PROXIMAL ALGORTHM FOR MONOTONE VARIATIONAL INEQUALITY PROBLEMS

Tran Van Thang*, Nguyen Minh Khoa, Phan Thi Tuyet
Electric Power University

ABSTRACT

In this paper, we introduce a new proximal algorithm for monotone variational inequality problems in a general form in space R^n with cost mappings are monotone, L-Lipschitz continuous on the whole space R^n. The proposed algorithm involves only one proximal operator periteration and combines proximal operators with the Halpern iteration technique. The strong convergence result of the iterative sequence generated by the proposed algorithm is established, under mild conditions, in space R^n.

Keywords: *Variational inequalities; lipschitz continuous; monotone; projection; proximal operator.*

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THUẬT TOÁN GẦN KỀ CHO BÀI TOÁN BẤT ĐẲNG THỨC BIẾN PHÂN ĐƠN ĐIỆU

Trần Văn Thắng*, Nguyễn Minh Khoa, Phan Thị Tuyết
Trường Đại học Điện lực

TÓM TẮT

Trong bài báo này, chúng tôi đưa ra thuật toán mới cho bài toán bất đẳng thức biến phân đơn điệu dưới dạng mở rộng trong không gian R^n với hàm giá là đơn điệu, liên tục L-Lipschitz trên toàn không gian R^n. Thuật toán mà chúng tôi đưa ra chỉ bao gồm một toán tử kề trong mỗi bước lặp và là sự kết hợp giữa toán tử kề với kỹ thuật lặp Halpern. Sự hội tụ mạnh của dãy lặp sinh bởi thuật toán được thiết lập với các giả thiết thông thường trong không gian R^n.

Từ khóa: *Bất đẳng thức biến phân; liên tục Lipschitz; đơn điệu; phép chiếu; toán tử kề*

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* Corresponding author. *Email: thangtv@epu.edu.vn*

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1. Introduction

Variational inequality problem in Hilbert space H is formulated by:

Find $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C, \qquad (1)$$

where C is a convex, closed, empty subset in H and $f : C \to H$ is a operator. This is an important problem that has a variety of theoretical and practical applications [1]. To solve this problem, many algorithms have been proposed, such as: single projection method, twice projection method [1], [2], modified projection method [3], ... Until now, variational inequality problem has been increasingly extended in more extended forms: multivalued variational problem, problem of finding a common solution of variational inequality problem and fix point problem, bielevel variational inequality problem,... This paper considers a problem of the variational inequality in a general form

$$\text{Find } x^* \in C \textit{ such that} \qquad (2)$$
$$\langle f(x^*), x - x^* \rangle + g(x) - g(x^*) \geq 0 \quad \forall x \in C,$$

where $g : C \to H$ is a proper, convex, continuous function. Recently, Y. Malitsky [4] presented proximal extrapolated gradient method for solving this problem. In this paper, we introduce a new proximal algorithm for solving problem (2). The proposeed algorithm combines the proximal operators with the Halpern iteration technique in paper [2]. We have proved that the algorithm is convergent under the assumption of the monotonicity and Lipschitz continuity of cost mappings.

2. Preliminaries

Throughout this paper, unless otherwise mentioned, let $\mathcal{H}$ denote a Hilbert space with inner product $\langle ., . \rangle$ and the induced norm $\|.\|$.

Definition 1. Let C be a nonempty closed convex subset in $\mathcal{H}$. The metric projection from $\mathcal{H}$ onto C is denoted by P_C and

$$P_C(x) = \text{argmin}\{\|x - y\| : \ y \in C\} \ x \in \mathcal{H}.$$

It is well known that the metric projection $Pr_C(\cdot)$ has the following basic property:

$$\langle x - P_C(x), y - P_C(x) \rangle \leq 0 \ \forall x \in \mathcal{H}, \ y \in C.$$

Definition 2. A mapping $f : \mathcal{H} \to 2^{\mathcal{H}}$ is called to be

(i) *monotone*, if

$$\langle f(x) - f(y), x - y \rangle \geq 0 \quad \forall x, y \in \mathcal{H};$$

