NUMERICAL RESULTS FOR THE PROBLEM OF FINDING A SOLUTION OF A SYSTEM OF ILL-POSED EQUATIONS

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ABSTRACT

Many issue s in reality result the problem of finding an unknown quantity $x \in H$ from the original data set $(f1, \dots, fN) \in HN$, $N \geq 1$, where H is a real Hilbert space. The data set $(f1, \dots, fN)$ which is often not exactly known, is just given approximately by $fi\delta \in H$. This problem is modeled by a system of operator equations. Therefore, we need to research and propose a stable solution for the above problem class. The purpose of this paper is to present an iterative regularization method in a real Hilbert space for the problem of finding a solution to a system of nonlinear ill-posed equations. We prove the strong convergence of this method; give an application of the optimal problem and two examples of numerical expressions are also given to illustrate the effectiveness of the proposed methods.
Keywords: *Ill-posed problem; system of nonlinear equations; monotone operator; Hilbert space; regularization method; iterative method.*

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KẾT QUẢ SỐ CHO BÀI TOÁN TÌM NGHIỆM CỦA HỆ PHƯƠNG TRÌNH TOÁN TỬ ĐẶT KHÔNG CHỈNH

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TÓM TẮT

Nhiều vấn đề của các lĩnh vực trong khoa học kỹ thuật cũng như kinh tế xã hội dẫn đến bài toán tìm một đại lượng $x \in H$ chưa biết từ bộ dữ kiện ban đầu $(f1, \dots, fN) \in HN$, $N \geq 1$, ở đây H là không gian Hilbert thực. Trên thực tế, bộ dữ liệu $(f1, \dots, fN)$ nhận được bằng việc đo đạc trực tiếp trên các tham số và thường không được biết chính xác, chỉ được cho xấp xỉ bởi $fi\delta \in H$. Bài toán này được mô hình hóa bởi hệ phương trình toán tử. Vì vậy, ta cần nghiên cứu và đề xuất phương pháp giải ổn định cho lớp bài toán trên. Trong bài báo này, chúng tôi đưa ra một phương pháp hiệu chỉnh lặp trong không gian Hilbert thực giải bài toán tìm nghiệm của hệ phương trình toán tử phi tuyến đặt không chỉnh. Đồng thời, chúng tôi chứng minh sự hội tụ mạnh của phương pháp, đưa ra một áp dụng giải bài toán tối ưu và hai ví dụ số minh họa cho sự hiệu quả của phương pháp được đề xuất.
Từ khóa: *Bài toán đặt không chỉnh; hệ phương trình toán tử phi tuyến; toán tử đơn điệu; không gian Hilbert; phương pháp hiệu chỉnh; phương pháp hiệu chỉnh lặp.*

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1. Introduction

The inverse problem we are interested in consists in determining an unknown physical quantity from a finite set of data in Hilbert spaces. In practical situations, we do not know the data exactly. Instead, we have only approximate measured data satisfying some conditions. The finite set of data mentioned above is obtained by indirect measurements of a parameter, this process being described by a model of system of nonlinear equations (SNEs) in Banach spaces, which is, in general, a typical ill-posed problem.

In 2006, in order to solve SNEs, Buong [1] presented a regularization method of Browder–Tikhonov (RMBT) when each mapping is monotone, hemicontinuous and potential. For a literature concerning RMBT, please refer to [2], [3], [4]. . . .

In what follows, we are interested in regularization methods for solving SNEs, where each equation in SNEs is ill-posed. The present work is motivated by interesting ideas on regularization for SNEs involving monotone mappings in [1].

The rest of this paper is divided into five sections. In Section 2, we recall some definitions and results that will be used in the proof of our main theorems. In Section 3 we present a method to construct approximate solutions and the last section we consider two examples of numerical expressions.

2. Preliminaries

Let H be a real Hilbert space. When $\{x_n\}$ is a sequence in H, $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x, and $x_n \to x$ means the strong convergence. In what follows, we collect some definitions on monotone operators and their useful properties. We refer the reader [5] for more details.

