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EXISTENCE AND UNIQUENESS OF SOLUTION FOR GENERALIZATION OF FRACTIONAL BESSEL TYPE PROCESS

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ABSTRACT
The real financial models such as the short term interest rates, the log-volatility in Heston model are very well modeled by a fractional Brownian motion. This fact raises a question of developing a fractional generalization of the classical processes such as Cox - Ingersoll - Ross process, Bessel process. In this paper, we are interested in the fractional Bessel process (Mishura, Yurchenko-Tytarenko, 2018). More precisely, we consider a generalization of the fractional Bessel type process. We prove that the equation has a unique positive solution. Moreover, we study the supremum norm of the solution.
Keywords: *Fractional stochastic differential equation; Fractional Brownian motion; Fractional Bessel process; Fractional Cox- Ingersoll- Ross process; Supremum norm.*

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SỰ TỒN TẠI VÀ DUY NHẤT NGHIỆM CỦA QUÁ TRÌNH DẠNG BESSEL PHÂN THỨ TỔNG QUÁT

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TÓM TẮT
Các mô hình tài chính thực tế như tỷ lệ lãi suất ngắn hạn, log- độ biến động trong mô hình Heston được mô hình hóa rất tốt bởi chuyển động Brown phân thứ. Điều này đặt ra câu hỏi về việc phát triển dạng phân thứ tổng quát cho các quá trình cổ điển như quá trình Cox- Ingersoll- Ross, quá trình Bessel. Trong bài báo này chúng tôi quan tâm tới quá trình Bessel phân thứ (Mishura, Yurchenko-Tytarenko, 2018). Cụ thể hơn, chúng tôi xét dạng tổng quát của quá trình Bessel phân thứ. Chúng tôi chứng minh sự tồn tại và duy nhất nghiệm dương của phương trình. Hơn nữa, chúng tôi đưa ra đánh giá cho chuẩn supremum của nghiệm.
Từ khóa: *Phương trình vi phân ngẫu nhiên phân thứ, Chuyển động Brown phân thứ, Quá trình Bessel phân thứ, Quá trình Cox- Ingersoll- Ross phân thứ, Chuẩn Supremum.*

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1 Introduction

The Cox- Ingersoll- Ross (CIR) process

$$r(t) = r(0)+\int_0^t (k-ar(s))ds+\int_0^t \sigma\sqrt{r(s)}dW_s,$$

$r(0), k, a, \sigma > 0$, W is a Brownian motion, was introduced and studied by Cox, Ingersoll, Ross in [1]-[3] to model the short term interest rates. This process is also used in mathematical finance to study the log-volatility in Heston model [4]. But the real financial models are often characterized by the so-called "memory phenomenon" [5]- [7] , while the standard Cox–Ingersoll– Ross process does not satisfy it. It is reasonable to develop a *fractional generalization* of the classical CIR process. In [8], Mishura and Yurchenko-Tytarenko introduced a fractional Bessel type process

$$dy(t) = \frac{1}{2}\Big(\frac{k}{y(t)} - ay(t)\Big)dt + \frac{1}{2}\sigma dB_t^H, \quad y_0 > 0, \tag{1.1}$$

where B^H is a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, and then showed that $x(t) = y^2(t)$ satisfied the SDEs

$$dx(t) = (k - ax(t))dt + \sigma\sqrt{x(t)} \circ dB_t^H, \quad t \geq 0,$$

where the integral with respect to fractional Brownian motion is considered as the pathwise Stratonovic integral.

In this paper, we study a generalization of the Bessel type process y given by (1.1). More precisely, we consider a process $Y = (Y(t))_{0\leq t\leq T}$ satisfying the following SDEs,

$$dY(t) = \left(\frac{k}{Y(t)} + b(t, Y(t))\right)dt + \sigma dB^H(t), \tag{1.2}$$

where $0 \leq t \leq T$, $Y(0) > 0$ and B^H is a fractional Brownian motion with the Hurst parameter $H > \frac{1}{2}$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t, t \in [0, T]\}$ satisfying the usual condition.

