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Research Article

VIABILITY FOR DELAY FRACTIONAL DIFFERENTIAL INCLUSIONS

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ABSTRACT

Viability theory designs and develops mathematical and algorithmic methods that can be found in many domains such as living beings, biological evolution, economics, environmental sciences, financial markets or control theory, and robotics. In the paper, we provide sufficient conditions assuring the existence of viable solutions of differential inclusions with fractional derivatives with delay:

$D_C^q x(t) \in F(t, x_t), 0 < q < 1, t \in I := [0, T].$

We were inheriting the ideas of Carja, Donchev, Rafaqat, and Ahmed (2014) and Girejko (2018), we give the condition of tangency and the concept of approximate solution, these are compatible with our problem. Thanks to Brezis-Browder Theorem, we prove the existence of an approximate solution to the interval [0,T]. Then, passing to the limit, the sequence of approximate solutions convergent to the viable solution. These results generalize the corresponding results by Carja et al. (2014), Girejko, Mozyrska, and Wyrwas (2011), Vasundgaradevi and Lakshmikantham (2009).

Keywords: delay; Fractional Derivative; Inclusion; Viable solution

1. Introduction

The first definition of the fractional derivative was introduced at the end of the 19th century by Liouville and Riemann. Still, the concept of non-integer derivative and integral, as a generalization of the traditional integer-order differential and integral calculus, was mentioned already in 1695 by Leibniz and L'Hospital. However, only in the late 1960s did engineers start to be interested in this idea when the fact that descriptions of some systems are more accurate in "fractional language" appeared. Since then, fractional calculus has been increasingly used to model behaviors of natural systems in various fields of science and engineering. Recently, several authors have reported new results concerning the solutions

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for nonlinear fractional differential equations (Diethelm, 2010; Kosmatov, 2009; Zhang, 2009), (Bonilla, Rivero, Rodriguez-Germá, & Trujillo, 2007), (Agarwal, Lakshmikantham, & Nieto, 2010), (Luchko, Rivero, Trujillo, & Velasco, 2010), (Wei, Li, & Che, 2010), (Diethelm & Ford, 2002).

Viability theory has its origin in the Nagumo theorem (Nagumo, 1942), in which necessary and sufficient condition was stated for a differential equation to have a viable solution. A viable solution means a solution initiated in a set of constraints and staying in this set for a certain amount of time. Nowadays, viability theory designs and develops mathematical and algorithmic methods that can be found in many domains such asliving beings, biological evolution, economics, environmental sciences, financial markets or control theory, and robotics. This theory joins fields of science that have been traditionally developed in isolation into one interdisciplinary investigation (Aubin, Bayen, & Saint-Pierre, 2011). All this gives a motivation to combine viability theory with fractional calculus. To the best of our knowledge, there are only a few papers devoted to this subject (Carja et al., 2014), (Girejko et al., 2011), (Mozyrska, Girejko, & Wyrwas, 2011), (Vasundgaradevi & Lakshmikantham, 2009). The viability problem for fractional differential equations was studied some studies (Girejko et al., 2011), (Mozyrska et al., 2011), (Vasundgaradevi & Lakshmikantham, 2009). Unfortunately, in both papers (Girejko et al., 2011), (Vasundgaradevi & Lakshmikantham, 2009), the proofs of the existence of viable solutions are not correct because the tangency conditions proposed by the authors are not appropriate. The viability property was first introduced by (Carja et al., 2014) for the Caputo derivative. The authors give proper tangency conditions that ensure viable solutions for a class of fractional differential inclusions.

$$D_C^q x(t) \in F\big(t, x(t)\big), 0 < q < 1, t \in I := [0, T], x(0) = x_0 \in \mathbb{R}^n.$$

Then (Girejko, 2018) inherits the previous tangency condition for Caputo-Fabrizio derivative.

We denote $\|\cdot\|$ as the norm on $\mathbb{R}^n$. This paper studies the viability properties of solutions to nonlinear fractional differential inclusions with delay

$$D_C^q x(t) \in F(t, x_t), 0 < q < 1, t \in I := [0, T], \quad (1)$$

satisfying the initial condition

$$x(t) = \psi(t), t \in [-r, 0],$$

where $T > 0, \psi \in C_r := C([-r, 0]; \mathbb{R}^n)$ with $\|\psi\|_0 = \sup\limits_{t\in[-r,0]} \|\psi(t)\|$

We consider $x: [-r, T] \to \mathbb{R}^n$. Fix $t \in [0, T]$, we set $x_t: [-r, 0] \to \mathbb{R}^n$ is defined by

$$x_t(\theta) = x(t + \theta), \theta \in [-r, 0].$$

The set valued map $F: [0, T] \times C_r \to P(\mathbb{R}^n)$ (2) is upper semicontinuous with nonempty convex and compact values. Further, there exists a constant $\alpha > 0$ such that

$\|F(t,v)\|_* = \sup\{\|z\|: z \in F(t,v)\} \le \alpha(1 + \|v\|_0)$
for all $(t,v) \in [0,T] \times C_r$.

