ON SOME PROPERTIES OF THE PSEUDO SUPPORT IN DIMENSION MORE THAN S

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ABSTRACT

Let (R, m) be a Noetherian local ring and M be a finitely generated Rmodule. Let s and i be integers such that $i \geq 0$ and $s \geq -1$. The i-th pseudo support in dimension more than s of M, denoted by Psupp$iR(M)$ $>s$, is defined by Psupp$iR(M)$ $>s$ = {p $\in$ Spec R | N-dim$R$p Hpi $-R$dimp $R/$ p(Mp) $>$ s}. The i-th pseudo support in dimension more than -1 of M is the i-th pseudo support of M introduced by Markus Brodmann and R. Y. Sharp, the i-th pseudo support in dimension more than 0 of M is the i-th length support of M introduced by Le Thanh Nhan, Nguyen Thi Kieu Nga and Pham Huu Khanh (2014). They are useful tools in studying Cohen-Macaulay loci. In this paper, we are going to present some properties of the i-th pseudo support in dimension more than s of M.

Key words: *the i-th pseudo support; catenary; universally catenary; attached primes; formal fibres.*

Received: 12/5/2020; Revised: 22/8/2020; Published: 27/8/2020

MỘT SỐ TÍNH CHẤT CỦA TẬP GIẢ GIÁ CHIỀU LỚN HƠN *S*

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TÓM TẮT

Cho (R, m) là một vành địa phương Noether và M là một R-môđun hữu hạn sinh. Cho $i \geq 0$ và $s \geq -1$ là các số nguyên. Giả giá thứ i chiều lớn hơn s của M, được ký hiệu bởi Psupp$iR(M)$ $>s$, và được định nghĩa là tập Psupp$iR(M)$ $>s$ = { p $\in$ Spec R | N-dim$R$p Hpi $-R$dimp $R/$ p(Mp) $>$ s}. Giả giá thứ i chiều lớn hơn -1 của M là giả giá thứ i của M được giới thiệu bởi Markus Brodmann và R. Y. Sharp, giả giá thứ i chiều lớn hơn 0 của M là giá suy rộng thứ i của M được giới thiệu bởi Lê Thanh Nhàn, Nguyễn Thị Kiều Nga và Phạm Hữu Khánh (2014). Các tập này là các công cụ hữu ích trong nghiên cứu các quỹ tích liên quan đến tính Cohen-Macaulay của môđun. Trong bài báo này, chúng tôi trình bày một số tính chất của tập giả giá thứ i chiều lớn hơn s của M.

Từ khóa: *Tập giả giá thứ i; tính catenary; tính catenary phổ dụng; tập các iđêan nguyên tố gắn kết; thớ hình thức.*

Ngày nhận bài: 12/5/2020; Ngày hoàn thiện: 22/8/2020; Ngày đăng: 27/8/2020

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https://doi.org/10.34238/tnu-jst.3119

1. Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module with $\dim M = d$. Let $i \geq 0$ and $s \geq -1$ be integers. Following M. Brodmann and R. Y. Sharp ([1]), *the i-th pseudo support of M*, denoted by $\mathrm{Psupp}^i_R(M)$, is the set

$$\{\mathfrak{p} \in \mathrm{Spec}\, R \mid H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p} R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0\}.$$

The pseudo supports of M play an important role in studying the dimension and multiplicity of local cohomologies with respect to the maximal ideal. Pseudo supports are also very useful in describing the non-Cohen-Macaulay locus $\mathrm{nCM}(M)$ of M (see [2]).

The notion of the length support was introduced by Nhan, Nga and Khanh in [3]. The *ith length support* of M, denoted by $\mathrm{Lsupp}^i_R(M)$, is the set

$$\{\mathfrak{p} \in \mathrm{Spec}(R) | \ell_{R_{\mathfrak{p}}}(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p} R_{\mathfrak{p}}}(M_{\mathfrak{p}})) = \infty\}.$$

The length supports are effective in investigating the non-generalized Cohen-Macaulay locus $\mathrm{nGCM}(M)$ of M (see [3]).

In [4], L. P. Thao introduced the notion of the i-th pseudo support in dimension more than s of M and described the non-Cohen-Macaulay locus in dimension more than s via the i-th pseudo support in dimension more than s of M.

Definition 1.1. *The i-th pseudo support in dimension more than s of M,* denoted by $\mathrm{Psupp}^i_R(M)_{>s}$, is the set $\{\mathfrak{p} \in \mathrm{Spec}\, R \mid \text{N-}\dim_{R_{\mathfrak{p}}} \big(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p} R_{\mathfrak{p}}}(M_{\mathfrak{p}})\big) > s\}$.