We first show that, the equation (1.2) has a unique positive solution. Moreover, we estimate the supremum norm of the solution.

2 The existence and uniqueness of the solution

Fix $T > 0$ and we consider equation (1.2) on the interval $[0, T]$. We suppose that $k > 0$ and the coefficient $b = b(t, x) : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ are mesurable functions and globally Lipschitz continuous with respect to x, linearly growth with respect to x. It means that there exists positive constants L, C such that the following conditions hold:

(i) $|b(t, x) - b(t, y)| = L|x - y|$, for all $x, y \in \mathbb{R}$ and $t \in [0, T]$;

(ii) $|b(t, x)| \leq C(1 + |x|)$, for all $x \in \mathbb{R}$ and $t \in [0, T]$;

Denote $a\vee b = \max\{a, b\}$ and $a\wedge b = \min\{a, b\}$. For each $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$f^{(n)}(s, x) = \frac{k}{x \vee \frac{1}{n}} + b(s, x) \vee \frac{-kn}{4}.$$

We consider the following fractional SDE

$$Y^{(n)}(t) = Y(0) + \int_0^t f^{(n)}(s, Y^{(n)}(s))ds + \sigma dB^H(s), \tag{2.1}$$

where $t \in [0, T], \quad Y(0) > 0$. Using the estimate $|a \vee c - b \vee c| \leq |a - b|$ we can prove that the coefficients of equation (2.1) satisfies the assumption of Theorem 2.1 in [9]. So equation (2.1) has a unique solution on the interval $[0, T]$.

Now, we set

$$\tau_n = \inf\{t \in [0, T] : |Y^{(n)}(t)| \leq \frac{1}{n}\} \wedge T.$$

In order to prove that equation (1.2) has a unique solution on $[0, T]$ we need the following lemma.

Lemma 2.1. *The sequence τ_n is non-decreasing, and for almost all $\omega \in \Omega$, $\tau_n(\omega) = T$ for n large enough.*

Proof. We will use the contradiction method as in Theorem 2 in [8]. It follows the result on the modulus of continuity of trajectories of fractional Brownian motion (see [10]) that for any $\epsilon \in (0, H - \frac{1}{2})$, there exists a finite random variable $\eta_{\epsilon,T}$ and an event $\Omega_{\epsilon,T} \in \mathcal{F}$ which do not depend on n, such that $\mathbb{P}(\Omega_{\epsilon,T}) = 1$, and

$$\left|\sigma(B^H(t,\omega) - B^H(s,\omega))\right| \leq \eta_{\epsilon,T}(\omega)|t-s|^{H-\epsilon}, \tag{2.2}$$

for any $\omega \in \Omega_{\epsilon,T}$ and $0 \leq s < t \leq T$. Assume that for some $\omega_0 \in \Omega_{\epsilon,T}$, $\tau_n(\omega_0) < T$ for all $n \in \mathbb{N}$. Denote

$$\kappa_n(\omega_0) = \sup\{t \in [0, \tau_n(\omega_0)] : Y^{(n)}(t,\omega_0) \geq \frac{2}{n}\}.$$

In order to simplify our notation, we will omit ω_0 in brackets in further formulas. We have

$$Y^{(n)}(\tau_n) - Y^{(n)}(\kappa_n) = -\frac{1}{n} =$$
$$= \int_{\kappa_n}^{\tau_n} f^{(n)}(s, Y^{(n)}(s))ds + \sigma(B^H(\tau_n) - B^H(\kappa_n)).$$

This implies

$$\left|\sigma(B^H(\tau_n) - B^H(\kappa_n))\right| =$$
$$\left|\frac{1}{n} + \int_{\kappa_n}^{\tau_n} \left(\frac{k}{Y^{(n)}(s) \vee \frac{1}{n}} + b(s, Y^{(n)}(s)) \vee \frac{-kn}{4}\right) ds.\right| \tag{2.3}$$