(ii) *L- Lipschitz continuous*, if

$$\|f(x) - f(y)\| \leq L\|x - y\| \ \forall x, y \in \mathcal{H},$$

Definition 3. Let $g : C \to \mathcal{R}$ be proper, convex and lower semicontinuous. The proximal operator of g on C is formulated as the follows:

$$prox_g(y) = \underset{x \in C}{\text{argmin}} \left\{ g(x) + \frac{1}{2}\|y - x\|^2 \right\}.$$

The following lemmas are useful in the sequel.

Lemma 1. *For all* $x, \ y \in \mathbb{R}^n$, *we have*

(i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2$;

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$.

Lemma 2. *Let $\{a_k\}$ be a sequence of nonnegative real numbers satisfying the following condition:*

$$a_{k+1} \leq (1-\alpha_k)a_k + \alpha_k\delta_k + \beta_k \ \forall k \geq 1,$$

where

(i) $\{\alpha_k\} \subset [0,1]$, $\sum_{k=0}^{\infty} \alpha_k = +\infty$;

(ii) $\limsup \delta_k \leq 0$;

(iii) $\beta_k \geq 0$, $\sum_{n=1}^{\infty} \beta_k < \infty$.

Then, $\lim_{k\to\infty} a_k = 0$.

3. Proximal Algorithm

In this paper, we consider the problem (2) with $H = \mathbb{R}^n$, the function $g : \mathbb{R}^n \to (-\infty, +\infty]$ is a proper, convex, continuous on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies following conditions:

(A_1) f is monotone and L-Lipschitz continuous on $\mathbb{R}^n$;

(A_2) the solution set of Problem (2) ($Sol(C, f)$) is nonempty.

Algorithm: Choose $x^0 \in \mathbb{R}^n$, sequences $\{\alpha_k\}$ and $\{\lambda_k\}$ such that

$$\begin{cases} \{\alpha_k\} \subset (0,1), \lim_{k\to\infty} \alpha_k = 0, \\ \sum_{k=0}^{\infty} \alpha_k = +\infty, \\ \{\lambda_k\} \subset [a,b] \subset (0, \frac{1}{L}) \subset (0,\infty). \end{cases} \quad (3)$$

Step 1. $(k = 0, 1, ...)$ Find $y^k \in C$:

$$y^k = Prox_{\lambda_k g}(x^k - \lambda_k f(x^k)).$$

If $x^k - y^k = 0$ then stop.

Step 2. Calculate $x^{k+1} = \alpha_k x^0 + (1 - \alpha_k)(x^k - \rho_k d^k)$, where $d^k := x^k - y^k - \lambda_k(f(x^k) - f(y^k))$ and $\{\rho_k\}$ is defined by

$$\rho_k = \begin{cases} \frac{\langle x^k - y^k, d^k\rangle}{\|d^k\|^2} & \text{if } d^k \neq 0 \\ 0 & \text{if } d^k = 0. \end{cases}$$

Step 3. Set $k := k + 1$, and go to Step 1.

Lemma 3. *Let $x^* \in Sol(C, f)$. Then,*

$$\|w^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \|w^k - x^k\|^2,$$

trong đó $w^k := x^k - \rho_k d^k$.

Proof. From definition of y^k in Step 1 and Theorem 2.1.3 in [1], there exists $p^k \in \partial g(y^k)$ such that

$$x^k - y^k - \lambda_k f(x^k) - \lambda_k p^k \in N_C(y^k).$$

It is equivalent to

$$\langle x^k - \lambda_k f(x^k) - y^k, x - y^k\rangle \leq \lambda_k\langle p^k, x - y^k\rangle,$$

for all x in C. Single $p^k \in \partial g(y^k)$, we have

$$\langle x^k - \lambda_k f(x^k) - y^k, x - y^k\rangle \leq \lambda_k[g(x) - g(y^k)],$$

for all x in C. Replacing x with x^*, we get

$$\langle y^k - x^*, x^k - y^k - \lambda_k f(x^k)\rangle \geq \lambda_k[g(y^k) - g(x^*)]. \quad (4)$$

