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*L*2 DECAY OF WEAK SOLUTIONS FOR THE NAVIER-STOKES EQUATIONS IN GENERAL DOMAINS

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ABSTRACT

Let *u* be a weak solution of the in-stationary Navier-Stokes equations in a completely general domain in R3. Firstly, we prove that the time decay rates of the weak solution *u* in the *L*2-norm like ones of the solutions for the homogeneous Stokes system taking the same initial value in which the decay exponent is less than 34 . Secondly, we show that under some additive conditions on the initial value, then *u* coincides with the solution of the homogeneous Stokes system when time tends to infinity. Our proofs use the theory about the uniqueness arguments and time decay rates of strong solutions for the Navier-Stokes equations in the general domain when the initial value is small enough.

Keywords: *Navier-Stokes equations, Decay , Weak solutions, Stokes equations, Uniqueness of solution.*

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DÁNG ĐIỆU TIỆM CẬN CỦA NGHIỆM YẾU CHO HỆ PHƯƠNG TRÌNH NAVIER-STOKES TRONG MIỀN TỔNG QUÁT VỚI CHUẨN *L*2

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TÓM TẮT

Giả sử *u* là một nghiệm yếu của hệ phương trình Navier-Stokes không dừng trong một miền tổng quát trong R3. Trước hết, chúng tôi chứng minh rằng tốc độ hội tụ theo thời gian của nghiệm yếu *u* với chuẩn *L*2 giống tốc độ hội tụ theo thời gian của nghiệm trong hệ Stokes thuần nhất với cùng giá trị ban đầu và số mũ hội tụ nhỏ hơn 34. Thứ hai, chúng tôi chỉ ra rằng với một số điều kiện của giá trị ban đầu thì *u* trùng với nghiệm của hệ Stokes thuần nhất khi thời gian dần tới vô cùng. Phần chứng minh các kết quả trong bài báo dựa trên lý thuyết về tính duy nhất và tốc độ hội tụ theo thời gian của nghiệm mạnh cho hệ phương trình Navier-Stokes trong miền tổng quát khi giá trị ban đầu đủ nhỏ.

Từ khóa: *Hệ phương trình Navier-Stokes, Dáng điệu tiệm cận, Nghiệm yếu, Hệ phương trình Stokes, Tính duy nhất nghiệm.*

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1 Introduction and main result

We consider the in-stationary problem of the Navier-Stokes system

$$\begin{cases} u_t - \Delta u + u\cdot\nabla u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u|_{\partial\Omega} = 0, \\ u(0,x) = u_0, \end{cases} \tag{1}$$

in a general domain $\Omega \subseteq \mathbb{R}^3$, i.e a non-empty connected open subset of $\mathbb{R}^3$, not necessarily bounded, with boundary $\partial\Omega$ and a time interval $[0,T), 0 < T \leq \infty$ and with the initial value u_0, where $u = (u_1, u_2, u_3)$; $u\cdot\nabla u = \operatorname{div}(uu), uu = (u_i u_j)_{i,j=1}$, if div $u = 0$.
In this paper we discuss the behavior as $t \to \infty$ of weak solutions of the Navier-Stokes equations in space $L^2(\Omega)$, which goes to zero with explicit rates. The L^2-decay problem for Navier-Stokes system was first posed by Leray [1] in $\mathbb{R}^3$. The first (affirmative) answer was given by Kato [2] in case $D = \mathbb{R}^n,\ n = 3,4$, through his study of strong solutions in general spaces L^p, see also [3, 4, 5]. The idea of Schonbek was then applied by [6, 7] to the case where D is a half-space of $\mathbb{R}^n,\ n \geq 2$ or an exterior domain of $\mathbb{R}^n,\ n \geq 3$. W. Borchers and T. Miyakawa [8] developed the method in [3, 6, 7] for the case of an arbitrary unbounded domain. They showed that if $\|e^{-tA}u_0\|_2 = O(t^{-\alpha})$ for some $\alpha \in (0, \frac{1}{2})$, then $\|u(t)\|_2 = O(t^{-\alpha})$. Our purpose in this paper is to improve and generalize the result of [8]. Firstly, we obtain the same result as that of them but under more general condition on α, in which the condition $\alpha \in (0, \frac{1}{2})$ is replaced by $\alpha \in (0, \frac{3}{4})$. Secondly, we obtain the stronger result than theirs by assuming some additive conditions on the initial value.
We recall some well-known function spaces, the definitions of weak and strong solutions to (1) and introduce some notations before describing the main results. Throughout the paper, we sometimes use the notation $A \lesssim B$ as an equivalent to $A \leq CB$ with a uniform constant C. The notation $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$. The expression $\langle\cdot,\cdot\rangle_\Omega$ denotes the pairing of functions, vector fields, etc. on Ω and $\langle\cdot,\cdot\rangle_{\Omega,T}$ means the corresponding pairing on $[0,T)\times\Omega$. For $1 \leq q \leq \infty$ we use the well-known Lebesgue and Sobolev $L^q(\Omega)$, $W^{k,p}(\Omega)$, with norms $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q$ and $\|\cdot\|_{W^{k,p}(\Omega)} = \|\cdot\|_{k,p}$. Further, we use the Bochner spaces $L^s\big(0,T;L^p(\Omega)\big)$, $1 \leq s,p \leq \infty$ with the norm