From the definition of τ_n, κ_n we have

$$\frac{1}{n} \leq Y^{(n)}(t) \leq \frac{2}{n}, \qquad \text{for all } t \in [\kappa_n, \tau_n].$$

Then for all $n > n_0 = \dfrac{2}{Y(0)}$, it follows from (2.3) that

$$\left|\sigma(B^H(\tau_n) - B^H(\kappa_n))\right| \geq \frac{1}{n} + \frac{kn}{4}(\tau_n - \kappa_n).$$

This fact together with (2.2) implies that

$$\eta_{\epsilon,T}|\tau_n - \kappa_n|^{H-\epsilon} \geq \frac{1}{n} + \frac{kn}{4}(\tau_n - \kappa_n), \tag{2.4}$$

for all $n \geq n_0$. Using the similar arguments in the proof of Theorem 2 in [8] we see that the inequality 2.4 fails for n large enough. Therefore $\tau_n(\omega_0) = T$ for n large enough.

Lemma 2.2. *If $(Y(t))_{0 \leq t \leq T}$ is a solution of equation (1.2) then $Y(t) > 0$ for all $t \in [0,T]$ almost surely.*

Proof. In order to prove this Lemma we will also use the contradiction method. Assume that for some $\omega_0 \in \Omega$, $\inf\limits_{t \in [0,T]} Y(t,\omega_0) = 0$. Denote $M = \sup_{t \in [0,T]} |Y(t,\omega_0)|$ and $\tau = \inf\{t : Y(t,\omega_0) = 0\}$. For each $n \geq 1$, we denote $\nu_n = \sup\{t < \tau : Y(t,\omega_0) = \frac{1}{n}\}$. Since Y has continuous sample paths, $0 < \nu_n < \tau \leq T$ and $Y(t,\omega_0) \in (0, \frac{1}{n})$ for all $t \in (\nu_n, \tau)$. We have

$$-\frac{1}{n} = Y(\tau) - Y(\nu_n) =$$
$$\int_{\nu_n}^{\tau} \left(\frac{k}{Y(s)} + b(s, Y(s))\right) ds + \sigma(B^H(\tau) - B^H(\nu_n)).$$

If $n > \frac{2C(1+M)}{k}$ then $|b(s, Y(s,\omega_0))| \leq C(1 + |Y(s,\omega_0)|) \leq C(1+M) \leq \frac{kn}{2}$, and

$$\left|\sigma(B^H(\tau,\omega_0) - B^H(\nu_n,\omega_0))\right| \geq \frac{1}{n} + \frac{kn}{2}(\tau - \nu_n). \tag{2.5}$$

Using the same argument as in the proof of Theorem 2 in [8] again, we see that the inequality (2.5) fails for all n large enough. This contradiction completes the lemma.

Theorem 2.3. *For each $T > 0$ equation (1.2) has a unique solution on $[0,T]$.*

Proof. We first show the existence of a positive solution. From Lemma 2.1, there exists a finite random variable n_0 such that $Y^{(n)}(t) \geq \dfrac{1}{n_0} > 0$ almost surely for any $t \in [0,T]$ and $i = 1, \ldots, d$. Since $|x \vee \frac{-kn}{4}| \leq |x|$ and $b(t,x)$ is linearly growth with respect to x, for all $n > n_0$ we have

$$|Y^{(n)}(t)| \leq |Y(0)| + n_0 T k + |\sigma| \sup_{s \in [0,T]} |B^H(s)| +$$
$$C \int_0^t \left(1 + |Y^{(n)}(s)|\right) ds.$$