Note that if $s = -1$ then the i-th pseudo support in dimension more than -1 of M is the i-th pseudo support of M. If $s = 0$, then the i-th pseudo support in dimension more than 0 of M is i-th length support of M.

In this paper, the i-th pseudo support in dimension more than s of M under completion, localization, and other its properties are investigated.

2. Main results

For each ideal I of R, denote by $\mathrm{Var}(I)$ the set of all prime ideals of R containing I. First, we recall the property (*) for an Artinian R-module A, which was considered firstly by N. T. Cuong and L. T. Nhan (see [5])

$$\mathrm{Ann}_R(0 :_A \mathfrak{p}) = \mathfrak{p} \text{ for all } \mathfrak{p} \in \mathrm{Var}(\mathrm{Ann}_R A).$$

If R is complete with respect to $\mathfrak{m}$-adic topology, it follows by Matlis duality that the property (*) is satisfied for all Artinian R-modules A. When R is universally catenary and all its formal fibres are Cohen-Macaulay, $H^i_{\mathfrak{m}}(M)$ satisfies the property (*) for any integer i, cf. [6].

Lemma 2.1. *The following statements are true:*

(i) For each integer $i \geq 0$, $\mathrm{Psupp}^i_R(M) \subseteq \mathrm{Var}(\mathrm{Ann}_R H^i_{\mathfrak{m}}(M))$. Furthermore $H^i_{\mathfrak{m}}(M)$ satisfies the property (*) if and only if $\mathrm{Psupp}^i(M) = \mathrm{Var}(\mathrm{Ann}_R H^i_{\mathfrak{m}}(M))$.

(ii) If the ring $R/\mathrm{Ann}_R M$ is universally catenary and all its formal fibres are Cohen-Macaulay then $H^i_{\mathfrak{m}}(M)$ satisfies the property (*) for all i.

The following lemma proved that the property (*) of local cohomology is unchanged under localization (see [7]).

Lemma 2.2. *Let $i \geq 0$ be an integer. Assume that $R/\mathrm{Ann}_R M$ is catenary. Then the following statements are equivalent:*

(i) $H^i_{\mathfrak{m}}(M)$ satisfies the property (*);

(ii) $H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p} R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ satisfies the property (*) for all $\mathfrak{p} \in \mathrm{Supp}(M)$.

The property (*) of an Artinian module A is a sufficient condition. It means that the

Krull dimension of A is equal to the Noetherian dimension of A (see [5], Proposition 4.6).

Lemma 2.3. *Let A be an Artinian module. If A satisfies the property (*) then*

$$\text{N-dim}_R A = \dim_R R/\operatorname{Ann}_R A.$$

Remark 2.4. *(i) If R is universally catenary and all its formal fibres are Cohen-Macaulay then by the Lemma 2.1, Lemma 2.2, and Lemma 2, we have $\operatorname{Psupp}^i_R(M)_{>s}$ is the set $\{\mathfrak{p} \in \operatorname{Spec} R \mid \dim_R R_\mathfrak{p}/\operatorname{Ann}_{R_\mathfrak{p}} H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) > s\}$.*
(ii) $\operatorname{Psupp}^i_R(M)_{>s} \subseteq \operatorname{Psupp}^i_R(M)$.

Now, we collect some properties of the pseudo support in the following lemma. These are known and can be found in [1], [2] and [7]. For each subset T of $\operatorname{Spec}(R)$ and each positive integer $k \geqslant 0$, we set $(T)_k = \{\mathfrak{p} \in T \mid \dim(R/\mathfrak{p}) = k\}$.

Lemma 2.5. *(i) If R is catenary then $\operatorname{Psupp}^i_R(M)$ is closed under specialization.*
(ii) $\big(\operatorname{Psupp}^i_R(M)\big)_i = (\operatorname{Ass}_R M)_i$.

The following lemma is a key one of the paper.

