\lambda}{\lambda_{k+1}} & \text{if } \lambda_k \leq \dfrac{\lambda}{a} < \lambda_{k+1}. \end{cases}$$

Hence we have

$$\begin{aligned}
&\frac{4}{\gamma_*}\mathcal{J}_{K,\lambda}(t_ju_j)\\
&\leq 4\mathcal{J}_{K,\lambda}(u_j) - <\mathcal{J}'_{K,\lambda}(u_j),u_j>\\
&+\int\limits_\Omega W(x)dx \to 4c + \int\limits_\Omega W(x)dx = \kappa < +\infty
\end{aligned}$$

as $j \to \infty$, thanks to (3.1) and (3.4). This contradicts with (3.22). Therefore, we get that the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0.

Case 2. Suppose that

$$v_\infty \not\equiv 0. \tag{3.25}$$

Then the set $\Omega' = \{x \in \Omega : v_\infty(x) \neq 0\}$ has positive Lebesgue measure and

$$|u_j(x)| = |v_j(x)|||u_j||_{X_0} \to +\infty \text{ on } \Omega' \tag{3.26}$$

as $j \to \infty$, thanks to (3.8), (3.9), (3.12) and (3.25). By (f_2) and (3.26), we have

$$\lim_{j\to\infty}\frac{f(x,u_j(x))}{u_j^3(x)} \to +\infty$$

in Ω'. Hence, by Fatou's Lemma, we get

$$\lim_{j\to\infty}\int\limits_{\Omega'}\frac{f(x,u_j(x))}{u_j^3(x)}|v_j(x)|^4dx \to +\infty \tag{3.27}$$

as $j \to +\infty$. On the other hand, taking into account that f is a continuous function, it is easy to see that

$$\int\limits_{\Omega\setminus\Omega'}\frac{f(x,u_j(x))}{u_j^3(x)}|v_j(x)|^4dx \geq -\frac{B_2}{||u_j||^4_{X_0}}|\Omega\setminus\Omega'|, \tag{3.28}$$

where $B_2 > 0$ is a constant. Therefore, from (3.28) and (3.27), we obtain

$$\lim_{j\to\infty}\int\limits_\Omega \frac{f(x,u_j(x))u_j(x)}{||u_j||^4_{X_0}}dx = +\infty. \tag{3.29}$$

By (3.8) and (3.6), we have $\dfrac{<\mathcal{J}'_{K,\lambda}(u_j),u_j>}{||u_j||^4_{X_0}} \to$ 0. This implies

$$\frac{a}{||u_j||^2_{X_0}} + b - \int\limits_\Omega \frac{f(x,u_j(x))u_j(x)}{||u_j||^4_{X_0}}dx = \frac{o(1)}{||u_j||^4_{X_0}} \tag{3.30}$$

as $j \to \infty$. From (3.29) and (3.30), we get a contradiction. Thus, the sequence $\{u_j\}$ is bounded in X_0.

Step 2. The property Cerami compactness condition of $\{u_j\}$. Since $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0 by Step 1 and X_0 is a reflexive space (being Hilbert space, by Lemma 7 in [3]), up to a subsequence, still denote by $\{u_j\}_{j\in\mathbb{N}}$, there exist $u_\infty \in X_0$ such that

$$\int_{\mathbb{R}^6} (u_j(x) - u_j(y))(\varphi(x) - \varphi(y))K(x-y)dxdy \to \int_{\mathbb{R}^6} (u_\infty(x) - u_\infty(y))(\varphi(x) - \varphi(y))K(x-y)dxdy \tag{3.31}$$

for any $\varphi \in X_0$ as $j \to \infty$. Moreover, by Lemma 1, up to a subsequence, we have

$$u_j \to u_\infty \text{ in } L^q(\mathbb{R}^3) \tag{3.32}$$
$$u_j \to u_\infty \text{ in } \mathbb{R}^3$$

as $j \to +\infty$ and apply to Lemma A.1 in [7], there exists $l \in L^q(\mathbb{R}^3)$ such that

$$\{|u_\infty(x)|, |u_j(x)|\} \le l(x) \tag{3.33}$$

for all $x \in \mathbb{R}^3$ and for any $j \in \mathbb{N}$. By (f_0) condition, (3.32)-(3.33), the fact that $t \mapsto f(.,t)$ is continuous in $t \in \mathbb{R}$ and the Dominated Convergence Theorem, we get

$$\int_\Omega f(x, u_j(x))(u_j(x) - u_\infty(x))dx \tag{3.34}$$
$$+ \lambda \int_\Omega u_j(x)(u_j(x) - u_\infty(x))dx \to 0. \tag{3.35}$$