$$\|\cdot\|_{L^s\left(0,T;L^p(\Omega)\right)} := \left(\int_0^T \|\cdot\|_p^s \,\mathrm{d}\tau\right)^{1/s} = \|\cdot\|_{p,s,T}.$$

To deal with solenoidal vector fields we introduce the spaces of divergence - free smooth compactly supported functions $C_{0,\sigma}^\infty(\Omega) = \{u \in C_0^\infty(\Omega), \operatorname{div}(u) = 0\}$, and the spaces $L^2_\sigma(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_2}$, $W_0^{1,2}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,2}}}$, and $W_{0,\sigma}^{1,2}(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{W^{1,2}(\Omega)}}$.

Let $\mathbb{P} : L^2(\Omega) \longrightarrow L^2_\sigma(\Omega)$ be the Helmholtz projection. Let the Stokes operator

$$A = -\mathbb{P}\Delta :\ \mathbb{D}(A) \longrightarrow L^2_\sigma(\Omega)$$

with the domain of definition

$$\mathbb{D}(A) = \{u \in W_{0,\sigma}^{1,2}(\Omega), \exists f \in L^2_\sigma(\Omega) : \\ \langle\nabla u, \nabla\varphi\rangle_\Omega = \langle f,\varphi\rangle_\Omega, \quad \forall\ \varphi \in W_{0,\sigma}^{1,2}(\Omega)\}$$

be defined as

$$Au = -\mathbb{P}\Delta u = f, u \in \mathbb{D}(A).$$

As in [9], we define the fractional powers

$$A^\alpha : \mathbb{D}(A^\alpha) \longrightarrow L^2_\sigma(\Omega),\ -1 \leq \alpha \leq 1.$$

We have $\mathbb{D}(A) \subset \mathbb{D}(A^\alpha) \subset L^2_\sigma(\Omega)$ for $\alpha \in (0,1]$. It is known that for any domain $\Omega \subseteq \mathbb{R}^3$ the operator A is self-adjoint and generates a bounded analytic semigroup $e^{-tA}, t \geq 0$ on $L^2_\sigma(\Omega)$.

The following embedding properties play a basic role in the theory of the Navier-Stokes system

$$\|A^{-\frac{\beta}{2}}\mathbb{P}u\|_2 \leq C\|u\|_q,\ u \in L^q_\sigma(\Omega) \tag{2}$$

where $\frac{1}{2} \leq \beta < \frac{3}{2},\ \frac{1}{q} = \frac{1}{2} + \beta$. Furthermore, we mention the Stokes semigroup estimates

$$\|A^\alpha e^{-tA}u\|_2 \leq t^{-\alpha}\|u\|_2, \tag{3}$$

with $u \in L^2_\sigma(\Omega), 0 \leq \alpha \leq 1$. Now we recall the definitions of weak and strong solutions to (1).