Combining $x^* \in Sol(C, f)$ with $y^k \in C$ and the monotony of f, we have

$$\lambda_k\langle f(y^k), y^k - x^*\rangle \geq -\lambda_k[g(y^k) - g(x^*)]. \quad (5)$$

From (4) and (5), it follows that

$$0 \leq \langle y^k - x^*, d^k\rangle.$$

Then, by definition of w^k we have

$$\begin{aligned} &\|w^k - x^*\|^2 \\ =&\|x^k - \rho_k d^k - x^*\|^2 \\ =&\|x^k - x^*\|^2 - 2\rho_k\langle x^k - x^*, d^k\rangle + \rho_k^2\|d^k\|^2 \\ \leq&\|x^k - x^*\|^2 - 2\rho_k\langle x^k - y^k, d^k\rangle + \rho_k^2\|d^k\|^2 \\ =&\|x^k - x^*\|^2 - \rho_k\langle x^k - y^k, d^k\rangle, \end{aligned}$$

and

$$\rho_k\langle x^k - y^k, d^k\rangle = \|\rho_k d^k\|^2 = \|w^k - x^k\|^2.$$

It follows that

$$\begin{aligned} \|w^k - x^*\|^2 \leq&\|x^k - x^*\|^2 - \rho_k\langle x^k - y^k, d^k\rangle \\ =&\|x^k - x^*\|^2 - \|w^k - x^k\|^2 \\ \leq&\|x^k - x^*\|^2 - \|w^k - x^k\|^2. \end{aligned}$$

□

Lemma 4. *Sequences $\{x^k\}$ and $\{w^k\}$ are bounded.*

Proof. Let $x^* \in Sol(C, f)$. By Lemma 3, we have

$$\begin{aligned}&\|x^{k+1} - x^*\|\\ =&\|\alpha_k x^0 + (1-\alpha_k)w^k - x^*\|\\ \leq&\alpha_k\|x^0 - x^*\| + (1-\alpha_k)\|w^k - x^*\|\\ \leq&\alpha_k\|x^0 - x^*\| + (1-\alpha_k)\|x^k - x^*\|\\ \leq&\max\left\{\|x^0 - x^*\|, \|x^k - x^*\|\right\}\\ &...\\ \leq&\|x^0 - x^*\| < +\infty,\end{aligned}$$

implies that $\{x^k\}$ is bounded. So $\{w^k\}$ is bounded by Lemma 3. □

Lemma 5. *Let $x^* \in Sol(C, f)$. Put $a_k = \|x^k - x^*\|^2$, $b_k = 2\langle x^0 - x^*, x^{k+1} - x^*\rangle$. Then,*

(i) $a_{k+1} \leq (1-\alpha_k)a_k + \alpha_k b_k$;

(ii) $-1 \leq \limsup_{k\to\infty} b_k < \infty$.

Proof. Using Lemma 1 (ii), we have

$$\begin{aligned}&\|x^{k+1} - x^*\|^2 =\\ &\|\alpha_k(x^0 - x^*) + (1-\alpha_k)(w^k - x^*)\|^2 \leq\\ &(1-\alpha_k)\|w^k - x^*\|^2 + 2\alpha_k\langle x^0 - x^*, x^{k+1} - x^*\rangle.\end{aligned}$$

This together with $\alpha_k \in (0, 1)$ and Lemma 3 implies that (i).