We see $\{u_j - u_\infty\}$ is bounded sequence in X_0, then we have

$$< \mathcal{J}'_{K,\lambda}(u_j), u_j - u_\infty > \to 0. \tag{3.36}$$

Therefore

$$\begin{aligned} 0 \leftarrow & < \mathcal{J}'_{K,\lambda}(u_j), u_j - u_\infty > \\ = & < a(u_j), u_j - u_\infty > \\ & - \int_\Omega f(x, u_j(x))(u_j(x) - u_\infty(x))dx \\ & - \lambda \int_\Omega u_j(x)(u_j(x) - u_\infty(x))dx, \end{aligned} \tag{3.37}$$

where

$$< a(u_j), u_j - u_\infty > \tag{3.38}$$
$$\begin{aligned} := & \Big(\int_{\mathbb{R}^6} |u_j(x) - u_j(y)|^2 K(x-y)dxdy \\ & - \int_{\mathbb{R}^6} (u_j(x) - u_j(y))(u_\infty(x) - u_\infty(y))K(x-y)dxdy \Big) \\ & \times \Big(a + b \int_{\mathbb{R}^6} |u_j(x) - u_j(y)|^2 K(x-y)dxdy \Big). \end{aligned} \tag{3.39}$$

Note that $a > 0$, combining (3.36)-(3.38) and (3.31), we have

$$\begin{aligned} & \lim_{j\to\infty} \int_{\mathbb{R}^6} |u_j(x) - u_j(y)|^2 K(x-y)dxdy \\ = & \lim_{j\to\infty} \int_{\mathbb{R}^6} (u_j(x) - u_j(y))(u_\infty(x) - u_\infty(y))K(x-y)dxdy \\ = & \int_{\mathbb{R}^6} |u_\infty(x) - u_\infty(y)|^2 K(x-y)dxdy. \end{aligned}$$

Then we obtain

$$||u_j||_{X_0} \to ||u_\infty||_{X_0} \tag{3.40}$$

as $j \to \infty$. Finally, from (3.40) and property continuous of scalar product $< .,. >$ in $X_0 \times X_0$, we have

$$\begin{aligned} & ||u_j - u_\infty||^2_{X_0} = < u_j - u_\infty, u_j - u_\infty > \\ & = ||u_j||^2_{X_0} + ||u_\infty||^2_{X_0} - 2 < u_j, u_\infty > \\ & \to 2||u_\infty||^2_{X_0} \\ & - 2 \int_{\mathbb{R}^3\times\mathbb{R}^3} |u_\infty(x) - u_\infty(y)|^2 K(x-y)dxdy = 0 \end{aligned}$$

as $j \to \infty$. This implies $u_j \to u_0$ strongly convergent in X_0. □

Following the notation used by Bartsch [8] (see Theorem 2.5), in the sequel for any $k \in \mathbb{N}$, we put

$$Y_k := \text{span}\{e_1, \dots, e_k\} \text{ and } Z_k := \overline{\text{span}\{e_k, e_{k+1}, \dots\}}.$$

Since Y_k is finite-dimensional, all norms on Y_k are equivalent. Therefore, there exist two positive constants $C_{k,q}$ and $\widetilde{C}_{k,q}$, depending on k, q, such that for any $u \in Y_k$

$$C_{k,q}||u||_{X_0} \leq ||u||_{L^q(\Omega)} \leq \widetilde{C}_{k,q}||u||_{X_0} \qquad (3.41)$$

for all $q \in [1, 2_s^*]$.

The Fountain Theorem provides the existence of an unbounded sequence of critical value for a smooth functional, under suitable compactness condition and geometry assumption on it, which, in our framework, read as follows:

(*i*) $a_k := \max\Big\{\mathcal{J}_{K,\lambda}(u) : u \in Y_k, ||u||_{X_0} = r_k\Big\} \leq 0,$

(*ii*) $b_k := \inf\Big\{\mathcal{J}_{K,\lambda}(u) : u \in Z_k, ||u||_{X_0} = \alpha_k\Big\} \to \infty$ as $k \to \infty$.

