LINEAR PURSUIT GAMES ON TIME SCALES WITH DELAY IN INFORMATION AND GEOMETRICAL CONSTRAINTS

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ABSTRACT

The time scales that enable the integration of the dynamics equations presented by differential equations and difference equations under the same model are the dynamic equations on time scales, and many other models are expressed by the dynamic equations on time scales as well. The pursuit problem is a basic problem of games theory, in which, the motions of two given objects (the pursuer and the evader) are described by the differential equations involved in the control variables. In this problem, the pursuer shall construct their control according to the information of the evader. And in reality, the information constructed by the pursuer is usually delayed for a certain amount of time. The paper considers some sufficient conditions for completing the linear pursuit games with delay in information. The controls of the pursuer and the evader are impacted by the geometrical constraints.

Key words: *The pursuit game, time scale, admissible control, constraints, delay in information.*

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VỀ TRÒ CHƠI ĐUỔI BẮT TUYẾN TÍNH TRÊN THANG THỜI GIAN VỚI THÔNG TIN CHẬM VÀ HẠN CHẾ HÌNH HỌC

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TÓM TẮT

Thang thời gian cho phép hợp nhất các hệ động lực mô tả bởi hệ phương trình vi phân và hệ phương trình sai phân dưới cùng một mô hình chung là hệ động lực trên thang thời gian, đồng thời nhiều mô hình khác của thực tế cũng được mô tả bởi hệ động lực trên thang thời gian. Bài toán đuổi bắt là một trong các bài toán cơ bản của lý thuyết trò chơi mà ở đó chuyển động của hai đối tượng (tạm gọi là người đuổi và người chạy) được mô tả bởi các hệ phương trình vi phân có tham gia biến điều khiển. Trong bài toán này, người đuổi sẽ xây dựng điều khiển của mình theo thông tin của người chạy và trong thực tế, thông tin mà người đuổi xây dựng thường bị chậm bởi một khoảng thời gian nào đó. Bài báo này trình bày một số điều kiện đủ kết thúc trò chơi cho bài toán trò chơi đuổi bắt tuyến tính trên thang thời gian với thông tin chậm. Điều khiển của người đuổi và người chạy chịu tác động bởi hạn chế hình học.

Từ khóa: *Trò chơi đuổi bắt, thang thời gian, điều khiển chấp nhận được, hạn chế, thông tin chậm.*

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1. Introduction

Stefan Hilger introduced in his doctoral thesis [1], the notion *time scale*. For the last 30 years, mathematical analysis on time scales and dynamic equations on time scales have emerged and increasingly developed, see, for example, [2]-[3]. The time scales that enable the integration of the dynamics equations presented by differential equations and difference equations under the same model are the dynamic equations on time scales, and many other models are expressed by the dynamic equations on time scales as well. There have been rising interests in the dynamic equations with controls on time scales in recent years, see, for example, [2]. The basic results of the differential equation (theory of qualitative, theory of stability, ...) and the theory of controls (controllability, optimal controls,...) have been rephrased according to the dynamic equations on time scales, see, for example, [4], [3], [5]. However, to our knowledge, the games on time scales (the dynamic equations under two controls in general with opposite targets), have yet to receive the attention they deserve.

The pursuit problem is a basic problem of games theory. This problem can be expressed as follows: The motions of two given objects (the pursuer and the evader) are described by the differential equations involved in the control variables. The goal of the pursuer is to catch up with the evader as quickly as possible. The goal of the evader is to keep themselves from the pursuer for as long and as distant as possible. Therefore, we can say that the pursuer needs minimize and evader needs maximumize the function of distance. The pursuer shall construct their control $u(t)$ according to the information about $v(t)$ of the evader, i.e., $u(t) := u(v(t))$. Persuit games, described by a differential equations or by a discrete equations has been studied in many papers, see, for example, [7], [8]. The persuit game on the time scales has been studied [6]. However, in reality, the information constructed by the pursuer is usually delayed for a certain amount of time. The article considers sufficient conditions for completing the linear pursuit games with delayed information. Results in the article are the combination of the acknowledged results in linear pursuit games with delayed information presented by differential equations and difference equations (see [7], [8] and references therein).