2. Viability for delay fractional differential inclusion

2.1. Preliminaries

Definition 2.1.1. **(**Caputo derivative) Let $x : [a,b] \to \mathbb{R}^n$ be continuous with Lebesgue integrable derivative. The Caputo fractional derivative $D_c^q x(t)$ of order $0 < q < 1$ is defined by

$$D_c^q x(t) = \frac{1}{\Gamma(1-q)} \int_0^t x'(\tau)(t-\tau)^{-q} d\tau \,, a < t < b.$$

Definition 2.1.2. **(**Upper semicontinuous) The multi-function $F: I \times C_r \to P(\mathbb{R}^n)$ is upper semicontinuous (u.s.c.) at $\xi \in I \times C_r$ if for every open neighborhood V of $F(\xi)$ there exists an open neighborhood U of ξ such that $F(\eta) \subset V$ for each $\eta \in U$.

Definition 2.1.3. **(**Mild solution) The continuous function $x \in AC([-r,T], \mathbb{R}^n)$ is called a mild solution of (1) if there exists a selection mapping $f_x(t) \in F(t, x_t)$ such that for every $t \in I$ we have

$$x(t) = \psi(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_x(s) ds \,, 0 \le t \le T$$

and $x(t) = \psi(t), t \in [-r, 0]$.

According to Lemma 3.1 (Zhou & Peng, 2016), when F satisfies (2), we have

$$\mathrm{Sel}_F(x) = \{f \in L^1(I, \mathbb{R}^n): f(t) \in F(t, x_t) \text{ for a.e. } t \in I\} \neq \emptyset, \forall x \in C([-r,T], \mathbb{R}^n).$$

Definition 2.1.4. (Viable solution) Let closed set $\Omega \subset \mathbb{R}^n$ and $K_\Omega = \{\psi \in C_r: \psi(0) \in \Omega\}$. We say K_Ω is viability if for every $\psi \in K_\Omega$, there exists $T > 0$ such that (1) has mild solution $x: [-r,T] \to \mathbb{R}^n$ satisfies $x_t \in K_\Omega, \forall t \in I$. We call x is a corresponding viable solution.

Noticing:

$x_t(\theta) = x(t+\theta)$

$x_t \in K_\Omega, \forall t \in I \Leftrightarrow x_t(0) \in \Omega, \forall t \in I \Leftrightarrow x(t) \in \Omega, \forall t \in I.$

2.2. Tangency condition

Definition 2.2.1. Suppose $\bar{t} \in I$. Given $E \subset \mathbb{R}^n$ and $g \in L^\infty(I, \mathbb{R}^n)$. We define $y_g: [-r,T] \to \mathbb{R}^n$ by

$$y_g(t) = \psi(0) + \int_0^t (t-s)^{q-1} g(s) ds \,, \forall t \in I.$$

We say the pair (g, E) is tangent to $I \times K_\Omega$ at $(\bar{t}, \varphi) \in I \times K_\Omega$ if $\left(y_g\right)_0 = \varphi$ and

$$\liminf_{h \to 0^+} h^{-q} \mathrm{dist}\left[\varphi(0) + \Phi(\bar{t}; g)(h) + \frac{h^q}{\Gamma(q+1)} E; \Omega\right] = 0, \quad (3)$$

with

$$\Phi(\bar{t}; g)(h) = \frac{1}{\Gamma(q)} \int_0^{\bar{t}} [(\bar{t} + h - s)^{q-1} - (\bar{t} - s)^{q-1}] g(s) ds.$$

Remark: If we replace C_r with $\mathbb{R}^n$ and x_t with $x(t)$ we get the exact tangency condition that was introduced in (Carja et al., 2014).

Proposition 2.2.2. (g, E) is tangent to $I \times K_\Omega$ at $(\bar{t}, \varphi) \in I \times K_\Omega$ if and only if: for each $\delta > 0$ and each $\varepsilon > 0$, there exist $h \in (0, \delta)$ and $v \in E, p \in \mathbb{R}^n$ satisfy $|p| < \varepsilon$ such that

$$\varphi(0) + \Phi\,(\bar{t}; g)(h) + \frac{h^q}{\Gamma(q+1)}(v + p) \in \Omega.$$

Proof.