Definition 1.1. *(See [9].) Let $u_0 \in L^2_\sigma(\Omega)$.*
1. A vector field

$$u \in L^\infty\big(0,T;L^2_\sigma(\Omega)\big)\cap L^2_{\mathrm{loc}}\big([0,T);W_{0,\sigma}^{1,2}(\Omega)\big) \tag{4}$$

is called a weak solution in the sense of Leray-Hopf of the Navier-Stokes system (1) *with the initial value* $u(0,x)=u_0$ *if the relation*

$$-\langle u, w_t\rangle_{\Omega,T}+\langle \nabla u,\nabla w\rangle_{\Omega,T}-\langle uu,\nabla w\rangle_{\Omega,T} = \langle u_0, w\rangle_{\Omega} \quad (5)$$

is satisfied for all test functions $w\in C_0^\infty\big([0,T);C_{0,\sigma}^\infty(\Omega)\big)$, *and additionally the energy inequality*

$$\frac{1}{2}\|u(t)\|_2^2+\int_0^t\|\nabla u(\tau)\|_2^2\mathrm{d}\tau\le\frac{1}{2}\|u_0\|_2^2 \quad (6)$$

is satisfied for all $t\in[0,T)$.

A weak solution u *is called a strong solution of the Navier-Stokes equation* (1) *if additionally local Serrin's condition*

$$u\in L^s_{\mathrm{loc}}\big([0,T);L^q(\Omega)\big) \quad (7)$$

is satisfied with $2<s<\infty$, $3<q<\infty$ *where* $\frac{2}{s}+\frac{3}{q}\le 1$.

As is well known, in the case the domain Ω is bounded, it is not difficult to prove the existence of a weak solution u as in Definition 1.1 which additionally satisfies the *strong energy inequality*

$$\frac{1}{2}\|u(t)\|_2^2+\int_{t'}^t\|\nabla u(\tau)\|_2^2\mathrm{d}\tau\le\frac{1}{2}\|u(t')\|_2^2 \quad (8)$$

for almost all $t'\in[0,T)$ and all $t\in[t',T)$, see [9], p. 340. For further results in this context for unbounded domains we refer to [10].

Now we can state our main results.

Theorem 1.1. *Let* $\Omega\subseteq\mathbb{R}^3$ *be a general domain,* $u_0\in L^2_\sigma(\Omega)$ *and* u *is a weak solution of the Navier-Stokes system* (1) *satisfying strong energy inequality* (8). *Then*

(a) *If* $\|e^{-tA}u_0\|_2=O(t^{-\alpha})$ *for some* $0\le\alpha<\frac{3}{4}$, *then* $\|u(t)\|_2=O(t^{-\alpha})$ *as* $t\to\infty$.

(b) *If* $\|e^{-tA}u_0\|_2=o(t^{-\alpha})$ *for some* $0\le\alpha<\frac{3}{4}$, *then* $\|u(t)\|_2=o(t^{-\alpha})$ *as* $t\to\infty$.

Theorem 1.2. *Let* $\Omega\subseteq\mathbb{R}^3$ *be a general domain,* $u_0\in L^2_\sigma(\Omega)$ *and* u *is a weak solution of the Navier-Stokes system* (1) *satisfying strong energy inequality* (8). *If* $u_0\in L^q(\Omega)\cap L^2_\sigma(\Omega)$, $1<q\le 2$, *then*

$$\|u(t)\|_2=o\Big(t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}\Big)\quad \text{as}\quad t\to\infty.$$