Single $\{x^k\}$ is bounded, we have

$$b_k \leq 2\|x^0 - x^*\|\|x^{k+1} - x^*\| < \infty,$$

and so $\limsup_{k\to\infty} b_k < \infty$. Assume by contradiction that $\limsup_{k\to\infty} b_k < -1$. There exists $k_0 \in \mathcal{N}$ such that $b_k < -1$ for all $k \geq k_0$. It follows from (i) that, for all $k \geq k_0$,

$$\begin{aligned}a_{k+1} \leq&(1-\alpha_k)a_k + \alpha_k b_k\\ \leq&a_k - \alpha_k.\end{aligned}$$

Consequently

$$a_{k+1} \leq a_{k_0} - \sum_{i=k_0}^{k}\alpha_i \ \forall k \geq k_0.$$

Taking the limit superior of both sides, we have

$$\limsup_{k\to\infty} a_k \leq a_{k_0} - \sum_{i=k_0}^{+\infty}\alpha_i + \sum_{i=k_0}^{+\infty}\beta_i = -\infty.$$

This contradicts the fact that $a_k \geq 0$ for all $k \in \mathcal{N}$. Therefore, $\limsup_{k\to\infty} b_k \geq -1$. □

Lemma 6. *Let $\|x^k - y^k\| \to 0$, and a subsequence $\{x^{k_i}\}$ of $\{x^k\}$ converge to p. Then, $p \in Sol(C, f)$.*

Proof. From (4), one has

$$\begin{aligned}&\langle x^{k_i} - \lambda_{k_i}f(x^{k_i}) - y^{k_i}, x - y^{k_i}\rangle\\ \leq&\lambda_{k_i}[g(x) - g(y^{k_i})] \quad \forall x \in C.\end{aligned}$$

It is equivalent to

$$\begin{aligned}&\langle x^{k_i} - y^{k_i}, x - y^{k_i}\rangle + \langle\lambda_{k_i}f(x^{k_i}), y^{k_i} - x^{k_i}\rangle\\ &\leq \langle\lambda_{k_i}f(x^{k_i}), x - x^{k_i}\rangle + \lambda_{k_i}[g(x) - g(y^{k_i})],\end{aligned}$$

forall $x \in C$. Since $\{x^k\}$ is bounded and $\lim_{i\to\infty}\|x^k - y^k\| = 0$, $\{y^k\}$ is also bounded and $y^{k_i} \to p$. By (A_1), $f(x^{k_i}) \to f(p)$. Letting $i \to \infty$ in the last inequality, we get

$$0 \leq \langle f(p), x - p\rangle + g(x) - g(p).$$

Hence, $p \in Sol(C, f)$. □

Theorem 1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping satisfying the assumptions $(A_1) - (A_2)$. Then, the sequence $\{x^k\}$ generated by the algorithm converges to a solution $z = P_{Sol(C,f)}(x^0)$.*

Proof. Putting $a_k := \|x^k - z\|^2$, In order to prove the strong convergence of the algorithm, we consider two following cases.

Case A. Assume $a_{k+1} \leq a_k$ for every $k \geq k_0,\ k_0 \in \mathbb{N}$. Then, one has

$$\lim_{k\to\infty} a_k \in [0, \infty).$$

From Step 3, Lemma 3 and Lemma 1 (ii), it follows that

$$\begin{aligned}
&\|x^{k+1} - z\|^2 \\
=&\|(1-\alpha_k)(w^k - z) + \alpha_k(x^0 - z)\|^2 \\
\leq&\|w^k - z\|^2 + 2\alpha_k\langle x^0 - z, x^{k+1} - z\rangle \\
\leq&\|w^k - z\|^2 + 2\alpha_k\langle x^0 - z, x^{k+1} - z\rangle \\
\leq&\|x^k - z\|^2 - \|w^k - x^k\|^2 + \alpha_k\Gamma_0,
\end{aligned}$$

where $\Gamma_0 := \sup\{2\langle x^0 - z, x^{k+1} - z\rangle : k = 0, 1, ...\} < \infty$. This implies that

$$a_{k+1} - a_k + \|w^k - x^k\|^2 \leq \alpha_k\Gamma_0 \ \forall k \geq 0. \quad (6)$$

Letting $k \to \infty$ in the above inequality, we obtain $\lim_{k\to\infty}\|w^k - x^k\| = 0$.

