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CONSTRUCT INTEGRAL INEQUALITY FROM ALGEBRAIC INEQUALITY

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ABSTRACT

Integral inequality is part of the inequality. This is a difficult content. It attracts many mathematicians interested in, research and development. Integral inequalities are widely used in optimization problems, calculus, differential equations, integral equations, etc. In this paper, using the order-preserving property of the limit, we describe the construction of some integral inequalities from some known algebraic inequalities.

Keywords: *Inequality; integral inequality; algebra inequality; integrable; limits.*

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XÂY DỰNG BẤT ĐẲNG THỨC TÍCH PHÂN TỪ BẤT ĐẲNG THỨC ĐẠI SỐ

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TÓM TẮT

Bất đẳng thức tích phân là một phần trong nội dung bất đẳng thức. Đây là một nội dung khó. Nó thu hút được nhiều nhà toán học quan tâm, nghiên cứu và phát triển. Bất đẳng thức tích phân được sử dụng nhiều trong các bài toán tối ưu, giải tích, phương trình vi phân, phương trình tích phân… Trong bài báo này, bằng việc sử dụng tính chất bảo tồn thứ tự của giới hạn, chúng tôi mô tả việc xây dựng một số bất đẳng thức tích phân từ một số bất đẳng thức đại số đã biết.

Từ khóa: *bất đẳng thức; tích phân; đại số; khả tích; giới hạn.*

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1 Method description

In this section, we describe the method to construct integral inequalities from algebraic inequalities. We recall the definition of integrable function on closed interval $[a;b]$.

Definition 1. Let $f(x)$ definite on $[a;b]$. By a partition of $[a;b]$, we mean a finite set of points $a = t_0 < t_1 < \cdots < t_n = b$. On each interval $[t_{i-1};t_i]$, we choose $x_i (i = 1, ..., n)$. Let $\delta_i = t_i - t_{i-1}$. Put

$$S = \sum_{i=1}^{n} f(x_i)\delta_i$$

and $\Delta = \max \delta_i$. If the limits $\lim\limits_{\Delta \to 0} S$ exits and indepent on all partitions of $[a;b]$ and x_i, $f(x)$ is called integrable on $[a;b]$. This limits is called integral of $f(x)$ on $[a;b]$. Denote $\int\limits_a^b f(x)dx$.

It is easy to see that if $f(x)$ is integrable on $[a;b]$, we have

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i) = \int\limits_a^b f(x)dx \tag{1}$$

here $x_i \in [a + \frac{i-1}{n}; a + \frac{i}{n}]$, $i = 1, ..., n$.

The method of constructing integral inequalities from known algebraic inequalities can be described as follows:

Assum that we have inequality

$$P(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i, ...) \geq 0. \tag{2}$$

Consider $a_i, b_i, ...$ as values of functions $f(x), g(x), ...$ at $x_i \in [a;b]$.

We convert the inequality (2) into a form

$$Q(\frac{b-a}{n} \sum_{i=1}^{n} a_i, \frac{b-a}{n} \sum_{i=1}^{n} b_i, ...) \geq 0. \tag{3}$$

Here P, Q is continuos functions. By preserving order properties of limits and (1), we have inequality (3) transformed into

$$Q(\int\limits_a^b f(x)dx, \int\limits_a^b g(x)dx, ...) \geq 0.$$

Now, we give some examples

Example 1. [1] Let $a_1, ..., a_n$ be positive real numbers. We have

$$\Big(a_1 + \cdots + a_n\Big)\Big(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\Big) \geq n^2. \tag{4}$$

Inequality (4) is equivalent to

$$(\frac{b-a}{n}\sum_{i=1}^{n}a_i)(\frac{b-a}{n}\sum_{i=1}^{n}\frac{1}{a_i})\geq(b-a)^2.$$

If we consider a_i as the values of integrable funtion $f(x)$, one have the following result.

Let $f(x)$ be positive and integrable on $[a;b]$, then (see [2])

$$\int_a^b f(x)dx\int_a^b\frac{dx}{f(x)}\geq(b-a)^2.$$

Example 2 (*Cauchy-Schwarz inequality*)**.** [1] Let $a_1,\cdots,a_n$ and $b_1,\cdots,b_n$ be real numbers. Then

$$\sum_{i=1}^{n}|a_ib_i|\leq\sqrt{\sum_{i=1}^{n}a_i^2}\sqrt{\sum_{i=1}^{n}b_i^2}. \tag{5}$$

For $a<b$, inequality (5) equivalent to

$$\frac{b-a}{n}\sum_{i=1}^{n}|a_ib_i|\leq\sqrt{\frac{b-a}{n}\sum_{i=1}^{n}a_i^2}\sqrt{\frac{b-a}{n}\sum_{i=1}^{n}b_i^2}$$

Consider a_i, b_i as the values of integrable funtions $f(x)$ and $g(x)$, we have.