Lemma 2.6. *If R is universally catenary and all its formal fibres are Cohen-Macaulay then $\operatorname{Psupp}^i_R(M)_{>s}$ is the complement of the set $\{\mathfrak{p} \in \operatorname{Psupp}^i_R(M) \mid \forall\, \mathfrak{q} \subseteq \mathfrak{p}, \mathfrak{q} \in \operatorname{Psupp}^i_R(M), \operatorname{ht}(\frac{\mathfrak{p}}{\mathfrak{q}}) \leq s\}$ in $\operatorname{Psupp}^i_R(M)$. In particular, $Psupp^i_R(M)$ is closed under specialization.*

Proof. Let $\mathfrak{p}$ on the left hand side of the above equation. Then there exits $\mathfrak{q} \in \operatorname{Psupp}^i_R(M)$ such that $\operatorname{ht}(\frac{\mathfrak{p}}{\mathfrak{q}}) > s$. Since $\mathfrak{q} \in \operatorname{Psupp}^i_R(M)$ then $H^{i-\dim(R/\mathfrak{q})}_{\mathfrak{q}R_\mathfrak{q}}(M_\mathfrak{q}) \neq 0$. Since R is catenary, $(i-\dim R/\mathfrak{p})-\dim R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}$ is equal to $(i-\dim R/\mathfrak{p}) - \operatorname{ht}\mathfrak{p}/\mathfrak{q}$, which is equal to $(i-\dim R/\mathfrak{p}) - (\dim R/\mathfrak{q} - \dim R/\mathfrak{p})$. Hence $(i-\dim R/\mathfrak{p})-\dim R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p} = i-\dim R/\mathfrak{q}$. So that $H^{i-\dim(R/\mathfrak{p})-\dim(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p})}_{\mathfrak{q}(R_\mathfrak{p})_{\mathfrak{q}R_\mathfrak{p}}}(M_\mathfrak{p})_{\mathfrak{q}R_\mathfrak{p}} \cong H^{i-\dim(R/\mathfrak{q})}_{\mathfrak{q}R_\mathfrak{q}}(M_\mathfrak{q}) \neq 0$. This implies $\mathfrak{q}R_\mathfrak{p} \in \operatorname{Psupp}^{i-\dim(R/\mathfrak{p})}_{R_\mathfrak{p}}(M_\mathfrak{p})$. We have $\mathfrak{q}R_\mathfrak{p} \supseteq \operatorname{Ann}_{R_\mathfrak{p}} H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})$ by the Lemma 2.4. On the other hand $\operatorname{ht}(\frac{\mathfrak{p}}{\mathfrak{q}}) > s$. Then

$$\dim\big(R_\mathfrak{p}/\operatorname{Ann}_{R_\mathfrak{p}} H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\big) > s.$$

So, $\mathfrak{p} \in \operatorname{Psupp}^i_R(M)_{>s}$.
Conversely, let $\mathfrak{p} \in \operatorname{Psupp}^i_R(M)_{>s}$. Since R is universally catenary and all its formal fibres are Cohen-Macaulay, $\dim_R R_\mathfrak{p}/\operatorname{Ann}_{R_\mathfrak{p}} H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) > s$. Then there exists $\mathfrak{q}R_\mathfrak{p} \in \operatorname{Att}_{R_\mathfrak{p}} H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})$ such that $\dim(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) > s$, i.e. $\mathfrak{q} \in \operatorname{Att}_R H^i_\mathfrak{m}(M)$ and $\operatorname{ht}(\mathfrak{p}/\mathfrak{q}) > s$. By Lemma 2.1, $\operatorname{Psupp}^i(M) = \operatorname{Var}(\operatorname{Ann}_R H^i_\mathfrak{m}(M))$. Hence $\mathfrak{q} \in \operatorname{Psupp}^i_R(M)$. This proves $\mathfrak{p} \in \operatorname{Psupp}^i_R(M) \setminus \{\mathfrak{p} \in \operatorname{Psupp}^i_R(M) \mid \forall\, \mathfrak{q} \subseteq \mathfrak{p}, \mathfrak{q} \in \operatorname{Psupp}^i_R(M), \operatorname{ht}(\frac{\mathfrak{p}}{\mathfrak{q}}) \leq s\}$.

Let $\mathfrak{q} \subseteq \mathfrak{p}$ such that $\mathfrak{q} \in \operatorname{Psupp}^i_R(M)_{>s}$. Then $\mathfrak{q} \in \operatorname{Psupp}^i_R(M)$. Since R is catenary, $\mathfrak{p} \in \operatorname{Psupp}^i_R(M)$. For all $\theta \in \operatorname{Psupp}^i_R(M)$, we have $\operatorname{ht}(\mathfrak{q}/\theta) > s$. Hence $\operatorname{ht}(\mathfrak{p}/\theta) > s$. This implies $\mathfrak{p} \in \operatorname{Psupp}^i_R(M)_{>s}$. □