By (A_1), we have

$$\begin{aligned}
&\langle x^k - y^k, d^k\rangle \\
&= \|x^k - y^k\|^2 - \lambda_k\langle x^k - y^k, f(x^k) - f(y^k)\rangle \\
&\geq \|x^k - y^k\|^2 - \lambda_k\|x^k - y^k\|\|f(x^k) - f(y^k)\| \\
&\geq (1 - b\bar{L})\|x^k - y^k\|^2. \qquad (7)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|d^k\| &= \|x^k - y^k - \lambda_k(f(x^k) - f(y^k))\| \\
&\leq \|x^k - y^k\| + \lambda_k\|f(x^k) - f(y^k)\| \\
&\leq (1 + \lambda_k\bar{L})\|x^k - y^k\| \\
&\leq (1 + b\bar{L})\|x^k - y^k\|. \qquad (8)
\end{aligned}$$

By (7) and (8), one has

$$\langle x^k - y^k, d^k\rangle \geq \frac{1 - b\bar{L}}{(1 + b\bar{L})^2}\|d^k\|^2.$$

This together with (7) and Step 2 implies that

$$\begin{aligned}
\|x^k - y^k\|^2 \leq&\frac{1}{(1 - b\bar{L})}\langle x^k - y^k, d^k\rangle \\
=&\frac{1}{(1 - b\bar{L})\rho_k}\|w^k - x^k\|^2 \\
\leq&\frac{(1 + b\bar{L})^2}{(1 - b\bar{L})^2}\|w^k - x^k\|^2.
\end{aligned}$$

From the last inequality and $\lim_{k\to\infty}\|w^k - x^k\| = 0$, it follows that $\lim_{k\to\infty}\|x^k - y^k\| = 0$ and

$$\|w^k - y^k\| \leq \|w^k - x^k\| + \|x^k - y^k\| \to 0$$

as $k \to \infty$. Using Step 2 and Lemma 4, We obtain

$$\|x^{k+1} - w^k\| = \alpha_k\|x^0 - w^k\| \leq \alpha_k\Gamma_1 \to 0$$

as $k \to \infty$, where $\Gamma_1 = \sup\{\|x^0 - w^k\| : k = 0, 1, ...\} < +\infty$. It follows that

$$\|x^{k+1} - x^k\| \leq \|x^{k+1} - w^k\| + \|w^k - x^k\| \to 0$$

as $k \to \infty$. Since $\{x^k\}$ is bounded, there exists subsequence $\{x^{k_i+1}\}$ of $\{x^k\}$ such that $x^{k_i+1} \to p$ as $i \to \infty$ and

$$\begin{aligned}
&\limsup_{k\to\infty}\langle x^0 - z, x^{k+1} - z\rangle \\
&= \lim_{i\to\infty}\langle x^0 - z, x^{k_i+1} - z\rangle.
\end{aligned}$$

From $\lim_{k\to\infty}\|x^k - y^k\| = 0$ and Lemma 6, it follows that $p \in Sol(C, f)$. Consequently,

$$\begin{aligned}
\limsup_{k\to\infty} b_k &= 2\limsup_{k\to\infty}\langle x^0 - z, x^{k+1} - z\rangle \\
&= 2\lim_{i\to\infty}\langle x^0 - z, x^{k_i+1} - z\rangle \\
&= 2\langle x^0 - z, p - z\rangle \leq 0. \qquad (9)
\end{aligned}$$

This together with Lemma 2 and Lemma 5 (i) implies that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty}\|x^k - z\|^2 = 0.$$