Theorem 1.3. *Let* $\Omega\subseteq\mathbb{R}^3$ *be a general domain,* $u_0\in L^2_\sigma(\Omega)$ *and* u *is a weak solution of the Navier-Stokes system* (1) *satisfying strong energy inequality* (8). *If there exist positive constants* t_0, C_1, *and* C_2 *such that*

$$C_1t^{-\alpha_1}\le\|e^{-tA}u_0\|_2\le C_2t^{-\alpha_2}\quad \text{for}\quad t\ge t_0,$$

where α_1, *and* α_2 *are constants satisfying*

$$0\le\alpha_2<\frac{1}{2}\quad\text{and}\quad\alpha_2\le\alpha_1<\alpha_2+\frac{1}{4},$$

then u *coincides with the solution of the homogeneous Stokes system with the initial value* u_0 *when time tends to infinity in the sense that*

$$\lim_{t\to\infty}\frac{\big\|u(t)-e^{-tA}u_0\big\|_2}{\|u(t)\|_2}=0. \quad (9)$$

2 Proof of main theorems

Let us construct a weak solution of the following integral equation

$$u(t)=e^{-tA}u_0-\int_0^t A^{\frac{1}{2}}e^{-(t-\tau)A}A^{-\frac{1}{2}}\mathbb{P}(u\cdot\nabla u)\mathrm{d}\tau. \quad (10)$$

We know that

$$u\in L^\infty\big(0,T;L^2_\sigma(\Omega)\big)\cap L^2_{\mathrm{loc}}\big([0,T);W^{1,2}_{0,\sigma}(\Omega)\big)$$

is a weak solution of the Navier-Stokes system (1) iff u satisfies the integral equation (10), see [9]. In order to prove the main theorems, we need the following lemmas.

Lemma 2.1. *Let* $\gamma,\theta\in\mathbb{R}$ *and* $t>0$, *then*

(a) *If* $\theta<1$, *then*

$$\int_0^{\frac{t}{2}}(t-\tau)^{-\gamma}\tau^{-\theta}\mathrm{d}\tau=K_1t^{1-\gamma-\theta}$$

where $K_1=\int_0^{\frac{1}{2}}(1-\tau)^{-\gamma}\tau^{-\theta}\mathrm{d}\tau<\infty$.

(b) *If* $\gamma<1$, *then*

$$\int_{\frac{t}{2}}^t(t-\tau)^{-\gamma}\tau^{-\theta}\mathrm{d}\tau=K_2t^{1-\gamma-\theta}$$

where $K_2=\int_{\frac{1}{2}}^1(1-\tau)^{-\gamma}\tau^{-\theta}\mathrm{d}\tau<\infty$.

The proof of this lemma is elementary and may be omitted. $\square$

Lemma 2.2. *Let $u \in L^2(\Omega)$ and $\nabla u \in L^2(\Omega)$. Then*

$$\left\|e^{-tA}\mathbb{P}(u\cdot\nabla u)\right\|_2 \lesssim t^{-\frac{\beta}{2}}\|u\|_2^{\beta-\frac{1}{2}}\|\nabla u\|_2^{\frac{5}{2}-\beta}$$

where β is positive constant such that $\dfrac{1}{2} \le \beta < \dfrac{3}{2}$.

Proof. Applying inequalities (6), (3), Holder inequality, interpolation inequality, and Lemma 2.1, we obtain

$$\begin{aligned}
\left\|e^{-tA}\mathbb{P}(u\cdot\nabla u)\right\|_2 &= \left\|A^{\frac{\beta}{2}}e^{-tA}A^{-\frac{\beta}{2}}\mathbb{P}(u\cdot\nabla u)\right\|_2\\
&\le t^{-\frac{\beta}{2}}\left\|A^{-\frac{\beta}{2}}\mathbb{P}(u\cdot\nabla u)\right\|_2 \lesssim t^{-\frac{\beta}{2}}\left\|u\cdot\nabla u\right\|_q\\
&\lesssim t^{-\frac{\beta}{2}}\|u\|_{\frac{3}{\beta}}\|\nabla u\|_2\\
&\lesssim t^{-\frac{\beta}{2}}\|u\|_2^{\beta-\frac{1}{2}}\|\nabla u\|_2^{\frac{3}{2}-\beta}\|\nabla u\|_2\\
&\lesssim t^{-\frac{\beta}{2}}\|u\|_2^{\beta-\frac{1}{2}}\|\nabla u\|_2^{\frac{5}{2}-\beta}.
\end{aligned}$$