We see (3) is equivalent to

$$\sup_{\delta>0} \inf_{h\in(0,\delta)} h^{-q} \mathrm{dist}\left[\varphi(0) + \Phi(\bar{t}; g)(h) + \frac{h^q}{\Gamma(q+1)} E; \Omega\right] = 0.$$

In its turn, this relation is equivalent to: for each $\delta > 0$ and each $\varepsilon > 0$, there exist $h \in (0, \delta)$ such that

$$\mathrm{dist}\left[\varphi(0) + \Phi(\bar{t}; g)(h) + \frac{h^q}{\Gamma(q+1)} E; \Omega\right] < \frac{\varepsilon h^q}{\Gamma(q+1)}.$$

Since $\mathrm{dist}(C, D) < \alpha$ if and only if there exist $x \in C$ and $y \in B(0, \alpha)$ such that $x + y \in D$. we finally deduce that (3) equivalent to: for each $\delta > 0$ and each $\varepsilon > 0$, there exist $h \in (0, \delta)$, $v \in E$ and $p \in \mathbb{R}^n$ satisfy $|p| < \varepsilon$ and

$$\varphi(0) + \Phi\,(\bar{t}; g)(h) + \frac{h^q}{\Gamma(q+1)}(v + p) \in \Omega. \quad \blacksquare$$

2.3. Approximate solution

We recall the Henry–Gronwall inequality (see Lemma 7.1.1 by Henry (1981)), which can be used in fractional differential equations and integral equations with a singular kernel.

Lemma 2.3.1. Let $u: [0, b] \to [0, \infty)$ be a real function and v be a nonnegative, locally integrable function on $[0, b]$. Suppose there are constants $a > 0$ and $0 < \alpha < 1$ such that

$$u(t) \le v(t) + a \int_0^t (t - s)^{\alpha-1} u(s) ds.$$

Then, there exists a constant $K = K(\alpha)$ such that

$$u(t) \le v(t) + Ka \int_0^t (t - s)^{\alpha-1} v(s) ds.$$

From now on, we fix the closed set $\Omega \subset \mathbb{R}^n$ and $\psi \in K_\Omega$.

Definition 2.3.2. Let $\varepsilon \in (0,1), 0 \le \theta \le T$. We say that a quarter (σ, f, g, y) is an ε-solution to (1) on the interval $[-r, \theta]$ if the non-decreasing function $\sigma: [0, \theta] \to [0, \theta]$, the measurable function $f: [0, \theta] \to \mathbb{R}^n$, the integrable function $g: [0, \theta] \to \mathbb{R}^n$ and continuous function $y: [-r, \theta] \to \mathbb{R}^n$ satisfy

(i) $t - \varepsilon \le \sigma(t) \le t$ for every $t \in [0, \theta]$;

(ii) $\|g(t)\| \le \varepsilon$ for every $t \in [0, \theta]$;

(iii) $y(\sigma(t)) \in \Omega$ for every $t \in [0, \theta\,]$ and $y(\theta) \in \Omega$;

(iv) $f(t) \in F(\sigma(t), y_{\sigma(t)})$ such that

$$y(t) = y(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(s) + g(s)]ds$$

for every $t \in [0, \theta]$;

(v) $y_0 = \psi$.

Lemma 2.3.3. There exists $N > 0$ such that for every $\varepsilon > 0$, $0 \le \theta \le T$, and every ε-solution (σ, f, g, y) to (1) on the interval $[-r, \theta]$, we have

$$\|y\|_{C([-r,\theta],\mathbb{R}^n)} \le N.$$

Proof.

Noticing,

$$y(t) := \psi(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f(s) + g(s)]ds,$$

where $f(s) \in F(\sigma(s), y_{\sigma(s)})$ and $y(t) = \psi(t), t \in [-r, 0]$. Thanking to (2), we have

$$\|y(t)\| \le |\psi(0)| + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left(\alpha + 1 + \alpha\|y_{\sigma(s)}\|_0\right)ds\,, 0 \le t \le \theta.$$

Put $w(t) = \sup\{\|y(s)\|: -r \le s \le t\}$. Using the above inequality and the definition of w, we have that

$$w(t) \le \|\psi\|_0 + \frac{\alpha + 1}{\Gamma(q+1)} + \frac{\alpha}{\Gamma(q)}\int_0^t (t-s)^{q-1}w(s)ds$$

Lemma 3 implies

$$w(t) \le \|\psi\|_0 + \frac{\alpha + 1}{\Gamma(q+1)} + \frac{K\alpha}{\Gamma(q)}\left(\|\psi\|_0 + \frac{\alpha + 1}{\Gamma(q+1)}\right)\int_0^t (t-s)^{q-1}ds$$

$$\le \|\psi\|_0 + \frac{\alpha + 1}{\Gamma(q+1)} + \frac{K\alpha T^q}{\Gamma(q+1)}\left(\|\psi\|_0 + \frac{\alpha + 1}{\Gamma(q+1)}\right) := N, \forall t \in [0, \theta].$$