Definition 1: (see [5]) A mapping $A : \mathcal{D}(A) \subset H \to H$ is called

(i) monotone if

$$\langle A(x) - A(y), x - y\rangle \geq 0 \; \forall x, y \in \mathcal{D}(A);$$

(ii) λ-inverse strongly monotone (or λ-cocoercive) if there exists a positive constant λ such that

$$\langle A(x) - A(y), x - y\rangle \geq \lambda\|A(x) - A(y)\|^2 \; \forall x, y \in \mathcal{D}(A).$$

Definition 2: (see [5]) A mapping $A : H \to H$ is called

(i) hemicontinuous at a point $x_0 \in \mathcal{D}(A)$ if

$$A(x_0 + tx) \rightharpoonup Ax_0$$

as $t \to 0$ for any x such that $x_0 + tx \in \mathcal{D}(A)$;

(ii) demicontinuous at a point $x_0 \in \mathcal{D}(A)$ if for any sequence $\{x_n\} \subset \mathcal{D}(A)$ such that $x_n \to x_0$, the convergence $Ax_n \rightharpoonup Ax_0$ holds (it is evident that hemicontinuity of A follows from its demicontinuity).

Lemma 1: (see [6]) Let $\{u_k\}, \{a_k\}, \{b_k\}$ be the sequences of positive number satisfying the following conditions:

(i) $u_{k+1} \leq (1 - a_k)u_k + b_k,\ 0 \leq a_k \leq 1$,

(ii) $\sum\limits_{k=1}^{\infty} a_k = +\infty,\ \lim\limits_{k\to+\infty} \frac{b_k}{a_k} = 0$.

Then, $\lim\limits_{k\to+\infty} u_k = 0$.

3. Main Results

In this paper, we consider the problem of finding a solution of a system of nonlinear ill-posed operator equations:

$$A_j(x) = f_j, \quad j = 1, \ldots, N, \tag{1}$$

where $N \geq 1$ is an integer, A_1 is monotone and hemicontinuous, the other mappings A_j, $j = 2, \ldots, N$, are λ_j-inverse strongly monotone with domain $\mathcal{D}(A_j) = H$, and $f_j \in H$ for all $j = 1, \ldots, N$. We are interested in the situation that the solution of (1) does not depend continuously on the data f_j. In addition, we assume that we are only given 'noisy data' $f_j^\delta \in H$ with known noise level $\delta > 0$, that is,

$$\|f_j - f_j^\delta\| \leq \delta \quad \forall j = 1, \ldots, N. \tag{2}$$

Denote by S_j the solution set of the j-th equation in (1), that is,

$$S_j = \{x \in H : A_j(x) = f_j\}.$$

Throughout this paper, we assume that

$$S := \bigcap_{j=1}^{N} S_j \neq \emptyset.$$

Now we consider the following iterative regularization method of zero order, where z_{n+1} is defined by

$$z_{n+1} = z_n - \beta_n\Big[(A_1(z_n) - f_1) + \sum_{j=2}^{N} \alpha_n^{\frac{1}{N+2-j}} (A_j(z_n) - f_j) + \alpha_n(z_n - x_*)\Big], \quad z_0 \in H, \tag{3}$$

where H is a real Hilbert space, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers, and $x_* \in H$.

Theorem 1: Suppose that $A_1 : \mathcal{D}(A_1) = H \to H$ is monotone and hemicontinuous, the other mappings $A_j : \mathcal{D}(A_j) = H \to H$, $j = 2, \dots, N$, are λ_j-inverse strongly monotone. Let $f_j^\delta \in H$ for all $\delta > 0$ and all $j = 1, \dots, N$. Assume that condition (2) holds. Then we have the following statements.

(i) For each $\alpha_n > 0$, problem

$$A_1(x_n) + \sum_{j=2}^{N} \alpha_n^{\frac{1}{N+2-j}} (A_j(x_n) - f_j) = f_1 \tag{4}$$

has a unique solution x_n.

(ii) If $0 < \alpha_n \leq 1$, $\alpha_n \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} x_n = x^0 \in S$ with x_*-minium norm.

Proof. See Theorem 2.4 in [4].

Theorem 2: Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ in the problem (3) satisfy the following conditions:

(i) $1 \geq \alpha_n \searrow 0$, $\beta_n \to 0$ as $n \to +\infty$;

(ii) $\lim\limits_{n\to+\infty} \dfrac{|\alpha_{n+1} - \alpha_n|}{\beta_n \alpha_n^2} = 0$, $\lim\limits_{n\to+\infty} \dfrac{\beta_n}{\alpha_n} = 0$;

(iii) $\sum\limits_{n=0}^{\infty} \beta_n \alpha_n = +\infty$.

Then, $\lim\limits_{n\to+\infty} z_n = x^0 \in S$ with x_*-minimum norm.