Applying Gronwall's inequality, we get

$$|Y^{(n)}(t)| \le C_1 e^{CT}, \quad \text{for any } t \in [0,T],$$

where

$$C_1 = |Y(0)| + n_0 Tk + |\sigma| \sup_{s\in[0,T]} |B^H(s)| + CT.$$

Note that C_1 is a finite random variable which does not depend on n. So

$$\sup_{0\le t\le T} |b(t, Y^{(n)}(t))| \le C(1 + \sup_{0\le t\le T} |Y^{(n)}(t)|) \le C(1 + C_1 e^{CT}).$$

Then for any $n \ge n_0 \vee \dfrac{4C(1+C_1e^{CT})}{k}$, $\inf\limits_{0\le t\le T} b(t, Y^{(n)}(t)) > \dfrac{-kn}{4}$. Therefore the process $Y^{(n)}(t)$ converges almost surely to a positive limit, called $Y(t)$ when n tends to infinity, and $Y(t)$ satisfies equation (1.2).

Next, we show that equation (1.2) has a unique solution in path-wise sense. Let $Y(t)$ and $\hat{Y}(t)$ be two solutions of equation (1.2) on $[0,T]$. We have

$$\begin{aligned} &|Y(t,\omega) - \hat{Y}(t,\omega)| \\ &\le \int_0^t \left| \frac{k}{Y(s,\omega)} - \frac{k}{\hat{Y}(s,\omega)} \right| ds + \\ &+ \int_0^t \left| b(s, Y(s,\omega)) - b(s, \hat{Y}(s,\omega)) \right| ds \end{aligned}$$

Using continuous property of the sample paths of $Y(t)$ and $\hat{Y}(t)$ and Lemma 2.2, we have

$$m_0 = \min_{t\in[0,T]} \left\{ Y(t,\omega), \hat{Y}(t,\omega) \right\} > 0.$$

Together with the Lipschitz condition of b we obtain

$$\begin{aligned} |Y(t,\omega) - \hat{Y}(t,\omega)| \le &\int_0^t \frac{k|Y(s,\omega) - \hat{Y}(s,\omega)|}{m_0^2} ds + \\ &+ \int_0^t L|Y(s,\omega) - \hat{Y}(s,\omega)| ds \end{aligned}$$

It follows from Gronwall's inequality that

$$|Y(t,\omega) - \hat{Y}(t,\omega)| = 0, \text{ for all } t \in [0,T].$$

Therefore, $Y(t,\omega) = \hat{Y}(t,\omega)$ for all $t \in [0,T]$. The uniqueness has been concluded.

The next result provides an estimate for the supremum norm of the solution in terms of the Hölder norm of the fractional Brownian motion B^H.

Theorem 2.4. *Assume that conditions $(A1)-(A2)$ are satisfied, and $Y(t)$ is the solution of equation* (1.2)*. Then for any $\gamma > 2$, and for any $T > 0$,*

$$\begin{aligned} &\|Y\|_{0,t,\infty} \le C_{1,\gamma,\beta,T,k,C,d}(|y_0 + 1) \times \\ &\times exp\left\{ C_{2,\gamma,\beta,T,k,C,d,\sigma} \left(\|B^H\|_{0,T,\beta}^{\frac{\gamma}{\beta(\gamma-1)}} + 1 \right) \right\}. \end{aligned}$$

Proof. Fix a time interval $[0,T]$. let $z(t) = Y^\gamma(t)$. Applying the chain rule for Young integral, we have

$$\begin{aligned} z(t) =& Y^\gamma(0) + \\ &+ \gamma \int_0^t \left(\frac{k}{z^{1/\gamma}(s)} + b(s, Y(s)) \right) z^{1-\frac{1}{\gamma}}(s) ds + \\ &+ \gamma \int_0^t \sigma z^{1-\frac{1}{\gamma}}(s) dB^H(s). \end{aligned}$$