2. Substance

2.1 The analysis and dynamic equations on time scales

Time scales

A time scale is an arbitrary nonempty closed subset of the real numbers. Throughout this paper we will denote a time scale by the symbol $\mathbb{T}$. If $\mathbb{T}=\mathbb{R}$ (the set of real numbers) we have *the continiuos time scales*. If $\mathbb{T}=\mathbb{N}$ (the set of natural numbers) or $\mathbb{T}=\mathbb{Z}$ (set of the integers) we have *the discrete time scales*. However, the general theory is of course applicable to many more time scales , for example, $\mathbb{T}=\bigcup_{k=0}^{\infty} T_k=\bigcup_{k=0}^{\infty}\left[2k,2k+1\right]$ is a time scale.

Let $\mathbb{T}$ be a time scale.

Definition 1 (see, i.e, [9]) For $t\in\mathbb{T}$ we define *the forward jump operator* $\sigma:\mathbb{T}\to\mathbb{T}$ by $\sigma(t):=\inf\{s\in\mathbb{T}, s>t\}$.

While the *backward jump operator* $\rho:\mathbb{T}\to\mathbb{T}$ by $\rho(t):=\sup\{s\in\mathbb{T}, s<t\}$.

The grainiess function $\mu:\mathbb{T}\to[0;\infty)$ is defined by $\mu(t):=\sigma(t)-t$.

In this definition we put $\inf\varnothing=\sup\mathbb{T}$ (i.e, $\sigma(t)=t$ if $\mathbb{T}$ has a maximum t) and $\sup\varnothing=\inf\mathbb{T}$ (i.e, $\rho(t)=t$ if $\mathbb{T}$ has a minimum t), where $\varnothing$ denotes the empty set.

Definition 2 Let $\mathbb{T}$ be a time scale.

For $t \in \mathbb{T}$. If $\sigma(t) > t$; we say that t is *right-scattered*. While $\rho(t) < t$; we say that t is *left-scattered*.

Points that are right-scattered and left-scattered at the same time are called *insolated.*

If $t < \sup \mathbb{T}$ and $\sigma(t) = t$; then t is called *right-dense*.

And If $t > \inf \mathbb{T}; \rho(t) = t$; then t is called *left-dense*.

Points that are right-dense and left-dense at the same time are called *dense.*

2.2 The Analysis on time scales

Topology on time scales

Since $\mathbb{T}$ has a topology inherited from the standard topology on the real line . We have the concept of neighborhood, limits and continuous functions by natural way.

The set $\mathbb{T}^k$ which is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left - scattered maximum M then $\mathbb{T}^k := \mathbb{T} \setminus \{M\}$ and $\mathbb{T}^k := \mathbb{T}$ otherwise .

Assume a function $f : \mathbb{T} \to \mathbb{R}$ is defined in $\mathbb{T}$, takes values in $\mathbb{R}$, and $t \in \mathbb{T}^k$.

Definition 3 A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided finite limit exits at all right-dense points in $\mathbb{T}$ and its left-sided finite limit exists at all left-dense points in $\mathbb{T}$.

Definition 4 A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left - sided finite limit exists at left-dense points in $\mathbb{T}$.

A $n \times n$ matrix $A(.)$ which is defined in $\mathbb{T}$ is called *rd-continuous* if all elements of $A(.)$ is rd-continuous on $\mathbb{T}$.

A rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is *called regressive* if $1 + \mu(t) f(t) \neq 0, \forall t \in \mathbb{T}$.

A $n \times n$ matrix $A(.)$ rd-continuous is called *regressive matrix* if $I + \mu(t) A(t)$ is a matrix invertible for all $t \in \mathbb{T}^k$. Where $I = I_n$ be unit matrix of order $n \times n$.

The set of regressive matrix will be denoted by $\mathfrak{R} = \mathfrak{R}(\mathbb{T}, \mathbb{R}^n)$.

2.3 Derivative on time scales

Definition 5 Assume $x : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $x^\Delta(t)$,

to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap D; \forall \delta > 0)$ such that:

$$|[x(\sigma(t) - x(s)] - x^\Delta(t)[\sigma(t) - s]| \leq \epsilon \left|\sigma(t) - s\right| \text{ for all } s \in U.$$

We call $x^\Delta(t)$ the *Delta (or Hilger) derivative* of x at t.

The Hilger derivative of vector function $x : \mathbb{T} \to \mathbb{R}^n$ is Hilger derivative vector of each coordinates.

If $\mathbb{T} \equiv \mathbb{R}$ then Hilger derivative is the usual derivative and Hilger derivative is the usual forward difference operator if $\mathbb{T} \equiv \mathbb{Z}$.

