

Research Article

THE ARTINIANESS AND (I, J) -STABLE OF LOCAL HOMOLOGY MODULE WITH RESPECT TO A PAIR OF IDEALS

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ABSTRACT

The concept of I -stable modules was defined by Tran Tuan Nam in (Tran, 2013) and the author used it to study about the representable of local homology modules. In this paper, we will introduce the concept of (I, J) -stable modules, which is an extension of the I -stable modules. We study the (I, J) -stable for local homology modules with respect to a pair of ideals, these modules have been studied in (Tran, & Do, 2020). We show some basic properties of (I, J) -stable modules and use it to study the artinianess of local homology modules with respect to a pair of ideals. Moreover, we also give the relationship between the artinianess, (I, J) -stable and the varnishing of local homology module with respect to a pair of ideals.

Keywords: artinian module; I -stable module; local homology

1. Introduction

Throughout this paper, (R, \mathfrak{m}) is a local noetherian ring with the maximal ideal \mathfrak{m} . Let I, J be ideals of R . In (Tran, & Do, 2020) we defined the local homology module $H_i^{I, J}(M)$ with respect to a pair of ideals (I, J) by

$$H_i^{I, J}(M) = \varprojlim_{\mathfrak{a} \in \tilde{W}(I, J)} \text{Tor}_i^R(R/\mathfrak{a}, M)$$

in which $\tilde{W}(I, J)$ the set of ideals \mathfrak{a} of R such that $I^n \subseteq \mathfrak{a} + J$ for some integer n . This definition is dual to the generalized local cohomology in (Takahashi, Yoshino, & Yoshizawa, 2009) and extension from the local homology module in (Nguyen, & Tran, 2001). We also studied some properties of these modules in (Tran, & Do, 2020), especially,

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we gave the relationship between these modules and local homology modules with respect to an ideal through the isomorphism $H_i^{I,J}(M) \cong \varinjlim_{\mathfrak{a} \in \tilde{W}(I,J)} H_i^{\mathfrak{a}}(M)$. In (Tran, 2013), T. T. Nam introduced the definition of I -stable modules and the author used it to study about the representable of local homology modules.

In this paper, we will introduce the concept of (I, J) -stable module, which is an extension from the concept I -stable in (Tran, 2013). Also, we show some properties of artinian and (I, J) -stable of local homology modules $H_i^{I,J}(M)$. The first main result is Proposition 2.2, there is a $\mathfrak{b} \in \tilde{W}(I, J)$ such that $\mathfrak{b} \subseteq \bigcap_{\mathfrak{p} \in \text{Coass}(M)} \mathfrak{p}$ where M is (I, J) -separated artinian R -module. Next, The Theorem 2.7 gives us the equivalent properties on artinianess of local homology module. And the last result gives the relationship between the artinianess, (I, J) -stable and the vanishing of local homology module $H_i^{I,J}(M)$.

2. Some properties

Lemma 2.1. *Let M be an artinian R -module. Then $H_0^{I,J}(M) = 0$ if and only if there is $x \in \mathfrak{b}$ such that $xM = M$ for some $\mathfrak{b} \in \tilde{W}(I, J)$.*

Proof. By (Tran, & Do, 2020), $H_0^{I,J}(M) \cong \Lambda_{I,J}(M)$ and by M is artinian so there is $\mathfrak{b} \in \tilde{W}(I, J)$ such that $\Lambda_{I,J}(M) \cong M / \mathfrak{b}M$. Therefore, $H_0^{I,J}(M) = 0$ if and only if $\mathfrak{b}M = M$ and by (Macdonald, 1973) if and only if $xM = M$ for $x \in \mathfrak{b}$.

We recall the concept of (I, J) -separated. Module M is called (I, J) -separated if

$$\bigcap_{\mathfrak{a} \in \tilde{W}(I, J)} \mathfrak{a}M = 0.$$

Proposition 2.2. *If M is (I, J) -separated artinian R -module. Then there is a $\mathfrak{b} \in \tilde{W}(I, J)$ such that $\mathfrak{b} \subseteq \bigcap_{\mathfrak{p} \in \text{Coass}(M)} \mathfrak{p}$.*

Proof. M is (I, J) -separated, by (Tran, & Do, 2020) $M \cong \Lambda_{I,J}(M) \cong M / \mathfrak{b}M$ for some $\mathfrak{b} \in \tilde{W}(I, J)$ hence $\mathfrak{b}M = 0$. It implies that $\bigcap \mathfrak{b}^t M = 0$, so M is \mathfrak{b} -separated. It follows from (Tran, 2013) that $\mathfrak{b} \subseteq \bigcap_{\mathfrak{p} \in \text{Coass}(M)} \mathfrak{p}$.

