A NEW HYBRID GRADIENT PROJECTION METHOD FOR THE PSEUDOMONOTONE VARIATIONAL INEQUALITY AND FIXED POINT PROBLEM IN BANACH SPACE

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Abstract: It is known that the variational inequality problem is an extension of the nonlinear compensation problem. It includes convex optimization problems, game theory, Kakutani fixed-point problems, polynomial and tensor variational inequalities, and direct applications in many fields such as operations research, Nash-Counot equilibrium model, and transportation. Methods for solving variational inequalities and fixed points problems have received much attention. However, most of recent methods for solving variational inequalities and fixed points problems require the monotone and Lipschitz continuous or inverse-strongly monotone assumptions of the cost mapping. Algorithms are mainly built on finite dimensional spaces or Hilbert spaces. In this paper, we propose a new hybrid gradient projection method to find a common element of the solution set of pseudo-monotone, Lipschitz continuous variational inequality problem with the fixed point set of relatively weak non-expansion mapping in Banach spaces. With the given assumptions, we obtain a strong convergence theorem of the algorithm.

Keywords: Hybrid gradient projection method, lipschitz continuous, pseudomonotone mapping, relatively weak nonexpansive mapping.

PHƯƠNG PHÁP CHIẾU ĐẠO HÀM KIỂU LAI GHÉP MỚI GIẢI BÀI TOÁN BẤT ĐẲNG THỨC BIẾN PHÂN GIẢ ĐƠN ĐIỆU VÀ BÀI TOÁN ĐIỂM BẤT ĐỘNG TRONG KHÔNG GIAN BANACH

Tóm tắt: Chúng ta biết rằng bài toán bất đẳng thức biến phân là bài toán mở rộng của bài toán bù phi tuyến. Nó bao gồm các bài toán tối ưu hóa lồi, lý thuyết trò chơi, các bài toán điểm bất động Kakutani, các bất đẳng thức biến phân đa thức và tenxơ, và những ứng dụng trực tiếp trong nhiều lĩnh vực như nghiên mô hình cân bằng Nash-Cournot, giao thông vận tải… Các phương pháp để giải bài toán điểm bất động và bài toán bất đẳng thức biến phân rất được chú trọng. Tuy nhiên hầu hết các phương pháp gần đây đều đòi hỏi tính đơn điệu và liên tục Lipschitz hoặc tính đơn điệu mạnh ngược của ánh xạ giá. Các thuật toán chủ yếu xây dựng trên không gian hữu hạn chiều hoặc không gian Hilbert. Trong bài báo này, chúng tôi giới thiệu một phương pháp chiếu đạo hàm kiểu lai ghép mới để tìm phần tử chung của tập nghiệm bài toán bất đẳng thức biến phân giả đơn điệu, liên tục Lipschitz với tập điểm bất động của ánh xạ không giãn yếu tương đối trong không gian Banach. Với các giả thiết cho trước, chúng tôi đạt được định lý hội tụ mạnh của thuật toán.

Từ khóa: Chiếu đạo hàm kiểu lai ghép, liên tục Lipschitz, ánh xạ giả đơn điệu, ánh xạ không giãn yếu tương đối.

1. Introduction

Let E be a real Banach space with norm $\|.\|$, and E^* be the dual of E. $\langle x, f\rangle$ denotes the duality pairing of E and E^*, C is a nonempty, closed and convex subset of E. The variational inequality problem is to find a point $x^* \in C$ such that

$$\left\langle y - x^*, F\left(x^*\right)\right\rangle \geq 0 \quad \forall x^* \in C \tag{1.1}$$

where $F: C \to E^*$ is a nonlinear mapping. The set of solutions of the variational inequality problem is denoted by $VI(C,F)$. Variational inequality was firstly introduced by Stampachia [1] in 1967. This problem has been intensively considered because it covers diverse disciplines such as partial differential equations, optimal control, optimization, mathematical programming, mechanics and finance (see [1-3]).

Let $F: C \to E^*$ be a single-valued mapping, F is called $L-$Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|Fx - Fy\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

F is called monotone if

$$\langle x - y, Fx - Fy\rangle \geq 0, \quad \forall x, y \in C.$$

F is called pseudomonotone if for any $x, y \in C$

$$\langle x - y, Fy\rangle \geq 0 \Rightarrow \langle x - y, Fx\rangle \geq 0.$$

A mapping $S: C \to C$ is said to be nonexpansive if

$\|Sx - Sy\| \le \|x - y\|, \ \forall x, y \in C$.