We say that $x(.)$ is $\Delta-$differentiable (or in short differentiable) on $\mathbb{T}^k$ if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$.

2.4 Dynamic equations on time scales

Definition 10

Let $f : \mathbb{T} \to \mathbb{R}^n$. We say that linear dynamic equations

$$x^{\Delta}(t) = A(t)x(t) + f(t), t \in \mathbb{T}, x(t_0) = x_0 \quad (2.1)$$

is *regressive* provided $A \in \Re$ and f is an rd-continuous function.

Clause (see [9]) Assume that $t_0 \in \mathbb{T}$ and $A \in \Re$ is a matrix of order $n \times n$. Then, the initial value problem $X^{\Delta}(t) = A(t)X(t), X(t_0) = I_n$, has a unique solution, where I_n be unit matrix of order $n \times n$, denoted by $\Phi_A(t, t_0)$.

Theorem 2 (see [9]) *Let* $A: \mathbb{T}^k \to \mathbb{R}^{n\times n}$ *and* $f: \mathbb{T}^k \times \mathbb{R}^n \to \mathbb{R}^n$ *be rd-continuous.*

If $x(t), t \geq t_0$ *is a solution of dynamic equation* $x^{\Delta}(t) = A(t)x(t), x(t_0) = x_0$, *then we have*

$$x(t) = \Phi_A(t, t_0)x_0 + \int_{t_0}^{t} \Phi_A(\tau, \sigma(\tau)) f(\tau)\Delta\tau \quad (2.2)$$

Vector function $x(.): \mathbb{T} \to \mathbb{R}^n$ *is differentiable on* $\mathbb{T}$ *satisfying (2.1) is called solution or trajectory of dynamic equations (2.1) on time scale* $\mathbb{T}$.

2.5 The linear pursuit game with delay in information and geometrical constraints on time scales

The linear pursuit process can be described as follows

$$z^{\Delta}(t) = A(t)z(t) - B(t)u(t) + C(t)v(t), \quad t \geq t_0, t, t_0 \in \mathbb{T}. \quad (2.3)$$

Where $z \in \mathbb{R}^n$; functions $u(.), u: \mathbb{T} \to \mathbb{R}^p$ is the pursuit control and $v(.), v: \mathbb{T} \to \mathbb{R}^q$ is the evasion control. $A(t), B(t), C(t)$ are matrices of orders $n\times n, n\times p, n\times q$ respectively.

The controls $u(t); v(t)$ are measurable functions satisfying *Geometrical constraints*:

$$u(t) \in P(t) \subseteq \mathbb{R}^p, t \in \mathbb{T}; \; v(t) \in Q(t) \subseteq \mathbb{R}^q, \; t \in \mathbb{T}. \quad (2.4)$$

In what follows such $u(.)$ and $v(.)$ satisfying (2.4) will be called admissible controls.

For each control $u(.)$ and $v(.)$ are chosen, takes into process (2.3) and uses (2.2) we have a solution of (2.3) as follows

$$z(t) = \Phi_A(t, t_0)z_0 - \int_{t_0}^{t} \Phi_A(\tau, \sigma(\tau))Bu(\tau)\Delta\tau + \int_{t_0}^{t} \Phi_A(\tau, \sigma(\tau))Cv(\tau)\Delta\tau.$$

Let $M \subseteq \mathbb{R}^n$. We shall say that the linear pursuit game (2.3), starting from $z(t_0) = z_0 \notin M$ is completed after the time $K \in \mathbb{T}$ if for any admissible control $v(.)$ of the evader, there exists an admissible control $u(.)$ of the pursuer such that the solution of the equation (2.3) satisfies $z(K) \in M$.

We assume that M of the form: $M = M_1 + M_2 \subseteq \mathbb{R}^N$, where $M_1 \subset \mathbb{R}^{N1} \subseteq \mathbb{R}^N$, $\mathbb{R}^{N_1}$ is a subspace of $\mathbb{R}^N$ and $M_2 \subseteq \mathbb{R}^{N_2}, \mathbb{R}^N = \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2}$.

Let π denote the orthogonal projection from $\mathbb{R}^n$ onto $\mathbb{R}^{N_2}$. Then the condition for the pursuit game to be completed $z(K) \in M$ equivalent $\pi z(K) \in M_2$.