Corollary 2.3. *Let M is an artinian R -module. If $H_i^{I,J}(M) \neq 0$, then there is a $\mathfrak{b} \in \tilde{W}(I, J)$ such that $\mathfrak{b} \subseteq \bigcap_{\mathfrak{p} \in \text{Coass}(H_i^{I,J}(M))} \mathfrak{p}$.*

Proof. By (Tran, & Do, 2020), $H_i^{I,J}(M)$ is (I, J) -separated and hence by Proposition 2.2, we have the conclusion.

Corollary 2.4. *Let M is an artinian R -module. If $H_i^{I,J}(M)$ is an artinian R -module, then there is an ideal $\mathfrak{b} \in \tilde{W}(I, J)$ such that $\mathfrak{b} \subseteq \sqrt{\text{Ann}_R H_i^{I,J}(M)}$.*

Proof. By (Brodmann, 1998), $\bigcap_{\mathfrak{p} \in \text{Att}(H_i^{I,J}(M))} \mathfrak{p} = \sqrt{\text{Ann}_R H_i^{I,J}(M)}$. In the other hands,

$H_i^{I,J}(M)$ is a representable, so $\text{Att}(H_i^{I,J}(M)) = \text{Coass}(H_i^{I,J}(M))$, by (Yassemi, 1995). Now the conclusion follows from Corollary 2.3.

The concept of I -stable modules was defined in (Tran, 2009). An R -module N is called I -stable if for each element $x \in I$, there is a positive integer n such that $x^t N = x^n N$ for all $t \geq n$. Now we will give an extension concept of the I -stable.

Định nghĩa 2.5. M is called (I, J) -stable if there is an ideal $\mathfrak{b} \in \tilde{W}(I, J)$ such that $\bigcap_{\mathfrak{a} \in \tilde{W}(I, J)} \mathfrak{a}M = \mathfrak{b}M$.

When $J = 0$, we have $\mathfrak{b}M = \bigcap_{\mathfrak{a} \in \tilde{W}(I, J)} \mathfrak{a}M \subseteq \bigcap_{t > 0} I^t M$. Since $\mathfrak{b} \in \tilde{W}(I, J)$ and $J = 0$,

there is n such that $I^n \subseteq \mathfrak{b}$. So $\bigcap_{t > 0} I^t M = I^n M$, then $I^t M = I^n M$ for all $t > n$.

Hence, when $J = 0$ then M is I -stable.

Bổ đề 2.6. *Let $0 \rightarrow M \rightarrow N \xrightarrow{g} P \rightarrow 0$ be a short exact sequence in which the modules M, N, P is (I, J) -separated. Then modules N is (I, J) -stable if and only if modules M, P are (I, J) -stable.*

Proof. Assume that N is (I, J) -stable. Then there is ideal $\mathfrak{b} \in \tilde{W}(I, J)$ such that $\mathfrak{b}N = \bigcap \mathfrak{a}N = 0$, N is (I, J) -separated, $\mathfrak{b}M \subseteq \mathfrak{b}N = 0$, so M is (I, J) -stable. We have $\mathfrak{b}P \cong (\mathfrak{b}N + \text{Ker } g) / \text{Ker } g = 0 = \bigcap \mathfrak{b}P$, so P is (I, J) -stable. Otherwise, suppose that M and P are (I, J) -stable, then there are ideals $\mathfrak{a}, \mathfrak{b}$ such that $\mathfrak{b}P = \mathfrak{a}M = 0$. Let $d = \mathfrak{a} \cap \mathfrak{b}$, then $dM = dP = 0$, by (Brodmann, 1998), $d^n N = 0$, so N is (I, J) -stable.

Mệnh đề 2.7. *Let M be an artinian R -module and t a positive integer. Then the following statements are equivalent*

- i) $H_i^{I,J}(M)$ is an artinian for all $i < t$;

ii) There is an ideal $\mathfrak{b} \in \tilde{W}(I, J)$ such that $\mathfrak{b} \subseteq \text{Rad}(\text{Ann}(H_i^{I,J}(M)))$ for all $i < t$.

Proof. (i \Rightarrow ii) $H_i^{I,J}(M)$ is artinian, hence by (Tran, & Do, 2020), there is $\mathfrak{b} \in \tilde{W}(I, J)$ such that

$$\mathfrak{b}H_i^{I,J}(M) = \bigcap_{\mathfrak{a} \in \tilde{W}(I, J)} \mathfrak{a}H_i^{I,J}(M) = 0. \text{ Therefore, } \mathfrak{b} \subseteq \text{Rad}(\text{Ann}(H_i^{I,J}(M))) \text{ for all}$$

$i < t$.