Corollary 2.7. *If R is universally catenary and all its formal fibers are Cohen-Macaulay then $\operatorname{Psupp}^i_R(M)_{>s}$ can be considered as the complement in $\operatorname{Psupp}^i_R(M)$ of one of the following sets*
(i) $\{\mathfrak{p} \mid \forall\, \mathfrak{q} \subseteq \mathfrak{p}, \mathfrak{q} \in \operatorname{Psupp}^i_R(M), \operatorname{ht}(\frac{\mathfrak{p}}{\mathfrak{q}}) \leq s\}$
(ii)$\{\mathfrak{p} \mid \forall\, \mathfrak{q} \subseteq \mathfrak{p}, \mathfrak{q} \in \min \operatorname{Psupp}^i_R(M), \operatorname{ht}(\frac{\mathfrak{p}}{\mathfrak{q}}) \leq s\}$
(iii)$\{\mathfrak{p} \mid \forall\, \mathfrak{q} \subseteq \mathfrak{p}, \mathfrak{q} \in \min \operatorname{Att}_R H^i_\mathfrak{m}(M), \operatorname{ht}(\frac{\mathfrak{p}}{\mathfrak{q}}) \leq s\}$.

Proof. It folows from the fact that $\operatorname{Psupp}^i(M) = \operatorname{Var}(\operatorname{Ann}_R H^i_\mathfrak{m}(M))$. □

Let $\widehat{R}$ and $\widehat{M}$ be the $\mathfrak{m}$-adic completion of R and M, respectively. The following result gives a new property of the i-th pseudo support in dimension more than s of M under completion.

Proposition 2.8. *Assume that R is universally catenary and all its formal fibers are Cohen-Macaulay. Then $\operatorname{Psupp}^i_R(M)_{>s}$ is the subset of $\{P \cap R \mid P \in \operatorname{Psupp}^i_{\widehat{R}}(\widehat{M})_{>s}\}$.*

Proof. Let $\mathfrak{p} \in \operatorname{Psupp}^i_R(M)_{>s}$. Then $\mathfrak{p} \in \operatorname{Psupp}^i_R(M)$. We have $\mathfrak{p} \supseteq \mathfrak{q} \in \min \operatorname{Psupp}^i_R(M)$. Hence $\operatorname{ht}(\frac{\mathfrak{p}}{\mathfrak{q}}) > s$ by Lemma 2.6. Let $P \in \operatorname{Ass}(\widehat{R}/\mathfrak{p}\widehat{R})$ such that $\dim(R/\mathfrak{p}) = \dim(\widehat{R}/P)$. Since $\mathfrak{p} \in \operatorname{Psupp}^i_R(M)$, $H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$. On the other hand, since $P \cap R = \mathfrak{p}$ and the homomorphism $R_\mathfrak{p} \to \widehat{R}_P$ is flat, we have $H^{i-\dim(\widehat{R}/P)}_{P\widehat{R}_P}(\widehat{M}_P) \cong \widehat{R}_P \otimes H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$. Hence $P \in \operatorname{Psupp}^i_{\widehat{R}}(\widehat{M})$. The homomorphism $R \to \widehat{R}$ satisfies the Going up property. It means that there exists $Q \in \operatorname{Spec}(\widehat{R}), Q \subseteq P$ such that $\operatorname{ht}(P/Q) \geq \operatorname{ht}(\mathfrak{p}/\mathfrak{q}) > s$ and $Q \cap R = \mathfrak{q}$. Since R is catenary, $\dim(R/\mathfrak{q}) \geq \dim(\widehat{R}/Q)$. Moveover,

$$\begin{aligned}\dim(\widehat{R}/Q) &= \dim(\widehat{R}/P) + \operatorname{ht}(P/Q) \quad (1)\\ &\geq \dim(R/\mathfrak{p}) + \operatorname{ht}(\mathfrak{p}/\mathfrak{q})\\ &= \dim(R/\mathfrak{q}).\end{aligned}$$

So $\dim(R/\mathfrak{q}) = \dim(\widehat{R}/Q)$. On the other hand, we have $H^{i-\dim(R/\mathfrak{q})}_{\mathfrak{q}R_\mathfrak{q}}(M_\mathfrak{q}) \neq 0$ and the homomorphism $R_\mathfrak{q} \to \widehat{R}_Q$ is flat, then $H^{i-\dim(\widehat{R}/Q)}_{Q\widehat{R}_Q}(\widehat{M}_Q) \neq 0$, then $Q \in \operatorname{Psupp}^i_{\widehat{R}}(\widehat{M})$. Since $\operatorname{ht}(P/Q) > s$, by Lemma 2.6, $P \in \operatorname{Psupp}^i_{\widehat{R}}(\widehat{M})_{>s}$. $\square$

The following example show that we can have the strict inclusion in Proposition 2.8.