A point $x \in C$ is a fixed point of S if $Sx = x$ and we denote by $Fix(S)$ the set of fixed points of S; that is, $Fix(S) = \{x \in C : Sx = x\}$. A point p in C is said to be asymptotic fixed point of S (see [4,5]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$.

The set of asymptotic fixed points of S will be denoted by $Fix(S)$. A mapping S from C into itself is called relatively nonexpansive if $Fix(S) = Fix(S)$ and $\phi(p, Sx) \le \phi(p, x)$ for $x \in C$ and $p \in Fix(S)$. A point p in C is said to be a strong asymptotic fixed point of S (see[10]) if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$. The set of strong asymptotic fixed points of S will be denoted by $Fix(S)$. A mapping S from C into itself is called relatively weak nonexpansive if $Fix(S) = Fix(S)$ and $\phi(p, Sx) \le \phi(p, x)$ for $x \in C$ and $p \in Fix(S)$.

In recent years, most of current methods for solving variational inequalities and fixed points problems require the monotone and Lipschitz continuous or inverse-strongly monotone assumptions of the cost mapping, see e.g., [4-7].

In 2018, Su and Qin in [8] introduced the monotone CQ method for nonexpansive semigroups and maximal monotone operators for Hilbert spaces. The advantage of this method is that the sequence generated by it is Cauchy sequence and we proceed without using of any weak topological techniques.

In order to weaken the inverse-strong monotonicity of A, Nakajo in [5] proposed the following hybrid gradient projection method:

$$\begin{cases} x_1 = x \in E, arbitrarily, \\ y_n = \prod_C J^{-1}\left(Jx_n - \lambda_n Fx_n\right), \\ z_n = Sy_n, \\ C_n = \left\{x^* \in C : \phi(x^*, z_n) \le \phi(x^*, x_n) - \phi(x_n, y_n) - 2\lambda_n \left\langle y_n - x^*, Fx_n - Fy_n \right\rangle\right\}, \\ Q_n = \left\{x^* \in C : \left\langle x_n - x^*, Jx - Jx_n \right\rangle \ge 0\right\}, \\ x_{n+1} = \prod_{C_n \cap Q_n} x, \end{cases} \qquad (1.2)$$

where E is a $2-$uniformly convex and uniformly smooth Banach spaces and F is only supposed to be monotone and Lipschitz continuous. He proved the sequence $\{x_n\}$ generated be (1.2) strongly converges to $\prod_D x$, where $D = VI(C,F) \cap Fix(S)$.

On the other hand, some authors also proposed some iterative algorithms to relax the assumption on F from monotonicity to pseudomonotonicity. However, these iterative algorithms are all confined to finite dimension spaces or Hilbert spaces, see [9,10] and the reference therein. There are not results for pseudomonotone variational inequalities in Banach spaces. In order to fill the gap, we will combine the hybrid projection algorithm introduced by Nakajo in [5] to propose a new hybrid gradient projection method for finding a common element of the set of solution of the variational inequalitiy (1.1) and the set of fixed points of a relatively weak nonexpansive mapping in Banach spaces. The advantage of the proposed algorithm in this paper is that strong convergence results only require the pseudomonotonicity of the cost mapping in a Banach space.

2. Preliminaries

Let $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$.

Let $U = \{x \in E, \|x\| = 1\}$. A Banach space E is said to be strictly convex if for any $x, y \in U$ and $x \neq y$ implies $\left\|\frac{x+y}{2}\right\| < 1$. It is also said to be uniformly convex if for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that for any $x, y \in U, \|x-y\| \geq \varepsilon$ implies $\left\|\frac{x+y}{2}\right\| < 1-\delta$. We define a function $\delta : [0,2] \to [0,1]$ called the modulus of convexity of E as follows :

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in U, \|x-y\| \geq \varepsilon\right\}$$

Then E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be $p-$*uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in (0,2]$. It is obvious that a $p-$uniformly convex Banach space is uniformly convex. A Banach space E is said to be smooth if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \qquad (2.1)$$

exist for all $x, y \in U$.

We denote by J the normalized duality from E to 2^{E^*}, defined by

$$Jx := \left\{v \in E^* : \langle x, y\rangle = \|y\|^2 = \|x\|^2\right\}, \quad \forall x \in E.$$

Let E be a smooth Banach space. We know the following functional studied in Alber [11] :

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,$$

for all $x, y \in E$. Clearly, we have from the definition of ϕ that

$$\left(\|x\| - \|y\|\right)^2 \le \phi(y, x) \le \left(\|x\| + \|y\|\right)^2, \quad \forall x, y \in E.$$

Remark 2.1. We have from Remark 2.1 in [12] that, if E is a strictly convex and smooth Banach space, then for $x, y \in E, \phi(x, y) = 0$ if and only if $x = y$.