(ii \Rightarrow i) We use induction on t . When $t = 1$, $H_0^{I,J}(M) \cong \Lambda_{I,J}(M) \cong M / \mathfrak{b}M$, so $H_0^{I,J}(M)$ is artinian. Let $t > 1$, by (Tran, & Do, 2020) we can replace M by $\bigcap_{\mathfrak{a} \in \tilde{W}(I, J)} \mathfrak{a}M$. As M is artinian, there is an $\mathfrak{b} \in \tilde{W}(I, J)$ such that $\mathfrak{b}M = \bigcap_{\mathfrak{a} \in \tilde{W}(I, J)} \mathfrak{a}M$.

Therefore, we can assume that $M = \mathfrak{b}M$, by (Macdonald, 1973), there is an element $x \in \mathfrak{b}$ such that $M = xM$. By the hypothesis, there is a positive integer s such that $x^s H_i^{I,J}(M) = 0$ for all $i < t$. Then the short exact sequence

$$0 \longrightarrow (0 :_M x^s) \longrightarrow M \xrightarrow{x^s} M \longrightarrow 0$$

gives rise to the exact sequence

$$0 \rightarrow H_{i+1}^{I,J}(M) \rightarrow H_i^{I,J}(0 :_M x^s) \rightarrow H_i^{I,J}(M) \rightarrow 0$$

for all $i < t - 1$. It follows from (Brodmann, 1998) that $\mathfrak{b} \subseteq \text{Rad}(\text{Ann}(H_i^{I,J}(0 :_M x^s)))$ and by the inductive hypothesis that $H_i^{I,J}(0 :_M x^s)$ is artinian for all $i < t - 1$. Thus $H_i^{I,J}(M)$ is artinian for all $i < t$.

We now recall the concept of *Noetherian dimension* of an R -module M denoted by $\text{Ndim } M$. Note that the notion of Noetherian dimension was introduced first by R. N. Roberts (Robert, 1975) by the name Krull dimension. Later, D. Kirby (Kirby, 1990) changed this terminology of Roberts and referred to Noetherian dimension to avoid confusion with well-known Krull dimension of finitely generated modules. Let M be an R -module. When $M = 0$ we put $\text{Ndim } M = -1$. Then by induction, for any ordinal α , we put $\text{Ndim } M = \alpha$ when (i) $\text{Ndim } M < \alpha$ is false, and (ii) for every ascending chain $M_0 \subseteq M_1 \subseteq \dots$ of submodules of M , there exists a positive integer m_0 such that $\text{Ndim}(M_{m+1} / M_m) < \alpha$ for all $m \geq m_0$. Thus M is non-zero and finitely generated if and only if $\text{Ndim } M = 0$.

Theorem 2.8. Let M be an artinian R -module and s an integer. Then the following statements are equivalent

- i) $H_i^{I,J}(M)$ is (I, J) -stable for all $i > s$;
- ii) $H_i^{I,J}(M)$ is artinian for all $i > s$;
- iii) $\text{Ass}(H_i^{I,J}(M)) \subseteq \{\mathfrak{m}\}$ for all $i > s$;
- iv) $H_i^{I,J}(M) = 0$ for all $i > s$.

Proof. ($i \Rightarrow ii$) We use induction on $d = \text{Ndim } M$. If $d = 0$, $H_i^{I,J}(M) = 0$ for all $i > 0$, so $H_i^{I,J}(M)$ is artinian. Let $d > 0$, we can replace M by $\bigcap_{\mathfrak{a} \in \tilde{W}(I,J)} \mathfrak{a}M$ and M is

artinian hence we may assume $M = \mathfrak{b}M$ for some $\mathfrak{b} \in \tilde{W}(I, J)$ and $\mathfrak{b}M$ is the minimum in the set $\{\mathfrak{a}M \mid \mathfrak{a} \in \tilde{W}(I, J)\}$. $H_i^{I,J}(M)$ is (I, J) -stable so there is an ideal $\mathfrak{c} \in \tilde{W}(I, J)$ such that $\mathfrak{c}H_i^{I,J}(M) = \bigcap_{\mathfrak{a} \in \tilde{W}(I,J)} \mathfrak{a}H_i^{I,J}(M) = 0$. Let $d = \mathfrak{c} \cap \mathfrak{b}$, then