Proof Theorem 2. First, we have $\|z_n - x^0\| \leq \|z_n - x_n\| + \|x_n - x^0\|$. The second term in right-hand side of this estimate tends to zero as $n \to \infty$, by Theorem 1. So we only have to proof that z_n approximates x_n as $n \to \infty$.

Let $\Delta_n = \|z_n - x_n\|$. Obviuously,

$$\begin{aligned}
\Delta_{n+1} &= \|z_{n+1} - x_{n+1}\| \\
&= \Big\| z_n - x_n - \beta_n \Big[A_1(z_n) - f_1 + \sum_{j=2}^{N} \alpha_n^{\frac{1}{N+2-j}} (A_j(z_n) - f_j) + \alpha_n(z_n - x_*) \Big] - (x_{n+1} - x_n) \Big\| \\
&\leq \Big\| z_n - x_n - \beta_n \Big[A_1(z_n) - f_1 + \sum_{j=2}^{N} \alpha_n^{\frac{1}{N+2-j}} (A_j(z_n) - f_j) + \alpha_n(z_n - x_*) \Big] \Big\| + \|x_{n+1} - x_n\|,
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
&\Big\| z_n - x_n - \beta_n \Big[A_1(z_n) - f_1 + \sum_{j=2}^{N} \alpha_n^{\frac{1}{N+2-j}} (A_j(z_n) - f_j) + \alpha_n(z_n - x_*) \Big] \Big\|^2 \\
&= \|z_n - x_n\|^2 + \beta_n^2 \Big\| A_1(z_n) - f_1 + \sum_{j=2}^{N} \alpha_n^{\frac{1}{N+2-j}} (A_j(z_n) - f_j) + \alpha_n(z_n - x_*) \Big\|^2 \\
&\quad - 2\beta_n \Big\langle z_n - x_n, A_1(z_n) - f_1 - (A_1(x_n) - f_1) \Big\rangle \\
&\quad - 2\beta_n \Big\langle z_n - x_n, \sum_{j=2}^{N} \alpha_n^{\frac{1}{N+2-j}} (A_j(z_n) - f_j) + \alpha_n(z_n - x_*) - \Big[\sum_{j=2}^{N} \alpha_n^{\frac{1}{N+2-j}} (A_j(x_n) - f_j) + \alpha_n(x_n - x_*) \Big] \Big\rangle \\
&= \|z_n - x_n\|^2 + \beta_n^2 \Big\| A_1(z_n) - f_1 + \sum_{j=2}^{N} \alpha_n^{\frac{1}{N+2-j}} (A_j(z_n) - f_j) + \alpha_n(z_n - x_*) \Big\|^2 \\
&\quad - 2\beta_n \Big\langle z_n - x_n, A_1(z_n) - A_1(x_n) \Big\rangle \\
&\quad - 2\beta_n \alpha_n \|z_n - x_n\|^2 - 2\beta_n \Big\langle z_n - x_n, \sum_{j=2}^{N} \alpha_n^{\frac{1}{N+2-j}} (A_j(z_n) - A_j(x_n)) \Big\rangle \\
&\leq (1 - 2\beta_n \alpha_n) \|z_n - x_n\|^2 + \beta_n^2 \Big\| A_1(z_n) - f_1 + \sum_{j=2}^{N} \alpha_n^{\frac{1}{N+2-j}} (A_j(z_n) - f_j) + \alpha_n(z_n - x_*) \Big\|^2.
\end{aligned} \tag{6}$$

Since A_j is λ_j-inverse-strongly monotone, A_j is Lipschitz continuous, $j = 2, \dots, N$,

$$\|A_j(z_n) - A_j(x_n)\|^2 \leq \frac{1}{\lambda_j} \langle A_j(z_n) - A_j(x_n), z_n - x_n \rangle \leq \Big(\frac{1}{\lambda_j} \|z_n - x_n\| \Big)^2, \quad m_{A_j} > 0$$