The proof of Lemma 2.2 is complete. $\square$

Lemma 2.3. *There exists a positive constant δ such that if $u_0 \in \mathbb{D}(A^{\frac{1}{4}})$ and $\|A^{\frac{1}{4}}u_0\|_2 \le \delta$, then the Navier-Stokes system (1) has a strong solution with the initial value u_0 satisfying $\|\nabla u(t)\|_2 \lesssim t^{-\frac{1}{2}}$ for all $t \ge 0$.*

Proof. See [11]. $\square$

Lemma 2.4. *Let u be a weak solution of the Navier-Stokes system (1) with the initial value $u_0 \in L^2_\sigma(\Omega)$. Then there exists the positive value t_0 large enough such that $\|\nabla u(t)\|_2 \lesssim t^{-\frac{1}{2}}$ for all $t \ge t_0$.*

Proof. Applying Holder inequality, we have

$$\begin{aligned}
\left\|A^{\frac{1}{4}}u\right\|_2^2 &= \int_0^\infty \lambda^{\frac{1}{2}}\mathrm{d}\|E_\lambda u\|_2^2\\
&\le \Big(\int_0^\infty \lambda\, \mathrm{d}\|E_\lambda u\|_2^2\Big)^{\frac{1}{2}}\Big(\int_0^\infty \mathrm{d}\|E_\lambda u\|_2^2\Big)^{\frac{1}{2}}\\
&= \|A^{\frac{1}{2}}u\|_2\|u\|_2.
\end{aligned} \tag{11}$$

Consider the weak solution of the Navier-Stokes system (1) satisfying the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_0}^t \|\nabla u(\tau)\|_2^2\mathrm{d}\tau \le \frac{1}{2}\|u(t_0)\|_2^2 \tag{12}$$

for all $t \in [0,\infty)\setminus N$ with N is a null set.
Let δ be a positive constant in Lemma 2.3. Since (11) and (12), it follows that there exists the large enough $t_0 \in [0,\infty)\setminus N$ such that $\|u(t_0)\|_{\mathbb{D}(A^{\frac{1}{4}})} \le \delta$.
Combining Lemma 2.3, inequality (12), and Serrin's uniqueness criterion [9, 12], we obtain

$$\|\nabla u(t)\|_2^2 \lesssim t^{-\frac{1}{2}} \text{ for all } \mathrm{t} \ge \mathrm{t}_0.$$

The proof of Lemma 2.4 is complete. $\square$

Proof of Theorem 1.1

(a) Consider the weak solution of the Navier-Stokes system (1), then u holds the integral equation

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A}\,\mathbb{P}\big(u\cdot\nabla u\big)\mathrm{d}s. \tag{13}$$

From Lemma 2.2, we have

$$\begin{aligned}
\|u(t)\|_2 &\lesssim \left\|e^{-tA}u_0\right\|_2\\
&\quad+\int_0^t (t-s)^{-\frac{\beta}{2}}\|u(s)\|_2^{\beta-\frac{1}{2}}\|\nabla u(s)\|_2^{\frac{5}{2}-\beta}\mathrm{d}s
\end{aligned}$$

for all $\dfrac{1}{2} \le \beta < \dfrac{3}{2}$. We divide the above integral into two different parts as follow

$$\begin{aligned}
I &= \int_0^t (t-s)^{-\frac{\beta}{2}}\|u(s)\|_2^{\beta-\frac{1}{2}}\|\nabla u(s)\|_2^{\frac{5}{2}-\beta}\mathrm{d}s\\
&= \int_0^{\frac{t}{2}} (t-s)^{-\frac{\beta}{2}}\|u(s)\|_2^{\beta-\frac{1}{2}}\|\nabla u(s)\|_2^{\frac{5}{2}-\beta}\mathrm{d}s\\
&+ \int_{\frac{t}{2}}^t (t-s)^{-\frac{\beta}{2}}\|u(s)\|_2^{\beta-\frac{1}{2}}\|\nabla u(s)\|_2^{\frac{5}{2}-\beta}\mathrm{d}s\\
&= I_1 + I_2.
\end{aligned}$$