Thus

$$\|y\|_{C([-r,\theta],\mathbb{R}^n)} \le N. \qquad \square$$

Proposition 2.3.4. Let $\varepsilon \in (0,1), 0 \le \theta \le T$ and let (σ, f, g, y) be ε −solution on $[-r, \theta]$ of (1). If the pair $\left(D_c^q y; F(\theta, y_\theta)\right)$ is tangent to $I \times K_\Omega$ at $(\theta, y_\theta) \in I \times K_\Omega$ then there exist $\delta > 0$ and an extension (σ_1, f_1, g_1, z) of (σ, f, g, y) which is ε −solution of (1) on $[-r; \theta + \delta]$.

Proof.

Proposition 6 implies that there exist $h \in (0, \varepsilon)$ and $v \in F(\theta, y_\theta), p \in \mathbb{R}^n$, $|p| < \varepsilon$ such that

$$y(\theta) + \Phi\left(\theta; D_c^q y\right)(h) + \frac{h^q}{\Gamma(q+1)}(v+p) \in \Omega. \quad (4)$$

We claim that there exists $\delta = h > 0$ such that z is defined on $[-r; \theta + \delta]$ by

$$z(t) = y(\theta) + \Phi\left(\theta; D_c^q y\right)(t-\theta) + \frac{(t-\theta)^q}{\Gamma(q+1)}(v+p), t \in [\theta; \theta+\delta] \quad (5)$$

and $z(t) = y(t), t \in [-r, \theta]$ is $\varepsilon-$solution. It is clear that Property (v) is satisfied. Denoting $\sigma_1(t) = \theta,\ g_1(t) = p,\ f_1(t) = v$ on $[\theta; \theta + \delta]$. We see propositions (i), and (ii) are satisfied.

Since y be $\varepsilon-$solution on $[-r, \theta]$ of $\left(P_\psi\right)$, we have $z\left(\sigma(t)\right) = y(\theta) \in \Omega$.

From (4) we see $z(\theta + \delta) = y(\theta) + \Phi\left(\theta; D_c^q y\right)(h) + \frac{h^q}{\Gamma(q+1)}(v+p) \in \Omega$.

Hence Property (iii) is satisfied.

Since y is $\varepsilon-$solution on $[-r, \theta]$ of (1), there exist non-decreasing function $\sigma: [0, \theta] \to [0, \theta]$, the integrable function $g: [0, \theta] \to \mathbb{R}^n$, and measurable selection $f_1(t) \in F\left(\sigma(t), y_{\sigma(t)}\right)$ such that

$$y(t) = y(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f_1(s) + g(s)]ds$$

for every $t \in [0, \theta]$. So $D_c^q y(t) = f_1(t) + g(t)$ for every $t \in [0, \theta]$.

From (5) we have

$$z(t) = y(\theta) + \frac{1}{\Gamma(q)}\int_0^\theta [(t-s)^{q-1} - (\theta-s)^{q-1}]D_c^q y(s)ds$$

$$+\frac{1}{\Gamma(q)}\int_\theta^t (t-s)^{q-1}[f_1(s) + g(s)]ds.$$

Furthermore,

$$y(\theta) = y(0) + \frac{1}{\Gamma(q)}\int_0^\theta (\theta-s)^{q-1}D_c^q y(s)ds.$$

So

$$z(t) = y(0) + \frac{1}{\Gamma(q)}\int_0^\theta (t-s)^{q-1}D_c^q y(s)ds + \frac{1}{\Gamma(q)}\int_\theta^t (t-s)^{q-1}[f_1(s) + g(s)]ds$$

$$= y(0) + \frac{1}{\Gamma(q)}\int_0^\theta (t-s)^{q-1}[f_1(s) + g(s)]ds + \frac{1}{\Gamma(q)}\int_\theta^t (t-s)^{q-1}[f_1(s) + g(s)]ds$$

$$= z(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[f_1(s) + g(s)]ds. \quad \blacksquare$$

We set

$\mathcal{M} = \{(\theta, \phi) \in I \times K_\Omega : \forall\, \varepsilon > 0, \exists\, \varepsilon - \text{solution } (\sigma, f, g, y) \text{ on } [0, \theta],\ y_\theta = \phi\}.$

Suppose $(\psi, F(0, \psi))$ is tangent to $I \times K_\Omega$ at $(0, \psi) \in I \times K_\Omega$. Then, according to Proposition 6, we see $\mathcal{M} \neq \emptyset$.