Lemma 2.1.(See [12].)Let E be a uniformly convex and smooth Banach space and let $\{y_n\}, \{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$, or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.

Let E be a reflexive, strictly convex and smooth Banach space. C denotes a nonempty, closed and convex subset of E. By Alber [11], for each $x \in E$, there exist a unique element $x_0 \in C$ (denote by $\prod_C(x)$) such that

$$\phi(x_0, x) = \min_{y\in C}\phi(y, x).$$

The mapping $\prod_C : E \to C$, defined by $\prod_C(x) = x_0$, is called the generalized projection operator from E onto C. Moreover, x_0 is called the generalized projection of x.

Lemma 2.2.(see [12]) Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \prod_C(x)$ if and only if

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.3.(see [12]) Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.4.(see [4]) Let E be a reflexive, strictly convex and smooth Banach space, let C be a closed convex subset of E, and let S be a relatively weak nonexpansive mapping from C into itself. Then $Fix(S)$ is closed and convex.

An operator A of C into E^* is said to be hemicontinuous if for all $x, y \in C$, the mapping f of $[0,1]$ into E^* defined by $f(t) = F(tx + (1-t)y)$ is continuous with respect to the weak* topology of E^*.

Lemma 2.5.(see [5]) Let E be a $2-$uniformly convex and smooth Banach space. Then, for every $x, y \in E, \phi(x, y) \geq c_1 \|x - y\|^2$, where $c_1 > 0$ is the $2-$uniformly convexity constant of E.

Lemma 2.6. Let C be a nonempty, closed and convex subset of a Banach space E and A a pseudomonotone, hemicontinuous operator of C into E^*. Then

$$VI(C, F) = \left\{ x^* \in C : \left\langle y - x^*, Fy \right\rangle \geq 0 \text{ for all } y \in C \right\}.$$

Proof. Let $D = \left\{ x^* \in C : \left\langle y - x^*, Fy \right\rangle \geq 0 \text{ for all } y \in C \right\}$. It is obvious from the pseudomonotonicity of F that $VI(C, F) \subset D$. Next, we show that $D \subset VI(C, F)$. Let $z \in D$, we have

$$\langle x - z, Fx \rangle \geq 0, \quad \forall x \in C. \qquad (2.2)$$

Putting $y_t = (1-t)z + tx$ for any $t \in (0,1)$ and $x \in C$. By convexity of $C, y_t \in C$. By substituting y_t for x in (2.2), we have $\langle y_t - z, Fy_t \rangle = t \langle x - z, Fy_t \rangle \geq 0$. This is $\langle x - z, Fy_t \rangle \geq 0$, for each $t \in (0,1)$. From $y_t \to z$ as $t \to 0$ and the hemicontinuous of F, we obtain

$$\langle x - z, Fz \rangle \geq 0 \quad \forall x \in C$$

Therefore, $D \subset VI(C, F)$. This implies that

$$VI(C,F)=\left\{x^*\in C:\left\langle y-x^*,Fy\right\rangle\geq 0\text{ for all } y\in C\right\}$$

3. Main results

Now, we consider the composition of a relatively weak nonexpansive mapping and the new hybrid gradient projection method for a pseudomonotone variational inequality and show a new strong convergence theorem.

In this section, we always assume the following conditions.

(A1) C is a nonempty closed convex subset of a $2-$uniformly convex and uniformly smooth Banach space E with the $2-$uniformly convexity constant c_1.

(A2) The mapping $F:C\to E^*$ is pseudomonotone and $L-$Lipschitz continuous such that $VI(C,F)\neq\varnothing$.

(A3) $S:C\to C$ is a relatively weak nonexpansive mapping such that $Fix(S)\neq\varnothing$.

(A4) $VI(C,F)\bigcap Fix(S)\neq\varnothing$.