We consider the following three cases:

$$0 \le \alpha \le \frac{1}{4}, \frac{1}{4} \le \alpha < \frac{1}{2}, \text{ and } \frac{1}{2} \le \alpha < \frac{3}{4}.$$

Case 1: $0 \le \alpha \le \dfrac{1}{4}$.
Applying the energy inequality and Holder inequality, we obtain

$$\begin{aligned}
I_1 &\lesssim \|u_0\|_2^{\beta-\frac{1}{2}} t^{-\frac{\beta}{2}} \int_0^{\frac{t}{2}} \|\nabla u(s)\|_2^{\frac{5}{2}-\beta} \mathrm{d}s \\
&\lesssim \|u_0\|_2^{\beta-\frac{1}{2}} t^{-\frac{\beta}{2}} (\int_0^{\frac{t}{2}} \mathrm{d}s)^{\frac{2\beta-1}{4}} (\int_0^{\frac{t}{2}} \|\nabla u(s)\|_2^2 \, \mathrm{d}s)^{\frac{5-2\beta}{4}} \\
&\lesssim \|u_0\|_2^{\beta-\frac{1}{2}} t^{-\frac{\beta}{2}} \left(\frac{t}{2}\right)^{\frac{2\beta-1}{4}} \|u_0\|_2^{\frac{5-2\beta}{4}} = O(t^{-\frac{1}{4}}).
\end{aligned}$$

From Lemma 2.4 and Lemma 2.1(b), we have

$$\begin{aligned}
I_2 &\lesssim \|u_0\|_2^{\beta-\frac{1}{2}} \int_{\frac{t}{2}}^t (t-s)^{-\frac{\beta}{2}} s^{-\frac{1}{2}\left(\frac{5}{2}-\beta\right)} \mathrm{d}s \\
&= O(t^{-\frac{1}{4}}) \ \text{ for } \ t \ge 2t_0
\end{aligned}$$

where t_0 is the constant in Lemma 2.4. It follows that

$$\begin{aligned}
\|u(t)\|_2 &\lesssim \left\|e^{-tA}u_0\right\|_2 + I \le O(t^{-\alpha}) + O(t^{-\frac{1}{4}}) \\
&= O(t^{-\alpha}) \ \text{ as } \ t \to \infty.
\end{aligned}$$

Case 2: $\dfrac{1}{4} \le \alpha < \dfrac{1}{2}$.
Applying the above inequality for $\alpha = -\dfrac{1}{4}$ and Holder inequality, we obtain

$$\begin{aligned}
I_1 &\lesssim t^{-\frac{\beta}{2}} \int_0^{\frac{t}{2}} (s^{-\frac{1}{4}})^{\beta-\frac{1}{2}} \|\nabla u(s)\|_2^{\frac{5}{2}-\beta} \mathrm{d}s \\
&\lesssim t^{-\frac{\beta}{2}} (\int_0^{\frac{t}{2}} s^{-\frac{1}{2}} \mathrm{d}s)^{\frac{2\beta-1}{4}} \Big(\int_0^{\frac{t}{2}} \|\nabla u(s)\|_2^2 \, \mathrm{d}s \Big)^{\frac{5-2\beta}{4}} \\
&\lesssim t^{-\frac{\beta}{2}} \big(t^{\frac{1}{2}}\big)^{\frac{2\beta-1}{4}} = O(t^{-\frac{\beta}{4}-\frac{1}{8}}).
\end{aligned}$$

On the other hand, from Lemma 2.4 and Lemma 2.1(b), we have

$$\begin{aligned}
I_2 &\lesssim \int_{\frac{t}{2}}^t (t-s)^{-\frac{\beta}{2}} \big(s^{-\frac{1}{4}}\big)^{\beta-\frac{1}{2}} \big(s^{-\frac{1}{2}}\big)^{\frac{5}{2}-\beta} \mathrm{d}s \\
&\lesssim \int_{\frac{t}{2}}^t (t-s)^{-\frac{\beta}{2}} s^{\frac{1}{8}-\frac{\beta}{4}} s^{\frac{\beta}{2}-\frac{5}{4}} \mathrm{d}s \\
&= O(t^{-\frac{\beta}{4}-\frac{1}{8}}) \ \text{ for } \ t \ge 2t_0.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|u(t)\|_2 &\lesssim \left\|e^{-tA}u_0\right\|_2 + I \\
&\le O(t^{-\alpha}) + O(t^{-\frac{\beta}{4}-\frac{1}{8}}) \ \text{ for } \ t \ge 2t_0.
\end{aligned}$$