In the pursuit games, we can assume that the information as follows (see [6]): We shall be interested in computing the value $u(t)$ of the pursuit control at each time t when the values $v(t)$

of the evasion control are known for all t, i.e., $u(t)=u(v(t))$. However, in the real life, the pursuer is taken delay in information after a finite interval of time. Consequently, according [7] and [8], we shall define the delay in information as follows.

Assume that there exists $\alpha \in \mathbb{T}, \alpha \geq 0$ and $r : \mathbb{T}_\alpha \to \mathbb{T}$ be increasing and delta-diffenrentiable function on $\mathbb{T}$ such that $r(t) \leq t$ for every $t \in \mathbb{T}_\alpha, \mathbb{T}_\alpha := \mathbb{T} \cap [\alpha; +\infty)$.

To formulate the admissible control $u(.)$, the pursuer are known the information of (2.3), the set M on which the game must be completed and specialy, the pursuer are known the information about the control of evader at moment $r(t)$ implies $u(t)=u(v(r(t)))$.

And, to solve the pursuit process, L. S. Pontriagin defined *the geometrical difference* of two sets (we shall call *Pontriagin geometrical difference*) as follows: Let $A, B \in \mathbb{R}^n$

$$A \overset{*}{-} B := \{z \in \mathbb{R}^N, z + B \subseteq A\}.$$

Theorem 3 *Assume* $K \in \mathbb{T}$ *is the smallest number of the* $t \geq t_0, t \in \mathbb{T}$ *such that the assumption are satisfying:*

There exists an admissible control $u^*(t)$ *on* $[t_0; \alpha]_{\mathbb{T}}$ *such that*

$$\pi\Phi_A(K,t_0)z_0 - \int_{t_0}^{\alpha} \pi\Phi_A(K,\sigma(s))B(s)u^*(s)\Delta s \in G(K) + \left(M_2 \overset{*}{-} H(K)\right), \quad (2.2.3)$$

where $H(K) = \left\{ \int_{t_0}^{r(\alpha)} \pi\Phi_A(K,\sigma(s))C(s)Q(s)\Delta s + \int_{r(K)}^{K} \pi\Phi_A(K,\sigma(s))C(s)Q(s)\Delta s \right\}$;

$$G(t) = \int_{\alpha}^{t} \pi\left[\Phi_A(t,\sigma(\tau))B(\tau)P(\tau) \overset{*}{-} \Phi_A(t,\sigma(r(\tau)))C(r(\tau))Q(r(\tau))r^{\Delta}(\tau)\right]\Delta\tau.$$

Then the linear pursuit game (2.2.1) with geometrical constraints (2.2.2) is completed after the time K.

Proof

Relying on (2.2.3), there exists vector $g(K) \in G(K)$ and $m_2(K) \in M_2 \overset{*}{-} H(K)$ such that

$$\pi\Phi_A(K,t_0)z_0 - \int_{t_0}^{\alpha} \pi\Phi_A(K,\sigma(s))B(s)u^*(s)\Delta s = g(K) + m_2(K).$$

Assume $v(\tau)$ is any admissible control of the evader, implies $v(\tau) \in Q(t), t \in \mathbb{T}$.

Then $$\int_{t_0}^{r(\alpha)} \pi\Phi_A(K,\sigma(s))C(s)v(s)\Delta s + \int_{r(K)}^{K} \pi\Phi_A(K,\sigma(s))C(s)v(s)\Delta s \in H(K).$$

according to the definition Pontriagin geometrical difference, there exists a vector $m_2 \in M_2$ such that

$$m_2(K) + \int_{t_0}^{r(\alpha)} \pi\Phi_A(K,\sigma(s))C(s)v(s)\Delta s + \int_{r(K)}^{K} \pi\Phi_A(K,\sigma(s))C(s)v(s)\Delta s = m_2.$$

Because of $g(K) \in G(K)$. By the definition Pontriagin geometrical difference, we get

$$g(K)=\int_{\alpha}^{K}\pi[\Phi_A(K,\sigma(\tau))B(\tau)u(\tau)-\Phi_A(K,\sigma(r(\tau)))C(r(\tau))v(r(\tau))r^{\Delta}(\tau)]\Delta\tau$$

$$=\int_{\alpha}^{K}\pi\Phi_A(K,\sigma(\tau))B(\tau)u(\tau)\Delta\tau-\int_{\alpha}^{K}\pi\Phi_A(K,\sigma(r(\tau)))C(r(\tau))v(r(\tau))r^{\Delta}(\tau)\Delta\tau$$