Example 2.9. Let $t > 0$ be an integer. Then there exists a Noetherian local ring $(R, \mathfrak{m})$ such that R is a quotion of a regular ring, $\mathfrak{p} \in \operatorname{Spec}(R)$, $P \in \operatorname{Spec}(\widehat{R})$, $P \cap R = \mathfrak{p}$ and $\dim(\widehat{R}_P/\mathfrak{p}\widehat{R}_P) = t$. Let $Q\widehat{R}_P \in \operatorname{Ass}_{\widehat{R}_P}(\widehat{R}_P/\mathfrak{p}\widehat{R}_P)$ so that $\dim(\widehat{R}_P/Q\widehat{R}_P) = t$. Then $Q \in \operatorname{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$, $Q \subset P$. Since $t > 0$, $Q \neq P$. Set $\dim(\widehat{R}/Q) = k$. Then $Q \in \operatorname{Psupp}^k_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$ by the Lemma 2.5. This implies $P \in \operatorname{Psupp}^k_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$ since $\operatorname{Psupp}^k_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$ is closed under specialization. Since $\operatorname{ht}(P/Q > 0)$, by the Lemma 2.6, $P \in \operatorname{Psupp}^i_R(M)_{>0}$. Moveover, we have $Q \in \operatorname{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$, $Q \cap R = \mathfrak{p}$. Hence $\mathfrak{p} \in \operatorname{Psupp}^k_R(R/\mathfrak{p})$ as $Q \in \operatorname{Psupp}^k_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$ and [7], Proposition 3.2. It deduces that $\mathfrak{p} \in \min \operatorname{Psupp}^k_R(R/\mathfrak{p})$. By the Lemma 2.6 $\mathfrak{p} \notin \operatorname{Psupp}^i_R(M)_{>0}$. Set $M := R/\mathfrak{p}$. We have $P \in \operatorname{Psupp}^i_R(M)_{>0}$ and $P \cap R = \mathfrak{p}$, however $\mathfrak{p} \notin \operatorname{Psupp}^i_R(M)_{>0}$.

Finally, we study the i-th pseudo support in dimension more than s of M under localization. The results is presented in the following theorem.

Theorem 2.10. *Assume that R is universally catenary and all its formal fibers are Cohen-Macaulay. Let $\mathfrak{p} \in \operatorname{Supp}_R(M)$. Then $\operatorname{Psupp}^{i-\dim(R/\mathfrak{p})}_{R_\mathfrak{p}}(M_\mathfrak{p})_{>s}$ is the set*

$$\{\mathfrak{q}R_\mathfrak{p} \mid \mathfrak{q} \in \operatorname{Psupp}^i_R(M)_{>s}, \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Proof. Let $\mathfrak{q}R_\mathfrak{p} \in \operatorname{Psupp}^{i-\dim(R/\mathfrak{p})}_{R_\mathfrak{p}}(M_\mathfrak{p})_{>s}$. Then $\mathfrak{q}R_\mathfrak{p} \in \operatorname{Psupp}^{i-\dim(R/\mathfrak{p})}_{R_\mathfrak{p}}(M_\mathfrak{p})$. By Corollary 2.7

$$\mathfrak{q}R_\mathfrak{p} \supseteq \mathfrak{q}_1 R_\mathfrak{p} \in \min \operatorname{Att}_{R_\mathfrak{p}}\left(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\right).$$

Since R is universally catenary and all its formal fibers are Cohen-Macaulay, by [7] and the Lemma 2.2, we have $\operatorname{Psupp}^{i-\dim(R/\mathfrak{p})}_{R_\mathfrak{p}}(M_\mathfrak{p})$ is the set $\{\mathfrak{q}R_\mathfrak{p} \mid$