If $f(x)$ and $g(x)$ are integrable on $[a;b]$, then (see [2])

$$\int\limits_a^b|f(x)g(x)|dx\leq\sqrt{\int\limits_a^b f^2(x)dx}\sqrt{\int\limits_a^b g^2(x)dx}.$$

2 Some problems

Problem 1. [1] Let $a_1,\cdots,a_n$ be positive real numbers. Then

$$\sum_{i=1}^{n}a_i\sum_{i=1}^{n}\frac{1-a_i}{a_i}\leq n\sum_{i=1}^{n}(1-a_i). \tag{6}$$

Construct the corresponding integral inequality.

Solution. For $a<b$, inequality (6) equivalent to

$$\Big(\frac{b-a}{n}\sum_{i=1}^{n}a_i\Big)\Big(\frac{b-a}{n}\sum_{i=1}^{n}\frac{1-a_i}{a_i}\Big)\leq\frac{b-a}{n}\Big(\sum_{i=1}^{n}(1-a_i)\Big).$$

Consider a_i as the values of integrable funtion $f(x)$, we have the following result. Let $f(x)$ be positive and integrable on $[a;b]$, then

$$\int\limits_a^b f(x)dx\int\limits_a^b\frac{1-f(x)}{f(x)}dx\leq(b-a)\int\limits_a^b(1-f(x))dx.$$

$\square$

Problem 2. [4] Let $0 < m \leq a_i \leq M$ for all $i = 1, ..., n$, we have

$$\Big(\sum_{i=1}^{n} a_i\Big)\Big(\sum_{i=1}^{n} \frac{1}{a_i}\Big) \leq \frac{n^2(M+m)^2}{4Mn} \tag{7}$$

Construct the corresponding integral inequality.

Solution. Inequality (7) is equivalent to

$$\Big(\frac{b-a}{n}\sum_{i=1}^{n} a_i\Big)\Big(\frac{b-a}{n}\sum_{i=1}^{n} \frac{1}{a_i}\Big) \leq (b-a)^2\frac{(M+m)^2}{4Mn}.$$

Consider a_i as the values of positive and integrable funtion $f(x)$, we have the following result.
If $f(x)$ is integrable on $[a; b]$ such that $0 < m \leq f(x) \leq M$, then (see [2])

$$\int\limits_a^b f(x)dx \int\limits_a^b \frac{1}{f(x)}dx \leq (b-a)^2\frac{(M+m)^2}{4Mn}$$

$\square$

Problem 3 (*Chebyshev inequality*)**.** [1] Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be real numbers such that $a_1 \leq \cdots \leq a_n$ and $b_1 \leq \cdots \leq b_n$ (or $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$), then

$$\Big(\sum_{i=1}^{n} a_i\Big)\Big(\sum_{i=1}^{n} b_i\Big) \leq n\sum_{i=1}^{n} a_i b_i. \tag{8}$$

Construct the corresponding integral inequality.

Solution. For $a < b$, the inequality (8) is equivalent to

$$\Big(\frac{b-a}{n}\sum_{i=1}^{n} a_i\Big)\Big(\frac{b-a}{n}\sum_{i=1}^{n} b_i\Big) \leq \frac{(b-a)^2}{n}\sum_{i=1}^{n} a_i b_i. \tag{9}$$

Consider a_i and b_i as values of funtions $f(x)$ and $g(x)$, we have the following result.
If $f(x)$ and $g(x)$ are either both increasing or both decreasing on $[a; b]$, then (see [2])

$$\int\limits_a^b f(x)dx \int\limits_a^b g(x)dx \leq (b-a)\int\limits_a^b f(x)g(x)dx.$$

$\square$

Problem 4 (*Hölder's inequality*)**.** [3] Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be real numbers and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{i=1}^{n} |a_i b_i| \leq \Big(\sum_{i=1}^{n} |a_i|^p\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^{n} |b_i|^q\Big)^{\frac{1}{q}} \tag{10}$$

Construct the corresponding integral inequality.