Definition 2.3.5. Inclusion (1) would satisfy tangency condition at $(\theta, \phi) \in \mathcal{M}$ if for each $\varepsilon > 0$, the pair $\left(D_c^q y; F(\theta, y_\theta)\right)$ is tangent to $I \times K_\Omega$ at $(\theta, y_\theta) \in I \times K_\Omega$, with (σ, f, g, y) is ε −solution on $[0, \theta]$.

Lemma 2.3.6. (Brezis-Browder Theorem) Let $\mathcal{S}$ be a nonempty set, $\preccurlyeq$ a preorder on $\mathcal{S}$, and let $\mathcal{M}: \mathcal{S} \to \mathbb{R} \cup \{+\infty\}$ be a function. Suppose that:

(i) for any increasing sequence $(\xi_k)_k \subset \mathcal{S}$, there exists some $\eta \in \mathcal{S}$ such that $\xi_k \preccurlyeq \eta$, for all $k \in \mathbb{N}$;

(ii) the function $\mathcal{M}$ is increasing.

Then for each $\xi \in \mathcal{S}$ there exists an $\mathcal{M}$ − maximal element $\bar{\xi} \in \mathcal{S}$ satisfying $\xi \preccurlyeq \bar{\xi}$.

Lemma 2.3.7. If $(\psi, F(0, \psi))$ is tangent to $I \times K_\Omega$ at $(0, \psi) \in I \times K_\Omega$ and (1) is tangent at every $(\theta, \phi) \in \mathcal{M}$, for each $\varepsilon \in (0,1)$, there exists ε −solution (σ, f, g, y) determined on the entire $[-r, T]$.

Proof.

Fix $\varepsilon \in (0,1)$. Let $\mathcal{S}$ be the set of all ε −approximate solutions to the initial value problem (1) defined on the interval $[0, c]$ with $c \in [0, T]$. On $\mathcal{S}$ we define the relation " $\preccurlyeq$ " by $(\sigma_1, f_1, g_1, x_1) \preccurlyeq (\sigma_2, f_2, g_2, x_2)$ if $[0, c_1] \subseteq [0, c_2]$ and the two ε −approximate solutions coincide on the common part of the domains.

Let $\left((\sigma_m, f_m, g_m, x_m)\right)_m$ be an increasing sequence defined on $[0, c_m]$, and let $c^* = \lim\limits_{m\to\infty} c_m$. Clearly, $c^* \in [0, T]$. Let us now prove the existence of $\lim\limits_{m\to\infty} x_m(c_m)$. Note that for each $m, k \in \mathbb{N}, m \leq k$, we have $\sigma_m(s) = \sigma_k(s)$, $g_m(s) = g_k(s)$ and $x_m(s) = x_k(s)$, $f_m(s) = f_k(s)$ for all $s \in [0, c_m]$. Moreover, $c_m - \varepsilon \leq \sigma_m(c_m) \leq c_m$.

Noticing, for every $m \in \mathbb{N}$,

$$x_m(t) := \psi(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f_m(s) + g(s)] ds,$$

where $f_m(s) \in F\left(\sigma_m(s), (x_m)_{\sigma_m(s)}\right)$ and $x_m(t) = \psi(t), t \in [-r, 0]$.

By virtue of Lemma 9, we have

$$\|x_m\|_{C([-r, c_m], \mathbb{R}^n)} \leq N. \quad (6)$$

This implies

$$\left\|(x_m)_{\sigma_m(s)}\right\|_0 \leq N, \forall s \in [0, c_m].$$

Furthermore $\|F(t, v)\| \leq \alpha(1 + \|v\|_0), \forall v \in C_r$ so

$$\left\|F\left(\sigma_m(s), (x_m)_{\sigma_m(s)}\right)\right\|_* \leq \alpha(1 + N) := M, \forall s \in [0, c_m].$$

We have

$$\|x_k(c_k) - x_m(c_m)\| = \frac{1}{\Gamma(q)}\left\|\int_0^{c_k}(c_k - s)^{q-1}\cdot[f_k(s) + g(s)]ds\right.$$

$$\left.-\int_0^{c_m}(c_m - s)^{q-1}\cdot[f_m(s) + g(s)]ds\right\|$$

$$= \frac{1}{\Gamma(q)}\left\|\int_0^{c_m}[(c_k - s)^{q-1} - (c_m - s)^{q-1}][f_m(s) + g(s)]ds\right.$$

$$\left.+\int_{c_m}^{c_k}(c_k - s)^{q-1}\cdot[f_k(s) + g(s)]ds\right\|$$

$$\le \frac{M+\varepsilon}{\Gamma(q)}\cdot\left|\int_0^{c_m}[(c_k - s)^{q-1} - (c_m - s)^{q-1}]ds + \int_{c_m}^{c_k}(c_k - s)^{q-1}ds\right|$$