Then

$$\begin{aligned} &|z(t) - z(s)| \\ &\le \left| \gamma \int_s^t \left(\frac{k}{z^{1/\gamma}(u)} + b(u, Y(u)) \right) z^{1-\frac{1}{\gamma}}(u) du \right| + \\ &+ \left| \gamma \int_s^t \sigma z^{1-\frac{1}{\gamma}}(u) dB^H(u) \right|. \end{aligned} \tag{2.6}$$

Together with the condition $(A2)$ we obtain

$$\begin{aligned} I_1 :=& \left| \int_s^t \left(\frac{k}{z^{1/\gamma}(u)} + b(u, Y(u)) \right) z^{1-\frac{1}{\gamma}}(u) du \right| \\ \le& \int_s^t \left(k|z^{1-\frac{2}{\gamma}}(u)| + C(1 + |z(u)|^{1/\gamma})|z^{1-\frac{1}{\gamma}}(u)| \right) du. \end{aligned}$$

Since $\gamma > 2$ then we have

$$I_1 \le \left[k\|z\|_{s,t,\infty}^{1-\frac{2}{\gamma}} + C\|z\|_{s,t,\infty}^{1-\frac{1}{\gamma}} + C\|z\|_{s,t,\infty} \right] (t-s). \tag{2.7}$$

Let $I_2 = \left| \int_s^t z^{1-\frac{1}{\gamma}}(u) dB^H(u) \right|$.
Following the argument in the proof of Theorem 2.3 in [11] we have

$$I_2 \le R\|B^H\|_{0,T,\beta} \times \\ \times \left(\|z\|_{s,t,\infty}^{1-\frac{1}{\gamma}} (t-s)^\beta + \|z\|_{s,t,\beta}^{1-\frac{1}{\gamma}} (t-s)^{\beta(2-\frac{1}{\gamma})} \right). \tag{2.8}$$

where R is a generic constant depending on α, β and T.
Substituting (2.7) and (2.8) into (2.6), we obtain

$$|z(t) - z(s)| \le \gamma \left[k\|z\|_{s,t,\infty}^{1-\frac{2}{\gamma}} + C\|z\|_{s,t,\infty}^{1-\frac{1}{\gamma}} + C\|z\|_{s,t,\infty} \right] \times \\ \times (t-s) + \sigma\gamma R \|B^H\|_{0,T,\beta} \times \\ \times \left[\|z\|_{s,t,\infty}^{1-\frac{1}{\gamma}} (t-s)^\beta + \|z\|_{s,t,\beta}^{1-\frac{1}{\gamma}} (t-s)^{\beta(2-\frac{1}{\gamma})} \right].$$

We choose Δ such that

$$\Delta = \left[\frac{1}{2\sigma\gamma R \|B^H\|_{0,T,\beta}} \right]^{\frac{\gamma}{\beta(\gamma-1)}} \wedge \frac{1}{8\gamma(k+C) + 8\gamma C} \wedge \\ \wedge \left(\frac{1}{8\sigma\gamma R \|B\|_{0,T,\beta}} \right)^{1/\beta}.$$

By following similar arguments in the proof of Theorem 2.3 in [11], for all $s, t \in [0, T], s \le t$ such that $t - s \le \Delta$, we have

$$\|z\|_{s,t,\infty} \le 2|z(s)| + 4\gamma(k+C)T + 4T^\beta. \tag{2.9}$$

It leads to

$$\|z\|_{0,T,\infty} \le \\ \le 2^{T\left[(2\sigma\gamma R\|B^H\|_{0,T,\beta})^{\frac{\gamma}{\beta(\gamma-1)}} \vee (8\gamma(k+C)+8\gamma C) \vee (8\sigma\gamma R\|B\|_{0,T,\beta})^{1/\beta} \right] + 1} \times \\ \times \left(|z(0)| + 4\gamma(k+C)T + 4T^\beta \right).$$

This fact together with the estimate

$$\|Y\|_{0,T,\infty} \le \|z\|_{0,T,\infty}^{1/\gamma},$$

we obtain the proof.

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