and

$$\begin{aligned}
&\Big\| A_1(z_n) - f_1 + \sum_{j=2}^{N} \alpha_n^{\frac{1}{N+2-j}} \left(A_j(z_n) - f_j\right) + \alpha_n(z_n - x_*) \Big\|^2 \\
&= \Big\| (A_1(z_n) - A_1(x_n)) + \sum_{j=2}^{N} \alpha_n^{\frac{1}{N+2-j}} \left(A_j(z_n) - f_j\right) + \alpha_n(z_n - x_*) \\
&\quad - \sum_{j=2}^{N} \alpha_n^{\frac{1}{N+2-j}} \left(A_j(x_n) - f_j\right) - \alpha_n(x_n - x_*) \Big\|^2 \\
&= \Big\| A_1(z_n) - A_1(x_n) + \sum_{j=2}^{N} \alpha_n^{\frac{1}{N+2-j}} \left(A_j(z_n) - A_j(x_n)\right) + \alpha_n(z_n - x_n) \Big\|^2 \\
&= \Big\| A_1(z_n) - A_1(x_n) + \sum_{j=2}^{N} \alpha_n^{\frac{1}{N+2-j}} \left(A_j(z_n) - A_j(x_n)\right) \Big\|^2 + \alpha_n^2 \|z_n - x_n\|^2 \\
&\quad + 2\alpha_n \Big\langle A_1(z_n) - A_1(x_n) + \sum_{j=2}^{N} \alpha_n^{\frac{1}{N+2-j}} \left(A_j(z_n) - A_j(x_n)\right), z_n - x_n \Big\rangle \\
&\le c\|z_n - x_n\|^2,
\end{aligned} \tag{7}$$

where c is positive constant. Combining (5)–(7), and Theorem 1 we have

$$\Delta_{n+1} \le \left[\Delta_n^2(1 - 2\beta_n\alpha_n + c\beta_n^2)\right]^{1/2} + M\frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n}.$$

By taking the squares of the both sides of the last inequality and then applying the elementary estimate (see [6])

$$(a+b)^2 \le (1 + \alpha_n\beta_n)a^2 + \left(1 + \frac{1}{\alpha_n\beta_n}\right)b^2$$

we obtain that

$$\begin{aligned}
\Delta_{n+1}^2 &\le \Delta_n^2(1 + \alpha_n\beta)(1 - 2\alpha_n\beta_n + c\beta_n^2) + \left(1 + \frac{1}{\alpha_n\beta_n}\right)M^2\frac{|\alpha_{n+1} - \alpha_n^2|^2}{\alpha_n^2} \\
&= \Delta_n^2(1 - \alpha_n\beta_n + c\beta_n^2 - 2\alpha_n^2\beta_n^2 + c\alpha_n\beta_n^3) + \left(1 + \frac{1}{\alpha_n\beta_n}\right)M^2\frac{|\alpha_{n+1} - \alpha_n^2|^2}{\alpha_n^2}.
\end{aligned} \tag{8}$$

The conditions of Lemma 1 for the numerical sequence $\{\Delta_n\}$ are true because of (8) and conditions $(i) - (iii)$ with

$$a_n = \alpha_n\beta_n - c\beta_n^2 + 2\alpha_n^2\beta_n^2 - c\alpha_n\beta_n^3, \quad b_n = \left(1 + \frac{1}{\alpha_n\beta_n}\right)M^2\frac{|\alpha_{n+1} - \alpha_n|^2|}{\alpha_n^2}.$$

The proof is completed.

□

Remark 1: The sequences $\alpha_n = (1+n)^{-p}$ with $0 < 2p < \dfrac{1}{N}$ and $\beta_n = (1+n)^{-1/2}$ satisfy all conditions in Theorem 2.

4. Numerical Results

To illustrate Theorem 1 and Theorem 2, we consider the following examples. We perform the iterative schemes in MATLAB 2020a running on a laptop with Intel(R) Core(TM) i7-8750H CPU @ 2.20GHz, 8GB RAM. Some signs in the result table:

n: Number of iterative steps.

z_0: The first approximation.

z_n: Solution in n-th step.

Now we consider the problem: find an element $x^0 \in H$ such that

$$\varphi_j(x^0) = \min_{x \in H} \varphi_j(x), \quad j = 1, \ldots, N \tag{9}$$

where φ_j is weakly lower semi-continuous proper convex function in a real Hilbert space H. We consider the case, when the function $\varphi_j(x)$ is defined by

$$\varphi_j(x) = \frac{1}{2}\langle A_j x, x\rangle.$$

Then x^0 is a solution to the problem (9) if and only if $x^0 \in S$ with $A_j x = \varphi_j'(x)$ where $A_j = B_j^T B_j$ is an $M \times M$ matrix, $B_j = (b_{lk}^j)_{l,k=1}^M$ is determined as follows.