$$= \frac{M+\varepsilon}{\Gamma(q)}\cdot\left|\int_0^{c_k}(c_k - s)^{q-1}ds - \int_0^{c_m}(c_m - s)^{q-1}ds\right|$$

$$= \frac{M+\varepsilon}{\Gamma(q+1)}\cdot\left|c_k^q - c_m^q\right|$$

for every $k, m \in \mathbb{N}$. We know that $\lim_{m\to\infty} c_m = c^*$, so $\lim_{m\to\infty} c_m^q = (c^*)^q$. Then for $\left|c_k^q - c_m^q\right| \le \frac{\Gamma(q+1)}{M+\varepsilon}\cdot\varepsilon_1$ we get $\|x_k(c_k) - x_m(c_m)\| \le \varepsilon_1$, what proves the existence of $\lim_{m\to\infty} x_m(c_m)$. Note that $x_m(c_m) \in \Omega$. The set Ω is closed, so $\lim_{m\to\infty} x_m(c_m) \in \Omega$.

All the functions in the set $\{\sigma_m : m \in \mathbb{N}\}$ are non-decreasing with values in $[0, c^*]$ and satisfy $\sigma_m(c_m) \le \sigma_k(c_k)$ for $m, k \in \mathbb{N}, m \le k$. Hence there exists $\lim_{m\to\infty} \sigma_m(c_m)$ and this limit belongs to $[0, c^*]$. Therefore the quartet of function $(\sigma^*, f^*, g^*, x^*): [0, c^*] \to [0, c^*] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ can be defined by

$$\sigma^*(t) = \begin{cases}\sigma_m(t) \text{ for } t \in [0, c_m], m \in \mathbb{N},\\ \lim_{m\to\infty} \sigma_m(c_m) \text{ for } t = c^*,\end{cases} \tag{7}$$

$$g^*(t) = \begin{cases}g_m(t) \text{ for } t \in [0, c_m], m \in \mathbb{N},\\ 0 \text{ for } t = c^*,\end{cases} \tag{8}$$

$$x^*(t) = \begin{cases}x_m(t) \text{ for } t \in [0, c_m], m \in \mathbb{N},\\ \lim_{m\to\infty} x_m(c_m) \text{ for } t = c^*.\end{cases} \tag{9}$$

$$f^*(t) = \begin{cases}f_m(t) \text{ for } t \in [0, c_m], m \in \mathbb{N},\\ \eta^* \text{ for } t = c^*.\end{cases} \tag{10}$$

where η^* is an arbitrary but fixed element in $F\big(c^*, x^*(\sigma^*(c^*))\big)$.

One can see that $(\sigma^*, f^*, g^*, x^*)$ is an ε-approximate solution for all $m \in \mathbb{N}$

$(\sigma_m, f_m, g_m, x_m) \preccurlyeq (\sigma^*, f^*, g^*, x^*)$.

Let us define the function $\mathcal{M}: \mathcal{S} \to \mathbb{R} \cup \{+\infty\}$ by $\mathcal{M}\big((\sigma, f, g, x)\big) = c$, where (σ, f, g, x) is defined on $[0, c]$. Then by the Brezis-Browder Theorem, $\mathcal{S}$ consists of at least one $\mathcal{M}$-maximal element $(\bar{\sigma}, \bar{f}, \bar{g}, \bar{x})$ defined on $[0, c]$, i.e. for every $(\tilde{\sigma}, \tilde{f}, \tilde{g}, \tilde{x}) \in \mathcal{S}$ such that $(\bar{\sigma}, \bar{f}, \bar{g}, \bar{x}) \preccurlyeq (\tilde{\sigma}, \tilde{f}, \tilde{g}, \tilde{x})$ we have $\mathcal{M}\left((\bar{\sigma}, \bar{f}, \bar{g}, \bar{x})\right) = \mathcal{M}\left((\tilde{\sigma}, \tilde{f}, \tilde{g}, \tilde{x})\right)$, which means that $\bar{c} = \tilde{c}$. By Proposition 10, we get $\bar{c} = T$. Therefore the existence of an ε −approximate solution defined on the whole interval $[0, T]$ was proved. ■

2.4. Proof of the main result

Before presenting proof of the main theorem, we will introduce some supporting results.

Definition 2.4.1. A subset $G \subset L^1(I, \mathbb{R}^n)$ is called uniformly integrable if for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_E \|f(t)\|\, dt < \varepsilon$$

for each measurable subset $E \subset I$, whose Lebesgue measure is more minor than δ, and uniformly for $f \in G$.