Case B. Assume that there doesnt exists $\bar{k} \in \mathbb{N}$ such that $\{a_k\}_{k=\bar{k}}^{\infty}$ is monotonically decreasing. By Remark 4.4 in [5], there is a subsequence $\{a_{\tau(k)}\}$ of $\{a_k\}$ and a integer number k_0 such that $\tau(k) \nearrow +\infty$,

$$0 \leq a_k \leq a_{\tau(k)+1}, a_{\tau(k)} \leq a_{\tau(k)+1} \quad \forall k \geq k_0,$$

where

$$\tau(k) = \max\{i \in \mathbb{N} : k_0 \leq i \leq k, a_i \leq a_{i+1}\}.$$

From $a_{\tau(k)} \leq a_{\tau(k)+1}$, $\forall k \geq k_0$ and (6), it follows that

$$\begin{aligned} 0 &\leq \|w^{\tau(k)} - x^{\tau(k)}\| \\ &\leq a_{\tau(k)+1} - a_{\tau(k)} + \|w^{\tau(k)} - x^{\tau(k)}\| \\ &\leq \alpha_{\tau(k)}\Gamma_0 \\ &\to 0 \text{ as } k \to \infty, \end{aligned}$$

and so $\lim_{k\to\infty} \|w^{\tau(k)} - x^{\tau(k)}\| = 0$. By arguments similar to the *Case A*, we can show that

$$\begin{aligned} &\lim_{n\to\infty} \|x^{\tau(k)+1} - x^{\tau(k)}\| = \lim_{n\to\infty} \|x^{\tau(k)} - y^{\tau(k)}\| \\ &= \lim_{n\to\infty} \|w^{\tau(k)} - y^{\tau(k)}\| = 0. \end{aligned}$$

Since $\{x^{\tau(k)}\}$ is bounded, there exists a subsequence of $\{x^{\tau(k)}\}$ convergeing to $p \in \mathbb{R}^n$, without lost general, we still denote by $\{x^{\tau(k)}\}$. By 6, we have $p \in Sol(C, f)$. By arguments similar to the *Case A*, we can prove that

$$\limsup_{k\to\infty} b_{\tau(k)} \leq 0. \tag{10}$$

Using 5 (i) and $a_{\tau(k)} \leq a_{\tau(k)+1}$, $\forall k \geq k_0$, one obtains

$$a_{\tau(k)} \leq b_{\tau(k)}.$$

This together with Lemma 5 and (10) implies that

$$\limsup_{k\to\infty} a_{\tau(k)} \leq \limsup_{k\to\infty} b_{\tau(k)} \leq 0.$$

It follows that $\lim_{k\to\infty} a_{\tau(k)} = 0$. This together with the inequality

$$\begin{aligned} \sqrt{a_{\tau(k)+1}} &= \|x^{\tau(k)+1} - z\| \\ &\leq \|x^{\tau(k)+1} - x^{\tau(k)}\| + \|x^{\tau(k)} - z\|, \end{aligned}$$

implies that $\lim_{k\to\infty} \sqrt{a_{\tau(k)+1}} = 0$. Consequently,

$$\lim_{k\to\infty} a_{\tau(k)+1} = 0.$$

Since $0 \leq a_k \leq a_{\tau(k)+1}$ for every $k \geq k_0$, we have $\lim_{n\to\infty} a_k = 0$, and so $\{x^k\}$ converges to z. □

4. Conclusions

In this paper, by using proximal operators and Halpern iteration technique, we introduced a new algorithm for solving variational inequality problem in a general form and proved the algorithm convergents under standard assumptions imposed on cost mappings.

References

[1]. F. Facchinei and J. S. Pang, *Finite-dimensional variational inequalities and complementary problems*. Springer-Verlag, NewYork, 2003.

[2]. Y. Shehu, Q. L. Dong and D. Jiang, "Single projection method for pseudomonotone variational inequality in Hilbert spaces," *Optimzation,* vol. 68 pp. 385-409, 2018.

[3]. D. K. Pham and T. V. Phan, "Modified projection method for strongly pseudomonotone variational inequalities", *Journal of Global Optimization,* vol. 58, no. 2, pp. 341-350, 2014.

[4]. Y. Malitsky, "Projected relected gradient methods for monotone variational inequalities," *SIAM Journal on Optimization,* vol. 25, pp. 502-520, 2015.

[5]. P. E. Maingé, "Projected subgradient techniques and viscosity methods for optimization with variational inequality constraints," *European Journal of Operational Research,* vol. 205, pp. 501-506, 2010.