Solution. The inequality (10) is equivalent to

$$\frac{b-a}{n}\sum_{i=1}^{n}|a_ib_i| \le \Big(\frac{b-a}{n}\sum_{i=1}^{n}|a_i|^p\Big)^{\frac{1}{p}}\Big(\frac{b-a}{n}\sum_{i=1}^{n}|b_i|^q\Big)^{\frac{1}{q}}$$

for $a<b$. We have the following result.
If functions $f(x)$, $g(x)$ are integrable on $[a;b]$ and $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, then (see [2])

$$\int_a^b |f(x)g(x)|dx \le \Big(\int_a^b |f(x)|^p dx\Big)^{\frac{1}{p}}\Big(\int_a^b |g(x)|^q dx\Big)^{\frac{1}{q}}.$$

□

Problem 5 (*Minkowski inequality*)**.** [3]For $a_1,\cdots,a_n$, $b_1,\cdots,b_n$ are real numbers and $p>1$, then

$$\Big(\sum_{i=1}^{n}|a_i+b_i|^p\Big)^{\frac{1}{p}} \le \Big(\sum_{i=1}^{n}a_i^p\Big)^{\frac{1}{p}}+\Big(\sum_{i=1}^{n}b_i^p\Big)^{\frac{1}{p}} \tag{11}$$

Construct the corresponding integral inequality.

Solution. The inequality (11) is equivalent to

$$\Big(\frac{b-a}{n}\sum_{i=1}^{n}|a_i+b_i|^p\Big)^{\frac{1}{p}} \le \Big(\frac{b-a}{n}\sum_{i=1}^{n}a_i^p\Big)^{\frac{1}{p}}+\Big(\frac{b-a}{n}\sum_{i=1}^{n}b_i^p\Big)^{\frac{1}{p}}.$$

forall $a<b$. We have the following result.
If $f(x)$ and $g(x)$ integrable on $[a;b]$ and $p>1$, then (see [2])

$$\Big(\int_a^b |f(x)+g(x)|^p dx\Big)^{\frac{1}{p}} \le \Big(\int_a^b |f(x)|^p dx\Big)^{\frac{1}{p}}+\Big(\int_a^b |g(x)|^p dx\Big)^{\frac{1}{p}}.$$

□

Problem 6. [4] Let $a_1,\cdots,a_n$, b_1,b_n are real numbers, $b_i\neq 0$ and $m\le\dfrac{a_i}{b_i}\le M$ for all $i=1,\cdots,n$, then

$$\sum_{i=1}^{n}a_i^2+mM\sum_{i=1}^{n}b_i^2 \le (M+m)\sum_{i=1}^{n}a_ib_i. \tag{12}$$

Construct the corresponding integral inequality.

Solution. For $a<b$, the inequality (12) is equivalent to

$$\frac{b-a}{n}\sum_{i=1}^{n}a_i^2+mM\frac{b-a}{n}\sum_{i=1}^{n}b_i^2 \le (M+m)\frac{b-a}{n}\sum_{i=1}^{n}a_ib_i.$$

We have the following result.
Let $f(x)$ and $g(x)$ be integrable functions on $[a;b]$, $g(x)\neq 0$ and $m\le\dfrac{f(x)}{g(x)}\le M$ for all $x\in[a;b]$, then

$$\int_a^b f^2(x)dx+Mm\int_a^b g^2(x)dx \le (M+m)\int_a^b f(x)g(x)dx.$$

□

Problem 7 (*Jensen's inequality*). [3] If $\varphi(x)$ is a convex function on $[\alpha, \beta]$, then for $\alpha_1, ..., \alpha_n \in [\alpha; \beta]$, we have

$$\varphi\Big(\frac{\alpha_1 + \cdots + \alpha_n}{n}\Big) \leq \frac{\varphi(\alpha_1) + \cdots + \varphi(\alpha_n)}{n} \tag{13}$$

Construct the corresponding integral inequality.

Solution. For $a < b$, the inequality (13) is equivalent to

$$\varphi\Big(\frac{1}{b-a}\frac{b-a}{n}\sum_{i=1}^{n}\alpha_i\Big) \leq \frac{1}{b-a}\frac{b-a}{n}\sum_{i=1}^{n}\varphi(\alpha_i).$$

Consider α_i as the values of integrable function $f(x)$ on $[a; b]$. We have the following result.
Suppose that $f(x)$ integrable $[a; b]$ and $m \leq f(x) \leq M$ for all $x \in [a; b]$. If φ is continuous and convex on $[m; M]$, then (see [2])

$$\varphi\Big(\frac{1}{b-a}\int\limits_a^b f(x)dx\Big) \leq \frac{1}{b-a}\int\limits_a^b \varphi(f(x))dx.$$

□

Summary: In this paper, by using order-preserving through limiting, we give the method to construct some integral inequalities from known algebraic inequalities.

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