Example 1: In this example, $N = 1$ and $M = 10$. We consider a equation $Ax = 0$ with the operator $A : \mathbf{R}^{10} \to \mathbf{R}^{10}$ is given by $A = B^T B$ with B is 10×10 matrix and $\det(B) = 0$

$$B = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since $\det(A) = \det(B^T B) = 0$, $Ax = 0$ is ill-posed problem. Consequently, the problem (9) in this case is ill-posed too. By selecting $x_* = (0 \ \ldots \ 0)^T$ in $\mathbf{R}^{10}$ easy to see $x^0 = (0 \ \ldots \ 0)^T \in \mathbf{R}^{10}$ is a solution x_*-minimal norm of $Ax = 0$.

We apply method (3) with $\alpha_n = (1+n)^{-p}$, p in $\left(0, \frac{1}{2}\right)$, $\beta_n = (1+n)^{-1/2}$, and $f = (0 \ \ldots \ 0)$ is given noise by $f_\delta = (\delta \ \ldots \ \delta)^T \in \mathbf{R}^{10}$ with $\delta = 0.001$, we obtain the Tables 1, 2, and 3.

Remark 2: Combining with three 3 calculation tables (Table 1 – Table 3), we have some remarks:

(1) The selection of the first approximation z_0 has an effect on the number of iterations to obtain a solution close to the correct solution.

(2) The selections of β_k and α_k also affects the number of iterations to obtain a solution close to the correct solution.

(3) By choosing α_n so that $p \sim 0$, $\{z_n\}$ converges to correct solution x^0 as quickly and converse, $p \sim \frac{1}{2}$, $\{z_n\}$ converges to correct solution x^0 as slowly.

Example 2: In this example, $N = 3$ and $M = 3$. We consider a system of linear algebraic equations $A_j x = 0$ $(j = 1, 2, 3)$ with the operator $A_j : \mathbf{R}^3 \to \mathbf{R}^3$ is given by $A_j = B_j^T B_j$ with B_j are 3×3 matrixs and $\det(B_j) = 0$

$$B_1 = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ -1 & 0 & 3 \end{pmatrix}; \ B_2 = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix};$$

$$B_3 = \begin{pmatrix} 0 & -1 & -1 \\ 2 & 3 & -3 \\ 1 & 2 & -1 \end{pmatrix}$$

Since $\det(A_j) = \det(B_j^T B_j) = 0$, $j = 1, 2, 3$, each equation in $A_j(x) = 0$ is ill-posed. Consequently, the problem (9) in this case is ill-posed too.

By selecting $x_* = (3 \ -1 \ 1)^T$, easy to see $x^0 = (3 \ -1 \ 1)^T \in \mathbf{R}^3$ is a solution x_*-minimal norm of $A_j x = 0$. We apply method (3) with $\alpha_n = (1+n)^{-p}$ with $0 < p < \frac{1}{6}$ and $\beta_n = (1+n)^{-1/2}$, we obtain the Tables 4, 5 and 6.

Remark 3: Combining with three 3 calculation tables (Table 4 – Table 6), we have some remarks: By choosing α_n so that $p \sim \frac{1}{12}$, $\{z_n\}$ converges to correct solution x^0 as quickly and converse, $p \sim \frac{1}{6}$, $\{z_n\}$ converges to correct solution x^0 as slowly.

Table 1. *The table with* $z_0 = (-4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5)^T \in \mathbf{R}^{10}$, $\alpha_n = (1+n)^{-0.001}$, $\beta_n = (1+n)^{-1/2}$

n	4	8	16	32
z_n^1	-0.0878	-0.0118	$-0.7610.10^{-3}$	$0.1231.10^{-3}$
z_n^2	-0.0084	0.0060	$0.5098.10^{-3}$	$0.0682.10^{-3}$
z_n^3	-0.0745	-0.0029	$-0.1518.10^{-3}$	$0.0685.10^{-3}$
z_n^4	-0.0301	0.0015	$0.1837.10^{-3}$	$0.0740.10^{-3}$
z_n^5	0.0001	-0.0006	$0.0151.10^{-3}$	$0.0701.10^{-3}$
z_n^6	0.0606	0.0005	$0.1004.10^{-3}$	$0.0723.10^{-3}$
z_n^7	0.1014	-0.0002	$0.0557.10^{-3}$	$0.0709.10^{-3}$
z_n^8	0.1216	0.0002	$0.0817.10^{-3}$	$0.0728.10^{-3}$
z_n^9	0.0983	-0.0001	$0.0605.10^{-3}$	$0.0662.10^{-3}$
z_n^{10}	0.0474	0.0002	$0.1001.10^{-3}$	$0.0966.10^{-3}$
$\|x^0 - z_n\|$	0.2343	0.0136	$9.6414.10^{-4}$	$2.5328.10^{-4}$