Algorithm 3.1. For any $x_0\in C$, we define a sequence $\{x_n\}$ iteratively by

$$\begin{cases} y_n=\prod_C J^{-1}\left(Jx_n-\lambda_n F\left(x_n\right)\right),\\ z_n=Sy_n,\\ C_0=C,\\ C_{n+1}=\left\{x^*\in C_n:\phi\left(x^*,z_n\right)\leq\phi\left(x^*,x_n\right)-\phi\left(y_n,x_n\right)-2\lambda_n\left\langle y_n-x^*,Fx_n-Fy_n\right\rangle\right\}\\ x_{n+1}=\prod_{C_{n+1}}x_0\end{cases}$$

(3.1)

where $\{\lambda_n\}$ is a sequence in $(0,\infty)$, which satisfies $0<\inf_{n\in N}\lambda_n\leq\sup_{n\in N}\lambda_n<\frac{c_1}{2L}$.

Theorem 3.1. Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then, under conditions (A1)-(A4), $x_n\to\prod_{VI(C,F)\bigcap Fix(S)}x_0$.

Proof. First, we show that C_n is closed and convex for all $n\in\mathbb{N}$. It is obvious that $C_0=C$ is closed and convex. Suppose C_k is closed and convex for some $k\in\mathbb{N}$. For $x^*\in C_k$, we obtain that

$$\phi\left(x^*, z_k\right) \le \phi\left(x^*, x_k\right) - \phi(y_k, x_k) - 2\lambda_k \left\langle y_k - x^*, Fx_k - Fy_k \right\rangle$$

is equivalent to

$$2\left\langle x^*, Jz_k - Jx_k \right\rangle - 2\lambda_k \left\langle y_k - x^*, Fx_k - Fy_k \right\rangle + \|x_k\|^2 - \|z_k\|^2 - \phi\left(y_k, x_k\right) \ge 0 .$$

It is easy so see that C_{k+1} is closed and convex. Then, for $n \in \mathbb{N}, C_n$ is closed and convex.

Next, we show that for $n \in \mathbb{N}, Fix(S) \bigcap VI(C,F) \subset C_0 = C$ is obvious. Suppose $Fix(S) \bigcap VI(C,F) \subset C_k$ for some $k \in \mathbb{N}$. Let $u \in Fix(S) \bigcap VI(C,F) \subset C_k$. By $u \in VI(C,F)$, we have $\left\langle y_k - x^*, F\left(x^*\right)\right\rangle \ge 0$. Since F is pseudomonotone, we have $\left\langle y_k - x^*, Fy_k \right\rangle \ge 0$. So, by Lemma 2.2, we get

$$\begin{aligned} \left\langle y_k - x^*, Jx_k - Jy_k \right\rangle &\ge \lambda_k \left\langle y_k - x^*, Fx_k \right\rangle \\ &= \lambda_k \left\langle y_k - x^*, Fx_k - Fy_k \right\rangle + \lambda_k \left\langle y_k - x^*, Fy_k \right\rangle \\ &\ge \lambda_k \left\langle y_k - x^*, Fx_k - Fy_k \right\rangle \end{aligned}$$

which implies that

$$\phi\left(x^*, y_k\right) \le \phi\left(x^*, x_k\right) - \phi(y_k, x_k) - 2\lambda_k \left\langle y_k - x^*, Fx_k - Fy_k \right\rangle . \tag{3.2}$$

On the other hand, by $u \in Fix(S)$, we have

$$\phi\left(x^*, z_k\right) = \phi\left(x^*, Sy_k\right) \le \phi\left(x^*, y_k\right) . \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\phi\left(x^*, z_k\right) \le \phi\left(x^*, x_k\right) - \phi(y_k, x_k) - 2\lambda_k \left\langle y_k - x^*, Fx_k - Fy_k \right\rangle$$

which shows that $x^* \in C_{k+1}$. This implies that $Fix\left(S\right) \bigcap VI\left(C,F\right) \subset C_n$ for all $n \in \mathbb{N}$. From $x_n = \prod_{C_n} x_0$, we have

$$\left\langle x_n - y, Jx_0 - Jx_n \right\rangle \ge 0, \quad \forall y \in C_n .$$

Since $Fix(S) \bigcap VI(C,F) \subset C_n$ for all $n \in \mathbb{N}$, we arrive at

$$\left\langle x_n - x^*, Jx_0 - Jx_n \right\rangle \geq 0, \quad \forall x^* \in Fix(S) \bigcap VI(C,F). \tag{3.4}$$