Now, we prove the Theorem 2. The idea consists in applying the Fountain Theorem. By Lemma 2, we see that $\mathcal{J}_{K,\lambda}$ satisfies the Palais-Smale condition and condition $f(x,-t) = -f(x,t)$ for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$ implies $\mathcal{J}_{K,\lambda}(-u) = \mathcal{J}_{K,\lambda}(u)$ for any $u \in X_0$. Then, it remains to show the geometry condition for the function $\mathcal{J}_{K,\lambda}$. To this purpose, let us proceed by steps.

Step 1. For any $k \in \mathbb{N}$, there exists $r_k > 0$ such that

$$a_k := \max\{\mathcal{J}_{K,\lambda}(u) : u \in Y_k, ||u||_{X_0} = r_k\} \leq 0.$$

By (f_2), for any $r > 0$, there exists $\delta_r > 0$ such that

$$f(x,t) \geq 4r|t|^3 \qquad (3.42)$$

for any $x \in \overline{\Omega}$, $|t| > \delta_r$. For $t > \delta_r$, we have

$$F(x,t) = \int_0^t f(x,\tau)d\tau \geq r|t|^4$$

for all $x \in \overline{\Omega}$. For $t < -\delta_r$, we have $-t > \delta_r$, and $F(x,-t) = F(x,t)$ since $f(x,-t) = -f(x,t)$, then we get

$$F(x,t) \geq r|t|^4$$

for all $x \in \overline{\Omega}$ and $|t| > \delta_r$. By the Weierstrass Theorem, we see that

$$F(x,t) \geq m_r := \min_{x\in\overline{\Omega}, |t|\leq\delta_r} F(x,t) \qquad (3.43)$$

for all $|t| \leq \delta_r$. Note that $m_r \leq 0$, since $F(x,0) = 0$ for any $x \in \overline{\Omega}$. Thus, from (3.42) and (3.43), we have

$$F(x,t) \geq r|t|^4 - B_r$$

for any $(x,t) \in \overline{\Omega} \times \mathbb{R}$, for suitable positive constant $B_r \geq r\delta_r^4 - m_r$. Hence, for any $u \in Y_k$, we have

$$\begin{aligned}\mathcal{J}_{K,\lambda}(u) &\leq \frac{a}{2}||u||_{X_0}^2 - \frac{\lambda}{2}||u||_{L^2(\Omega)}^2 + \frac{b}{4}||u||_{X_0}^4 \\ &- r||u||_{L^4(\Omega)}^4 + B_r|\Omega| \\ &\leq D_{k,\lambda}||u||_{X_0}^2 + \frac{b}{4}||u||_{X_0}^4 - r||u||_{L^4(\Omega)}^4 + B_r|\Omega| \\ &\leq D_{k,\lambda}||u||_{X_0}^2 + (\frac{b}{4} - rC_{k,4})||u||_{X_0}^4 + B_r|\Omega|,\end{aligned}$$

where $D_{k,\lambda}$ is a constant depending on k, λ. We choose r large enough such that $\frac{b}{4} - rC_{k,4} < 0$, we get that for any $u \in Y_k$ with $||u||_{X_0} = r_k$,

$$\mathcal{J}_{K,\lambda}(u) \leq 0$$

for r_k large enough, where r_k depends on r.

Step 2.[9, 10] Let $1 \leq q < 2^*$ and, for any $k \in \mathbb{N}$, let

$$\beta_k := \sup\{||u||_{L^q(\Omega)} : u \in Z_k, ||u||_{X_0} = 1\}.$$

Then $\beta_k \to 0$ as $k \to \infty$.