Theorem 2.4.2. (Theorem 1.3.7 by Dunford-Pettis theorem) $G \subset L^1([0, T], \mathbb{R}^n)$ is weakly compact if and only if it is uniformly integrable.

Now we give the proof of the main result.

Theorem 2.4.3. If (2) is satisfied, $\big(\psi, F(0, \psi)\big)$ is tangent to $I \times K_\Omega$ at $(0, \psi) \in I \times K_\Omega$ and (1) satisfies tangency condition at every $(\theta, \phi) \in I \times K_\Omega$, K_Ω is viable.

Proof.

Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a decreasing sequence such that $\varepsilon_k \in (0, 1)$ and $\lim_{k\to\infty} \varepsilon_k = 0$. Let $\big((\sigma_k, f_k, g_k, x_k)\big)_{k\in\mathbb{N}}$ be a sequence of ε_k-approximate solutions defined on the interval $[0, T]$. From (i) and (ii) of definition 8, we get the following uniform convergence on $[0, T]$:

$$\lim_{k\to\infty} \sigma_k(t) = t, \tag{11}$$

$$\lim_{k\to\infty} g_k(t) = 0. \tag{12}$$

As a result of Lemma 13, the sequence $(x_k)_{k\in\mathbb{N}}$ is uniformly bounded on $[0, T]$. Moreover, for $0 \le t_1 \le t_2$, with $f_k(s) \in F\left(\sigma_k(s), x_k\big(\sigma_k(s)\big)\right)$, we get

$$\|x_k(t_1) - x_k(t_2)\| = \|x_k(t_1) - \psi(0) - x_k(t_2) + \psi(0)\|$$
$$= \frac{1}{\Gamma(q)}\left\|\int_0^{t_1}(t_1 - s)^{q-1}\cdot[f_k(s) + g(s)]ds - \int_0^{t_2}(t_2 - s)^{q-1}\cdot[f_k(s) + g_k(s)]ds\right\|$$
$$= \frac{1}{\Gamma(q)}\left\|\int_0^{t_1}[(t_1 - s)^{q-1} - (t_2 - s)^{q-1}]\cdot[f_k(s) + g_k(s)]ds\right.$$
$$\left. - \int_{t_1}^{t_2}(t_2 - s)^{q-1}\cdot[f_k(s) + g(s)]ds\right\|$$
$$\leq \frac{M + \varepsilon_k}{\Gamma(q)}\cdot\left|\int_0^{t_1}[(t_1 - s)^{q-1} - (t_2 - s)^{q-1}]ds + \int_{t_1}^{t_2}(t_2 - s)^{q-1}ds\right|$$
$$\leq \frac{M+1}{\Gamma(q+1)}\cdot\left|t_1^q - t_2^q + 2(t_2 - t_1)^q\right|$$
$$\leq \frac{M+1}{\Gamma(q+1)}\cdot 3(t_2 - t_1)^q < \varepsilon$$

provided $|t_2 - t_1| \leq \delta = \left[\frac{\varepsilon\Gamma(q+1)}{3(M+1)}\right]^{\frac{1}{q}}$. Hence the sequence $(x_k)_{k\in\mathbb{N}}$ is equicontinuous on $[0, T]$. Since the sequence $(x_k)_{k\in\mathbb{N}}$ is bounded and equicontinuous, it has a uniformly convergent subsequence and keeps the same notations by the Arzelà–Ascoli Theorem. Hence $\left(x_{k_i}\right)_i$ is uniformly convergent on $[0, T]$ to a function $x: [0, T] \to \mathbb{R}^n$. Taking into account the fact that Ω is closed, $x_k\left(\sigma_k(t)\right) \in \Omega, x_k(T) \in \Omega$. Since $(x_k)_{k\in\mathbb{N}}$ is equicontinuous and condition (11) with we deduce that $\lim_{k\to\infty} x_k\left(\sigma_k(t)\right) = x(t)$. This implies that $x(t) \in \Omega$ for every $t \in I$.

By cause of $(x_k)_{k\in\mathbb{N}}$ is uniformly bounded on I and (2), we get $\{f_k\}_k$ is uniformly integrable in $L^1(I, \mathbb{R}^n)$. As long as Theorem 15, we take a subsequence of $\{f_k\}_k$ and keeping the same notations, we may assume that it converges weakly in $L^1(I, \mathbb{R}^n)$ to some $f \in L^1(I, \mathbb{R}^n)$. By the Mazur lemma, there exist $\lambda_i^n \geq 0, i = n, \dots, k(n)$, such that $\sum_{i=n}^{k(n)}\lambda_i^n = 1$, and the sequence $h_n \coloneqq \sum_{i=n}^{k(n)}\lambda_i^n f_i$ converges to f in $L^1(I, \mathbb{R}^n)$. By a classical result due to Lebesgue, we know that there exists a subsequence $\left(h_{n_j}\right)_j$ converges to f almost everywhere. Hence for every $t \in I$,

$$\lim_{j\to\infty}\int_0^t(t - s)^{q-1}h_{n_j}(s)ds = \int_0^t(t - s)^{q-1}f(s)ds.$$