By Lemma 2.3, we have

$$\phi(x_n, x_0) = \phi\left(\Pi_{C_n} x_0, x_0\right) \leq \phi\left(x^*, x_0\right) - \phi\left(x^*, x_n\right) \leq \phi\left(x^*, x_0\right)$$

for each $u \in Fix(S) \bigcap VI(C,F) \subset C_n$ and for all $n \in \mathbb{N}$. Therefore, the sequence $\left\{\phi\left(x_n, x_0\right)\right\}$ is bounded. By the definition of ϕ, we have $\{x_n\}$ is also bounded. Since F is Lipschitz continuous, we have $\{Fx_n\}$ is also bounded. From $\phi\left(y_n, J^{-1}\left(Jx_n - \lambda_n Fx_n\right)\right) \leq \phi\left(x^*, J^{-1}\left(Jx_n - \lambda_n Fx_n\right)\right)$, we have $\{y_n\}$ is bounded. Noting that $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have $\phi\left(x_n, x\right)_0 \leq \varphi\left(x_{n+1}, x_0\right)$, for all $n \in \mathbb{N}$. Therefore, $\left\{\phi\left(x_n, x_0\right)\right\}$ is nondecreasing. It follows that the limit of $\left\{\phi\left(x_n, x_0\right)\right\}$ exists. By the construction of C_n, we have that $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for any positive integer $m \geq n$. It follows that

$$\begin{aligned} \phi\left(x_m, x_n\right) &= \phi\left(x_m, \Pi_{C_n} x_0\right) \\ &\leq \phi\left(x_m, x_0\right) - \phi\left(\Pi_{C_n} x_0, x_0\right) \\ &= \phi\left(x_m, x_0\right) - \phi\left(x_n, x_0\right). \end{aligned} \tag{3.5}$$

Let $m, n \to \infty$ in (3.5), we have that

$$\phi(x_m, x_n) \to 0. \tag{3.6}$$

It follows from Lemma 2.1 that $x_m - x_n \to 0$ as $m, n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Thus we get that $x_n \to p \in C$, as $n \to \infty$.

Next, we show that $p \in Fix(S) \bigcap VI(C,F)$. By taking $m = n+1$ in (3.6), we have that

$$\lim_{n\to\infty} \phi\left(x_{n+1}, x_n\right) = 0 \tag{3.7}$$

and hence

$$\lim_{n\to\infty} \left\| x_{n+1} - x_n \right\| = 0 . \tag{3.8}$$

From $x_{n+1} \in C_{n+1}$ and Lemma 2.5, we get

$$\begin{aligned}\phi(x_{n+1},z_n) &\le \phi(x_{n+1},x_n)-\phi(y_n,x_n)-2\lambda_n\langle y_n-x_{n+1},Fx_n-Fy_n\rangle\\ &=\phi(x_{n+1},x_n)-\phi(y_n,x_n)-2\lambda_n\langle y_n-x_n,Fx_n-Fy_n\rangle-2\lambda_n\langle x_n-x_{n+1},Fx_n-Fy_n\rangle\\ &\le\phi(x_{n+1},x_n)-c_1\|y_n-x_n\|^2+2\lambda_nL\|y_n-x_n\|^2+2\lambda_nL\|x_n-x_{n+1}\|\|x_n-y_n\|\\ &=\phi(x_{n+1},x_n)+(2\lambda_nL-c_1)\|y_n-x_n\|^2+2\lambda_nL\|x_n-x_{n+1}\|\|x_n-y_n\|.\end{aligned}$$

(3.9)

Applying (3.7), (3.8) and $\sup_{n\in\mathbb{N}}\lambda_n<\frac{c_1}{2L}$ to (3.9), we have

$$\lim_{n\to\infty}\phi(x_{n+1},z_n)=\lim_{n\to\infty}\|x_n-y_n\|=0. \quad (3.10)$$

By Lemma 2.1, we also have

$$\lim_{n\to\infty}\|x_{n+1}-z_n\|=0 \ . \quad (3.11)$$

Since $\|z_n-y_n\|\le\|z_n-x_{n+1}\|+\|x_{n+1}-x_n\|+\|x_n-y_n\|$, combining (3.11), (3.10) with (3.8), we obtain that

$$\lim_{n\to\infty}\|z_n-y_n\|=0. \quad (3.11)$$

It follows from $x_n\to p$ and (3.10) that

$$y_n\to p,\ \ as\ n\to\infty \ . \quad (3.12)$$

Since $z_n=Sy_n$ and S is relatively weak nonexpansive, by (3.12) and (3.11), we have