$dM = \mathfrak{b}M = M$, hence there is $x \in d$ such that $xM = M$, and $xH_i^{I,J}(M) = 0$. We have the short exact sequence $0 \rightarrow (0 :_M x) \rightarrow M \rightarrow M \rightarrow 0$ gives arise the exact sequence

$$0 \rightarrow H_{i+1}^{I,J}(M) \rightarrow H_i^{I,J}(0 :_M x) \rightarrow H_i^{I,J}(M) \rightarrow 0.$$

Because $H_i^{I,J}(M)$ is (I, J) -stable for all $i > s$, so $H_i^{I,J}(0 :_M x)$ is (I, J) -stable for all $i > s - 1$. By the induction hypothesis $H_i^{I,J}(0 :_M x)$ is artinian for all $i > s - 1$. Therefore, $H_i^{I,J}(M)$ is artinian for all $i > s$.

($ii \Rightarrow iii$) By the (Yassemi, 1995), $\text{Supp}(H_i^{I,J}(M)) \supseteq \text{Cosupp}(H_i^{I,J}(M)) \cap \text{Max}(R) = \{\mathfrak{m}\}$. Hence $\text{Ass}(H_i^{I,J}(M)) \subseteq \{\mathfrak{m}\}$.

($iii \Rightarrow iv$) We use induction on $d = \text{Ndim } M$. When $d = 0$, by (Tran, & Do, 2020), $H_i^{I,J}(M) = 0$ for all $i > 0$. Now, let $d > 0$, we may assume that $M = xM$ for $x \in \mathfrak{b}$ and $\mathfrak{b} \in \tilde{W}(I, J)$. From the short exact sequence $0 \rightarrow (0 :_M x) \rightarrow M \rightarrow M \rightarrow 0$ gives arise the exact sequence

$$H_{i+1}^{I,J}(M) \rightarrow H_i^{I,J}(0 :_M x) \rightarrow H_i^{I,J}(M) \rightarrow H_i^{I,J}(M)$$

$\text{Ass}(H_i^{I,J}(0 :_M x)) \subseteq \{\mathfrak{m}\}$ and $\text{Ndim}(0 :_M x) \leq d - 1$. By the induction hypothesis $H_i^{I,J}(0 :_M x) = 0$ for all $i > s$. From that, we have the exact sequence $0 \rightarrow H_i^{I,J}(M) \xrightarrow{\cdot x} H_i^{I,J}(M)$. If $H_i^{I,J}(M) \neq 0$, for all $i > s$, then

$\text{Ass}(H_i^{I,J}(M)) = \{\mathfrak{m}\}$, then there is an element $a \in H_i^{I,J}(M)$ such that $\mathfrak{m} = \text{Ann}(a)$ it implies that $a\mathfrak{m} = 0$, so $xa = 0$, hence $a = 0$, it is a contraction. Therefore, $H_i^{I,J}(M) = 0$ for all $i > s$.

(iv \Rightarrow i) It is clearly.

3. Conclusion

In this paper, we gave the concept of the (I, J) -stable module. We studied about the properties of the (I, J) -stable of local homology module with respect to a pair of ideals (I, J) . Moreover, we showed the relationship between of the artinianess and the (I, J) -stable of local homology module with respect to a pair of ideals.

❖ **Conflict of Interest:** Authors have no conflict of interest to declare.

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TÍNH ARTIN VÀ TÍNH (I, J) -ỔN ĐỊNH CỦA MÔĐUN ĐỒNG ĐIỀU ĐỊA PHƯƠNG TƯƠNG ỨNG VỚI MỘT CẶP IDEAN

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TÓM TẮT

Khái niệm về môđun I -ổn định được đưa ra bởi Tran Tuan Nam trong bài báo (Tran, 2013) và tác giả đã sử dụng nó như một công cụ để nghiên cứu tính biểu diễn được của lớp môđun đồng điều địa phương. Trong bài báo này, chúng tôi sẽ giới thiệu về lớp môđun (I, J) -ổn định, đây được xem như là một khái niệm mở rộng thực sự từ khái niệm I -ổn định. Chúng tôi nghiên cứu tính (I, J) -ổn định cho lớp môđun đồng điều địa phương theo một cặp idean, lớp môđun này đã được chúng tôi nghiên cứu trong (Tran, & Do, 2020). Các tính chất cơ bản về môđun (I, J) -ổn định đã được nghiên cứu và sử dụng nó để nghiên cứu tính artin của lớp môđun đồng điều địa phương theo một cặp idean. Hơn nữa, chúng tôi cũng đưa ra mối liên hệ giữa tính artin, tính (I, J) -ổn định và tính triệt tiêu của lớp môđun đồng điều địa phương theo một cặp idean.

Từ khóa: môđun artin; môđun I -ổn định; đồng điều địa phương