It is not difficult to show that there exists a number β such that

$$\frac{\beta}{4} + \frac{1}{8} \ge \alpha \ \text{ and } \ \frac{1}{2} \le \beta < \frac{3}{2}.$$

Therefore, choose one of such β, it follows that

$$\|u(t)\|_2 = O(t^{-\alpha}) \ \text{ as } \ t \to \infty.$$

Case 3: $\dfrac{1}{2} \le \alpha < \dfrac{3}{4}$.
Applying Case 2 of part (a), we have

$$\|u(t)\|_2 \lesssim t^{-\gamma} \ \text{ for } \ t \ge 0, \tag{14}$$

where γ is a constant such that $0 \le \gamma < \dfrac{1}{2}$. Applying inequality (14) and Holder inequality, we obtain

$$\begin{aligned}
I_1 &\lesssim t^{-\frac{\beta}{2}} \int_0^{\frac{t}{2}} (s^{-\gamma})^{\beta-\frac{1}{2}} \|\nabla u(s)\|_2^{\frac{5}{2}-\beta} \mathrm{d}s \\
&\lesssim t^{-\frac{\beta}{2}} \Big(\int_0^{\frac{t}{2}} s^{-2\gamma} \mathrm{d}s \Big)^{\frac{2\beta-1}{4}} \Big(\int_0^{\frac{t}{2}} \|\nabla u(s)\|_2^2 \, \mathrm{d}s \Big)^{\frac{5-2\beta}{4}} \\
&\lesssim t^{-\frac{\beta}{2}} \big(t^{-2\gamma+1}\big)^{\frac{2\beta-1}{4}} = O(t^{\frac{\gamma}{2}-\gamma\beta-\frac{1}{4}}).
\end{aligned}$$

Moreover, from Lemma 2.4 and Lemma 2.1(b), we have

$$\begin{aligned}
I_2 &\lesssim \int_{\frac{t}{2}}^t (t-s)^{-\frac{\beta}{2}} \big(s^{-\gamma}\big)^{\beta-\frac{1}{2}} \big(s^{-\frac{1}{2}}\big)^{\frac{5}{2}-\beta} \mathrm{d}s \\
&\lesssim t^{-\frac{\beta}{2}-\gamma(\beta-\frac{1}{2})-\frac{1}{2}(\frac{5}{2}-\beta)+1} \int_{\frac{t}{2}}^t (1-s)^{-\frac{\beta}{2}} s^{-\gamma(\beta-\frac{1}{2})} \mathrm{d}s \\
&= O(t^{\frac{\gamma}{2}-\gamma\beta-\frac{1}{4}}) \ \text{ for } \ t \ge 2t_0.
\end{aligned}$$

It follows that

$$\|u(t)\|_2 \lesssim \left\|e^{-tA}u_0\right\|_2 + I \le O(t^{-\alpha}) + O(t^{\frac{\gamma}{2}-\gamma\beta-\frac{1}{4}})$$

for $t \ge 2t_0$. Similar to the above case, it is not difficult to show that there exist γ and β such that

$$\frac{\gamma}{2} - \gamma\beta - \frac{1}{4} \le -\alpha, \ \frac{1}{2} \le \beta < \frac{3}{2}, \ \text{and} \ \ 0 \le \gamma < \frac{1}{2}.$$

Choose ones of such γ and β, we conclude that $\|u(t)\|_2 = O(t^{-\alpha})$ as $t \to \infty$.
(b) This is deduced from the proof of part (a).
The proof of Theorem is complete.