$$p\in Fix(S). \quad (3.13)$$

Since $y_n=\prod_C J^{-1}(Jx_n-\lambda_nFx_n)$, we have $\langle y_n-x^*,Jx_n-Jy_n\rangle\ge\lambda_n\langle y_n-x^*,Fx_n\rangle$ for all $n\in\mathbb{N}$ and $x^*\in C$. Hence,

$$\langle y_n-x^*,Jx_n-Jy_n\rangle-\lambda_n\langle y_n-x^*,Fx_n-Fy_n\rangle\ge\lambda_n\langle y_n-x^*,Fy_n\rangle \quad (3.14)$$

for all $n\in\mathbb{N}$ and $x^*\in C$. From (3.10), we get

$$\lim_{n\to\infty}\|Jx_n-Jy_n\|=0. \quad (3.15)$$

From (3.14), we have that

$$\left\|y_n - x^*\right\|\left(\left\|Jx_n - Jy_n\right\| + \lambda_n L\left\|x_n - y_n\right\|\right) \ge \lambda_n \left\langle y_n - x^*, Fy_n\right\rangle \ge \left(\inf_{n\in\mathbb{N}} \lambda_n\right)\left\langle y_n - x^*, Fy_n\right\rangle$$

for all $n \in \mathbb{N}$ and $x^* \in C$. Letting $n \to \infty$, we obtain that

$$\left\langle p - x^*, Fp\right\rangle \le 0, \quad \forall x^* \in C.$$

This show that

$$p \in VI(C,F). \tag{3.16}$$

(3.13) and (3.16) imply that $p \in Fix(S)\bigcap VI(C,F)$. Finally, we prove that $p = \prod_{Fix(S)\bigcap VI(C,F)} x_0$. By taking a limit in (3.4), we have

$$\left\langle p - x^*, Jx_0 - Jp\right\rangle \ge 0, \quad \forall x^* \in Fix(\mathrm{S})\bigcap VI(C,F).$$

At this point, in view of Lemma 2.2, we have that $p = \prod_{Fix(S)\bigcap VI(C,F)} x_0$.

4. Conclusions

In this paper, we combine the hybrid projection algorithm introduced by Nakajo in [5] to propose a new hybrid gradient projection method for finding a common element of the set of solution of the variational inequalitiy (1.1) and the set of fixed points of a relatively weak nonexpansive mapping in Banach spaces. The advantage of the proposed algorithm in this paper is that strong convergence results only require the pseudomonotonicity of the cost mapping in a Banach space.

REFERENCES

1. Lions, J.L., Stampacchia, G (1967), Variational inequalities, *Comm. Pure Appl. Math*, 20, 493-517.

2. Browder, F.E (1965), Nonlinear monotone operators and convex sets in Banach spaces, *Bill. Am. Math. Soc*, 71, 780-785.

3. Browder, F.E (1965), Nonexpansive nonlinear operators in Banach spaces, *Proc. Natl. Acad. Sci, USA* 54, 1041-1044.

4. Zegeye H.,Shahzad N (2009), Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings, *Nonlinear Analysis* 70, 2707-2716.

5. Kazuhide Nakajo (2015), Strong convergence for gradient projection method and relatively nonexpansive mappings in Banach spaces, *Applied Mathematics and Computation,* 271, 251-258.

6. Chen J, Zhang L, Fan T (2007), Viscosity approximation methods for nonexpansive mappings and motonone mappings, *J. Math. Anal. Appl*. 334, 1450-1461.

7. Liu, Y (2009), Strong convergence theorems for relatively nonexpansive mappings and inverse-strongly-monotone mapping in a Banach space, *Applied Mathematics and Mechanics*, 30, 925-932.

8. Su, Y.F., Qin, X.L (2008), Monotone CQ iteration processes for nonexpansive semigroups and maximal motone operators, *Nonliear Analysis*, 68, 3657-3664.

9. Solodov, M.V., Svaiter, B.F (1999), A new projection method for variational inequality problems, *SIAM J. Control Optim*. 37, 765-776.

10. Ceng, L.C. Teboulle, M., Yao, J.C (2010), Weak Convergence of an Iterative Method for Pseudomonotone Variational Inequalities and Fixed Point Problems, *J Optim Theory Appl*, 146, 19-31.

11. Alber, Ya.I., Reich, S (1994), An iterative method for solving a class of nonlinear operator equations in Banach spaces, *Panamer. Math. J*. 4, 39-54.

Alber, Ya.I (1996), Metric and generalized projection operators in Banach spaces: Properties and applications, *Theory and Applications of Nonliear Operators of Accretive and Monotone Type*, 15-50.