Since $(x_k)_k$ converges uniformly to x, for every $t \in I$ we get

$$x(t) = \lim_{j\to\infty} \sum_{i=n_j}^{k(n_j)} \lambda_i^{n_j} x_i(t)$$

$$= \lim_{j\to\infty}\left(\psi(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[\sum_{i=n_j}^{k(n_j)} \lambda_i^{n_j} f_i(s) + \sum_{i=n_j}^{k(n_j)} \lambda_i^{n_j} g_i(s)\right] ds\right)$$

$$= \psi(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s)ds.$$

Put $x(t) = \psi(t), \forall t \in [-r, 0]$.

To end the proof, it is enough to show that $\mathbf{f(s)} \in \mathbf{F(s, x_s)}$ almost everywhere in $s \in [\mathbf{0}, \mathbf{T}]$**.**

Let **E** be an open half-space in $\mathbb{R}^n$ including $\mathbf{F(s, x_s)}$**.** Since $(x_k)_k$ is uniformly convergent on $[0, T]$ to x and $\lim_{k\to\infty} \sigma_k(s) = s$, we have $(x_k)_{\sigma_k(s)}$ converges to x_s in C_r. Since **F** is u.s.c. at $(\mathbf{s}, \mathbf{x_s})$**,** there exists k(E) belonging to $\mathbb{N}$, such that $F\big(\sigma_k(s), (x_k)_{\sigma_k(s)}\big) \subset E$ for each $k \geq k(E)$. From the relation above, taking into account that $f_k(s) \in F\big(\sigma_k(s), (x_k)_{\sigma_k(s)}\big)$, for each $k \in \mathbb{N}$ and a.e. for $s \in [0; T]$, we can conclude that

$$h_{n_j}(s) \in \overline{co}\big(\cup_{k\geq k(E)} F\big(\sigma_k(s), (x_k)_{\sigma_k(s)}\big)\big)$$

for each $j \in \mathbb{N}$ with $n_j \geq k(E)$. Passing to the limit for $j \to \infty$ in the relation above, we deduce that $f(s) \in \overline{E}$. Since $\mathbf{F(s, x_s)}$ is closed and convex, it is the intersection of all closed half-spaces which include it. So, in as much as E was arbitrary, we finally get $f(s) \in \mathbf{F(s, x_s)}$ almost everywhere in $s \in [\mathbf{0}, \mathbf{T}]$**.** ■

3. Conclusion

In this paper, we inherit the existing schemas to consider the viability of delay fractional differential inclusions. The new results presented in this paper include:

- Give a suitable tangency condition for this problem.
- Propose the concept of $\epsilon -$ solution and apply it to prove the existence of a viable solution.

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TÍNH CHẤT VIABLE CHO BAO HÀM THỨC VI PHÂN BẬC KHÔNG NGUYÊN CÓ ĐỐI SỐ LỆCH

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TÓM TẮT

Lí thuyết viability ra đời và phát triển, được ứng dụng trong nhiều lĩnh vực như sinh vật sống, tiến hóa sinh học, kinh tế học, khoa học môi trường, thị trường tài chính hoặc lí thuyết điều khiển và robotics. Trong bài báo này, chúng tôi đưa ra một điều kiện đủ cho sự tồn tại nghiệm vible của bao hàm thức vi phân bậc không nguyên có đối số lệch dạng

$D_C^q x(t) \in F(t, x_t), 0 < q < 1, t \in I \coloneqq [0, T].$

Kế thừa những ý tưởng của (Carja, Donchev, Rafaqat, & Ahmed, 2014), (Girejko, 2018), chúng tôi đưa ra điều kiện tiếp xúc và khái niệm nghiệm xấp xỉ phù hợp với cấu trúc bài toán. Theo định lí Brezis-Browder, chúng tôi thu được nghiệm xấp xỉ trên toàn bộ đoạn [0,T]. Bằng cách cho qua giới hạn, dãy nghiệm xấp xỉ hội tụ về nghiệm viable. Kết quả này tổng quát các kết quả đã có trong các bài báo (Carja et al., 2014), (Girejko, Mozyrska, & Wyrwas, 2011), (Vasundgaradevi & Lakshmikantham, 2009).

Từ khóa: nghiệm viable; đối số lệch; bao hàm thức; đạo hàm bậc không nguyên