Corollary 2.1. *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain. Given u_0 and u as in Theorem 1.1. If $\|u(t)\|_2 = o(t^{-\gamma})$ for some $\gamma \in [0, \frac{1}{2})$, then $\|u(t) - e^{-tA}u_0\|_2 = o(t^{-(\gamma+\theta)})$ for all $\theta \in [0, \frac{1}{4})$.*

Proof. The proof is derived directly from the proof of Case 3 of Theorem 1.1. $\square$

Proof of Theorem 1.2

Theorem 1.2 is an immediate consequence of Theorem 1.1(b) and the following lemma.

Lemma 2.5. *Let $u_0 \in L^2_\sigma(\Omega)$. Then*
(a) $\|e^{-tA}u_0\|_2 \to 0$ *as $t \to \infty$.*
(b) *If $u_0 \in L^2_\sigma(\Omega) \cap L^q(\Omega)$ for some $1 < q \leq 2$, then*

$$\left\|e^{-tA}u_0\right\|_2 = o\Big(t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}\Big) \text{ as } t \to \infty. \tag{15}$$

Proof. (a) See Lemma 1.5.1 in [9], p. 204.
(b) Applying inequality (3), we obtain

$$\begin{aligned}\left\|e^{-tA}u_0\right\|_2 &= \left\|e^{\frac{-tA}{2}}e^{\frac{-tA}{2}}u_0\right\|_2 \\ &= \left\|A^{\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}e^{\frac{-tA}{2}}e^{\frac{-tA}{2}}A^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}u_0\right\|_2 \\ &\lesssim t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}\left\|e^{\frac{-tA}{2}}A^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}u_0\right\|_2.\end{aligned} \tag{16}$$

On the other hand, using inequality (2), we get

$$A^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}u_0 \in L^2_\sigma(\Omega) \tag{17}$$

Property 15 is deduced from Lemma 2.5(a), (16), and (17). $\square$

Proof of Theorem 1.3

Proof. Applying Corollary 2.1 for $\gamma = \alpha_2,\ \theta = \dfrac{\alpha_1-\alpha_2}{2} + \dfrac{1}{8}$, there exists a positive constant M_1 such that

$$\begin{aligned}\left\|u(t) - e^{-tA}u_0\right\|_2 &\leq M_1 t^{-(\alpha_2 + \frac{\alpha_1-\alpha_2}{2} + \frac{1}{8})} \\ &= M_1 t^{-(\frac{\alpha_1+\alpha_2}{2} + \frac{1}{8})} \text{ for } t \geq t_0.\end{aligned}$$

It follows from the above inequality that

$$\begin{aligned}\|u(t)\|_2 &\geq \|u(t)\|_2 - \left\|u(t) - e^{-tA}u_0\right\|_2 \\ &\geq C_1 t^{-\alpha_1} - M_1 t^{-(\frac{\alpha_1+\alpha_2}{2}+\frac{1}{8})} \\ &\geq \Big(C_1 - M_1 t^{-(\frac{\alpha_2-\alpha_1}{2}+\frac{1}{8})}\Big)t^{-\alpha_1} \\ &\geq \frac{C_1}{2}t^{-\alpha_1} \text{ for } t \geq t_1,\end{aligned}$$

where

$$t_1 = \max\Big\{t_0, \Big(\frac{2M_1}{C_1}\Big)^{\frac{8}{4(\alpha_2-\alpha_1)+1}}\Big\}.$$

From the above two estimates, we obtain that

$$\begin{aligned}\frac{\left\|u(t) - e^{-tA}u_0\right\|_2}{\|u(t)\|_2} &\leq \frac{M_1 t^{-(\frac{\alpha_1+\alpha_2}{2}+\frac{1}{8})}}{\frac{C_1}{2}t^{-\alpha_1}} \\ &= \frac{2M_1}{C_1}t^{-\left(\frac{\alpha_2-\alpha_1}{2}+\frac{1}{8}\right)} \to 0 \text{ as } t \to \infty.\end{aligned}$$

The proof of Theorem is complete. $\square$

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