Substituting variable under the integral, let $\theta=r(\tau),\ \Delta\theta=r^{\Delta}(\tau)\Delta\tau.$

Hence

$$\int_{\alpha}^{K}\Phi_A(K,\sigma(r(\tau)))C(r(\tau))v(r(\tau))r^{\Delta}(\tau)]\Delta\tau=\int_{r(\alpha)}^{r(K)}\Phi_A(K,\sigma(\theta))C(\theta)v(\theta)\Delta\theta.$$

We obtain $$g(K)=\int_{\alpha}^{K}\pi\Phi_A(K,\sigma(\tau))B(\tau)u(\tau)\Delta\tau-\int_{r(\alpha)}^{r(K)}\pi\Phi_A(K,\sigma(\theta))C(\theta)v(\theta)\Delta\theta.$$

To construct the admissible control of the pursuer

$$u(\tau)=\begin{cases} u^*(\tau), & t_0\le\tau\le\alpha;\\ \bar{u}(\tau), & \alpha\le\tau\le K.\end{cases}$$

Where $\bar{u}(\tau)=u\left(v\left(r(\tau)\right)\right)$. Then we have:

$$\pi\Phi_A(K,t_0)z_0-\int_{t_0}^{\alpha}\pi\Phi_A(K,\sigma(s))B(s)u^*(s)\Delta s=g(K)+m_2(K).$$

$$=\int_{\alpha}^{K}\pi\Phi_A(K,\sigma(\tau))B(\tau)\bar{u}(\tau)\Delta\tau-\int_{r(\alpha)}^{r(K)}\pi\Phi_A(K,\sigma(\theta))C(\theta)v(\theta)\Delta\theta$$

$$+m_2-\int_{t_0}^{r(\alpha)}\pi\Phi_A(K,\sigma(s))C(s)v(s)\Delta s-\int_{r(K)}^{K}\pi\Phi_A(K,\sigma(s))C(s)v(s)\Delta s$$

$$=\int_{\alpha}^{K}\pi\Phi_A(K,\sigma(\tau))B(\tau)\bar{u}(\tau)\Delta\tau-\int_{t_0}^{K}\pi\Phi_A(K,\sigma(\theta))C(\theta)v(\theta)\Delta\theta+m_2$$

$$\Leftrightarrow\pi\Phi_A(K,t_0)z_0-\int_{t_0}^{\alpha}\Phi_A(K,\sigma(s))B(s)u^*(s)\Delta s-\int_{\alpha}^{K}\pi\Phi_A(K,\sigma(\tau))B(\tau)\bar{u}(\tau)\Delta\tau+$$
$$\int_{t_0}^{K}\pi\Phi_A(K,\sigma(\theta))C(\theta)v(\theta)\Delta\theta=m_2$$

$$\Leftrightarrow\pi\Phi_A(K,t_0)z_0-\int_{t_0}^{K}\pi\Phi_A(K,\sigma(s))B(s)u(s)\Delta s+\int_{t_0}^{K}\pi\Phi_A(K,\sigma(\theta))C(\theta)v(\theta)\Delta\theta=m_2$$

So: $$\pi[\Phi_A(K,t_0)z_0-\int_{t_0}^{K}\Phi_A(K,\sigma(s))B(s)u(s)\Delta s+\int_{t_0}^{K}\Phi_A(K,\sigma(\theta))C(\theta)v(\theta)\Delta\theta]=m_2$$

or $\pi z(K)=m_2$.

Therefore, the game is completed after the time K.

Means of the coditions in theorem 3 as follows:

1) On $[t_0;\alpha]$ we can't find the information about the evader so we choesed any control $u^*(t)$ such that (2.2.3).

2) Because we can't construct the control $u(t)$ of the pursuer correspond to the control of the evader $v(t)$ with $t_0 \le t \le r(\tau)$ and $r(K) \le t \le K$ so we get "inclution condition", implies the control is taken to $M_2 : M_2$ is large enough to "inclution". We have $M_2 + H(K) \ne \varnothing$.

3. Conclusion

In this paper, we have stated and demonstratedthe linear pursuit game on time scales with delayed information and the geometrical constraint. This theorem presented in the article enabled the integration of several known results in linear pursuit games expressed by continuous dynamics systems and discrete dynamics systems with the geometrical constrain

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