Step 3. There exists $\alpha_k > 0$ such that

$$b_k := \inf\{\mathcal{J}_{K,\lambda}(u) : u \in Z_k, ||u||_{X_0} = \alpha_k\} \to +\infty$$

as $k \to +\infty$. We have

$$\begin{aligned}\mathcal{J}_{K,\lambda}(u) &= \frac{a}{2}||u||_{X_0}^2 + \frac{b}{4}||u||_{X_0}^4 \\ &- \frac{\lambda}{2}\int_\Omega |u(x)|^2dx - \int_\Omega F(x,u(x)) \\ &\geq \frac{a}{2}||u||_{X_0}^2 + \frac{b}{4}||u||_{X_0}^4 - \frac{\lambda}{2}\int_\Omega |u(x)|^2dx \\ &- h\int_\Omega |u(x)|^2dx - C_2\int_\Omega |u(x)|^q dx.\end{aligned}$$

Then

$$\begin{aligned} &\mathcal{J}_{K,\lambda}(u) \qquad (3.44)\\ &\geq C^*_{K,\lambda}||u||^2_{X_0} + \frac{b}{4}||u||^4_{X_0} - C_2\int_\Omega |u(x)|^q dx\\ &\geq C^*_{K,\lambda}||u||^2_{X_0} - C_2||u||^q_{L^q(\Omega)}\\ &= C^*_{K,\lambda}||u||^2_{X_0} - C_2||\frac{u}{||u||_{X_0}}||^q_{L^q(\Omega)}||u||^q_{X_0}. \end{aligned}$$

Hence, we get

$$\mathcal{J}_{K,\lambda}(u) \geq C^*_{K,\lambda}||u||^2_{X_0} - C_2{\beta_k}^q||u||^q_{X_0}, \quad (3.45)$$

where β_k is defined in Step 2. We define α_k as

$$\alpha_k = \Big(\frac{2C^*_{K,\lambda}}{qC_2\beta_k^q}\Big)^{1/(q-2)}.$$

Therefore $\alpha_k \to +\infty$ as $k \to \infty$. Note that $q > 2$, then for any $||u||_{X_0} = \alpha_k$, we have

$$\begin{aligned} \mathcal{J}_{K,\lambda}(u) &\geq ||u||^2_{X_0}(C^*_{K,\lambda} - C_2{\beta_k}^q||u||^{q-2}_{X_0})\\ &= (1-\frac{2}{q})C^*_{K,\lambda}\alpha_k^2 \to +\infty \end{aligned}$$

as $k \to \infty$. Hence, all the geometric features of the Fountain Theorem are satisfied, then $\mathcal{J}_{K,\lambda}$ has an unbounded sequence of critical values which are solutions of problems (1.1).

Conclusion. In this paper, we prove the existence of weak solution to a Kirchhoff problem involving fractional Laplace. Our nonlinear function does not satisfy the Ambrosetti-Rabinowitz condition. In order to prove our result, we use the Fountain Theorem and the technique of variational method in fractional Sobolev space.

References

[1] R. Servadei and E. Valdinoci, "Lewy-Stampacchia type estimates for variational inequalities driven by (non) local operators", *Rev. Mat. Iberoam*, vol. 29, pp. 1091-1126, 2013.

[2] R. Servadei and E. Valdinoci, "Variational methods for non-local operators of elliptic type", *Discrete Contin. Dyn. Syst*, vol. 33, pp. 2105-2137, 2013.

[3] R. Servadei and E. Valdinoci, "Mountain Pass solutions for non-local elliptic operators", *J. Math. Anal. Appl*, vol. 389, pp. 887-898, 2012.

[4] R. Servadei and E. Valdinoci, "The Brezis-Nirenberg result for the fractional Laplacian", *Trans. Amer. Math. Soc*, vol. 367, pp. 67-102, 2015.

[5] G. Cerami, "An existence criterion for the critical points on unbounded manifolds (in Italian)", *Ist. Lombardo Accad. Sci. Lett. Rend. Sez. A*, vol. 112, pp. 332-336, 1978.

[6] G. Cerami, "On the existence of eigenvalues for a nonlinear boundary value problem (in Italian)", *Ann. Mat. Pura Appl*, vol. 124, pp. 161-179, 1980.

[7] M. Willem, *Minimax Theorems*, Birkhäuser, Basel, 1996.

[8] T. Bartsch, "Infinitely many solutions of a symmetric Dirichlet problem", *Nonlinear Anal*, vol. 20, pp. 1205-1216, 1993.

[9] G. M. Bisci, D. Repovs and R. Servadei, "Nontrivial solutions of superlinear nonlocal problems", *Forum. Math*, vol. 28, pp. 1095-1110, 2016.

[10] Z. Binlin, G. M. Bisci and R. Servadei, "Superlinear nonlocal fractional problems with infinitely many solutions", *Nonlinearity*, vol. 28, pp. 2247-2264, 2015.