Table 2. *The table with* $z_0 = (-4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5)^T \in \mathbf{R}^{10}$, $\alpha_n = (1+n)^{-0.049}$, $\beta_n = (1+n)^{-1/2}$

n	4	8	16	32
z_n^1	-0.1060	-0.0189	-0.0019	$-0.0689.10^{-3}$
z_n^2	0.0190	0.0096	0.0011	$0.1048.10^{-3}$
z_n^3	-0.0602	-0.0046	-0.0004	$0.0552.10^{-3}$
z_n^4	-0.0145	0.0023	0.0003	$0.0872.10^{-3}$
z_n^5	0.0001	-0.0010	-0.0001	$0.0696.10^{-3}$
z_n^6	0.0450	0.0007	0.0001	$0.0788.10^{-3}$
z_n^7	0.0701	-0.0003	0.0000	$0.0737.10^{-3}$
z_n^8	0.0857	0.0004	0.0001	$0.0780.10^{-3}$
z_n^9	0.0651	-0.0002	0.0001	$0.0685.10^{-3}$
z_n^{10}	0.0330	0.0003	0.0001	$0.1050.10^{-3}$
$\|x^0 - z_n\|$	0.1872	0.0219	0.0023	$2.5433.10^{-4}$

Table 3. *The table with* $z_0 = (-4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5)^T \in \mathbf{R}^{10}$, $\alpha_n = (1+n)^{-0.049}$, $\beta_n = (1+n)^{-1/2}$

n	4	8	16	32
z_n^1	0.2854	0.0228	0.0018	$0.1942.10^{-3}$
z_n^2	0.4575	-0.0102	-0.0007	$0.0330.10^{-3}$
z_n^3	0.8620	0.0032	0.0004	$0.0854.10^{-3}$
z_n^4	1.1247	0.0010	-0.0000	$0.0665.10^{-3}$
z_n^5	1.1551	-0.0032	0.0000	$0.0728.10^{-3}$
z_n^6	1.1983	0.0046	0.0002	$0.0721.10^{-3}$
z_n^7	0.9024	-0.0049	-0.0001	$0.0700.10^{-3}$
z_n^8	0.7531	0.0050	0.0002	$0.0742.10^{-3}$
z_n^9	0.3684	-0.0042	-0.0001	$0.0649.10^{-3}$
z_n^{10}	0.2001	0.0032	0.0002	$0.0976.10^{-3}$
$\|x^0 - z_n\|$	2.5742	0.0273	0.0020	$2.9177.10^{-4}$

Table 4. *The table with* $z_0 = (1 \quad 1 \quad 1)^T \in \mathbf{R}^3$, $\alpha_n = (1+n)^{-1/12}, \beta_n = (1+n)^{-1/2}$

n	10	20	30	40
z_n^1	2.9856	2.9984	2.9997	2.9999
z_n^2	-0.9952	-0.9995	-0.9999	-1.0000
z_n^3	0.9952	0.9995	0.9999	1.0000
$\lVert x^0 - z_n \rVert$	0.0159	0.0018	$3.5059.10^{-4}$	$9.4340.10^{-5}$

Table 5. *The table with* $z_0 = (1 \quad 1 \quad 1)^T \in \mathbf{R}^3$, $\alpha_n = (1+n)^{-1/7}, \beta_n = (1+n)^{-1/2}$

n	10	20	30	40
z_n^1	2.9750	2.9960	2.9989	2.9996
z_n^2	-0.9917	-0.9987	-0.9996	-0.9999
z_n^3	0.9917	0.9987	0.9996	0.9999
$\lVert x^0 - z_n \rVert$	0.0276	0.0044	0.0012	$4.1781.10^{-4}$

Table 6. *The table with* $z_0 = (10 \quad -10 \quad 20)^T \in \mathbf{R}^3$, $\alpha_n = (1+n)^{-1/12}, \beta_n = (1+n)^{-1/2}$

n	10	20	30	40
z_n^1	3.0883	3.0097	3.0019	3.0005
z_n^2	- 1.0294	-1.0032	-1.0006	-1.0002
z_n^3	1.0294	1.0032	1.0006	1.0002
$\lVert x^0 - z_n \rVert$	0.0976	0.0107	0.0021	$5.7783.10^{-4}$

5. Conclusion

The paper has given the following issues:

- We prove the strong convergence of the iterative method.
- We give an application for the optimization problem and calculates two numerical examples that illustrate the convergence of methods in a Hilbert space.

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