$\mathfrak{q} \in \mathrm{Psupp}^i_R(M), \mathfrak{q} \subseteq \mathfrak{p}\}$ and $\min \mathrm{Att}_{R_\mathfrak{p}} \left(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p} R_\mathfrak{p}}(M_\mathfrak{p})\right)$ is equal to the set $\{\mathfrak{q} R_\mathfrak{p} \mid \mathfrak{q} \in \min \mathrm{Att}_R H^i_\mathfrak{m}(M), \mathfrak{q} \subseteq \mathfrak{p}\}$. These mean that $\mathfrak{q} \in \mathrm{Psupp}^i_R(M)$ and $\mathfrak{q}_1 \in \min \mathrm{Att}_R H^i_\mathfrak{m}(M)$. Since $\mathfrak{q} R_\mathfrak{p} \in \mathrm{Psupp}^{i-\dim(R/\mathfrak{p})}_{R_\mathfrak{p}}(M_\mathfrak{p})_{>s}$, $\mathrm{ht}(\mathfrak{q} R_\mathfrak{p}/\mathfrak{q}_1 R_\mathfrak{p}) > s$, we have $\mathrm{ht}(\mathfrak{q}/\mathfrak{q}_1) > s$. This follows that $\mathfrak{p} \in \mathrm{Psupp}^i_R(M)_{>s}$.

Conversely, let $\mathfrak{q} \in \mathrm{Psupp}^i_R(M)_{>s}$ and $\mathfrak{q} \subseteq \mathfrak{p}$. Then $\mathfrak{q} \in \mathrm{Psupp}^i_R(M)$. Hence $\mathfrak{q} R_\mathfrak{p} \in \mathrm{Psupp}^{i-\dim(R/\mathfrak{p})}_{R_\mathfrak{p}}(M_\mathfrak{p})$ by [7], Lemma 2.1. This implies

$$H^{i-\dim(R/\mathfrak{p})-\dim(R_\mathfrak{p}/\mathfrak{q} R_\mathfrak{p})}_{\mathfrak{q}(R_\mathfrak{p})_{\mathfrak{q} R_\mathfrak{p}}}(M_\mathfrak{p})_{\mathfrak{q} R_\mathfrak{p}} \neq 0.$$

Since R is catenary, we have $\dim R/\mathfrak{q} = \dim R/\mathfrak{p} + \mathrm{ht}\,\mathfrak{p}/\mathfrak{q} = \dim R/\mathfrak{p} + \dim R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}$. On the other hand, we have $(M_\mathfrak{p})_{\mathfrak{q} R_\mathfrak{p}} \cong M_\mathfrak{q}$ and $\mathfrak{q}(R_\mathfrak{p})_{\mathfrak{q}R_\mathfrak{p}} \cong \mathfrak{q} R_\mathfrak{q}$. These follow that $H^{i-\dim(R/\mathfrak{q})}_{\mathfrak{q} R_\mathfrak{q}}(M_\mathfrak{q}) \cong H^{i-\dim(R/\mathfrak{p})-\dim(R_\mathfrak{p}/\mathfrak{q} R_\mathfrak{p})}_{\mathfrak{q}(R_\mathfrak{p})_{\mathfrak{q} R_\mathfrak{p}}}(M_\mathfrak{p})_{\mathfrak{q} R_\mathfrak{p}} \neq 0$. Hence, $\mathfrak{q} R_\mathfrak{p} \in \mathrm{Psupp}^{i-\dim(R/\mathfrak{p})}_{R_\mathfrak{p}}(M_\mathfrak{p})$.

There exists $\mathfrak{q}_1 \in \mathrm{Spec}(R)$ such that $\mathfrak{q} \supseteq \mathfrak{q}_1 \in \min \mathrm{Att}_R H^i_\mathfrak{m}(M)$. Since $\mathfrak{q} \in \mathrm{Psupp}^i_R(M)_{>s}$, $\mathrm{ht}(\mathfrak{q}/\mathfrak{q}_1) > s$. By [7], Theorem 1.2 (i), we have $\mathfrak{q}_1 R_\mathfrak{p} \in \min \mathrm{Att}_{R_\mathfrak{q}} H^{i-\dim(R/\mathfrak{q})}_{\mathfrak{q} R_\mathfrak{q}}(M_\mathfrak{q})$. It deduces that $\mathrm{ht}(\mathfrak{q} R_\mathfrak{p}/\mathfrak{q}_1 R_\mathfrak{p}) = \mathrm{ht}(\mathfrak{q}/\mathfrak{q}_1) > s$. Then $\mathfrak{q} R_\mathfrak{p} \in \mathrm{Psupp}^{i-\dim(R/\mathfrak{p})}_{R_\mathfrak{p}}(M_\mathfrak{p})_{>s}$. $\square$

3. **Conclusion**

Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finitely generated R-module. Let s and i be integers such that $i \geq 0$ and $s \geq -1$. In this paper, we have present some new properties of the i-th pseudo support in dimension more than s of M under completion and localization.

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