CONSIDER SOME CONCEPTS IN ALGEBRA AND CATEGORY THROUGH THE TERMINAL AND INITIAL IN CATEGORY THEORY

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Abstract: *The article shows that some concepts in algebra and category are the terminal object or initial object of a certain category. In other words, some algebraic concepts can be defined through the terminal object or initial object in category theory.*

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1. INTRODUCTION

The initial and terminal are concepts that are mentioned a lot in each category. These concepts have many special properties and many new concepts that have been built from the initials or terminals.

In this article, I consider some algebraic concepts such as the kernel, cokernel, image, coimage, tensor product,... through the concept of the initial or terminal in category theory.

2. CONTENT

Firstly, we will review category, the initial and terminal in the category theory.

2.1. Initial object and terminal object

Definition 1. A category $\mathcal{C}$ is given by a class $\mathcal{C}_1$ of objects and a class $\mathcal{C}_2$ of arrows which have the following structure:

i) Each arrow has a domain and a codomain which are objects; one writes $f: X \to Y$ or $X \xrightarrow{f} Y$ if X is the domain and Y is the codomain of the arrow f. One also writes $X = dom(f)$, $Y = cod(f)$ and $Hom_{\mathcal{C}}(X, Y)$ is the set of arrows $f: X \to Y$ in $\mathcal{C}$.

ii) Giving two arrows f and g such that $cod(f) = dom(g)$, the composition of f and g written gf is defined and has domain $dom(f)$ and codomain $cod(g)$:

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

iii) The composition is associative that is given $f: X \to Y, g: Y \to Z$ and $h: Z \to W$, $h(gf) = (hg)f$.

iv) For every object X, there is an identity arrow id_X, satisfying $id_X g = g$ for every $g: Y \to X$ and $f id_X = f$ for every $f: X \to Y$.

Definition 2. The dual category of a category $\mathcal{C}$, denoted by $\mathcal{C}^o$, has the same class of objects as $\mathcal{C}$, and with two objects X, Y in $\mathcal{C}^o$ we have $Hom_{\mathcal{C}^o}(X, Y) = Hom_{\mathcal{C}}(Y, X)$. The composition fg in $\mathcal{C}^o$ is defined as the composition gf in $\mathcal{C}$.

Definition 3. Let $\mathcal{C}$ be a category. An object T of $\mathcal{C}$ is called *terminal* if for any other object X there is exactly one morphism $X \to T$ in the category $\mathcal{C}$. An object I of $\mathcal{C}$ is called *initial* if for any other object X there is exactly one morphism $I \to X$ in the category $\mathcal{C}$.

Definition 4. Let $\mathcal{C}$ be a category. A morphism $\alpha: A \to B$ is called a *monomorphism* if $\alpha f = \alpha g$ implies that $f = g$ for all pairs of morphisms f, g with codomain A. A morphism $\alpha: A \to B$ is called an *epimorphism* if $f\alpha = g\alpha$ implies that $f = g$.

An object 0 is called a *null object* for $\mathcal{C}$ if $Hom_{\mathcal{C}}(A, 0)$ has precisely one element for each A in $\mathcal{C}$. In the dual category 0 becomes a *conull object*. We say that 0 is a *zero object* for $\mathcal{C}$ if it is at once a null object and a conull object. In this case, we will call a morphism $A \to B$ a *zero morphism* if it factors through 0.

Example 5. In the category $\underline{Set}$ whose objects are sets and arrows are maps. The set $T = \{a\}$ with one element is the terminal.

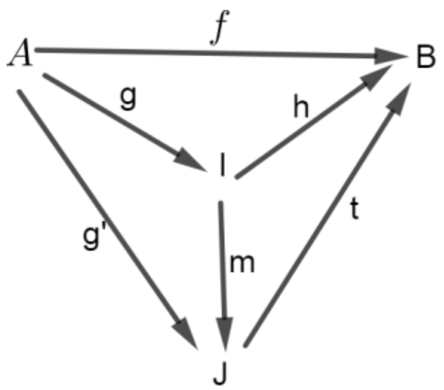
In the category $\underline{Gr}$ whose objects are groups and arrows are homomorphisms. The group $B = \{e\}$ are both the terminal and the initial.

We will now look at some algebraic concepts from the perspective of initial and terminal.

2.2. Image and coimage

Definition 6. The *image* of a morphism $f: A \to B$ is defined as the smallest subobject of B which f factors through. That is, a monomorphism $u: I \to B$ is the image of f if $f = uf'$ for some $f': A \to I$, and if u precedes any other monomorphism into B with the same property. The object I will sometimes be denoted by $Im(f)$.

Remark 7. Let $f: A \to B$ is an arrow in category $\mathcal{C}$. We consider a category $\mathcal{A}$ with: Objects of $\mathcal{A}$ are monomorphisms $h: I \to B$ such that there is exist an arrow $g: A \to I$ satisfy $f = hg$. For two objects $h: I \to B$ and $t: J \to B$ correspond to arrows $g: A \to I, g': A \to J$ satisfying $f = hg = tg'$, an arrow from object h to object t is an arrow $m: I \to J$ in the category $\mathcal{C}$ such that $g' = mg$ and $h = tm$.

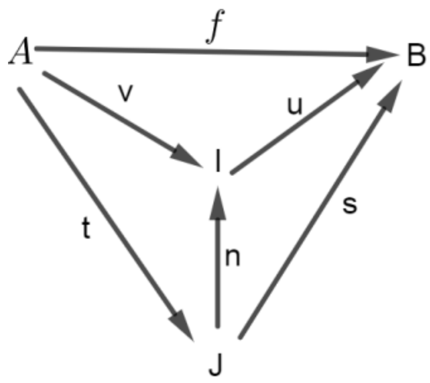


The composition of arrows in $\mathcal{A}$ is the composition of arrows in $\mathcal{C}$.

According to definition 6, we see that the image of f is the initial of category $\mathcal{A}$.

Definition 8. We call an epimorphism $A \to I$ the *coimage* of a morphism $f: A \to B$ if it is the image of f in the dual category. In this case we denote the object I by $Coim(f)$.

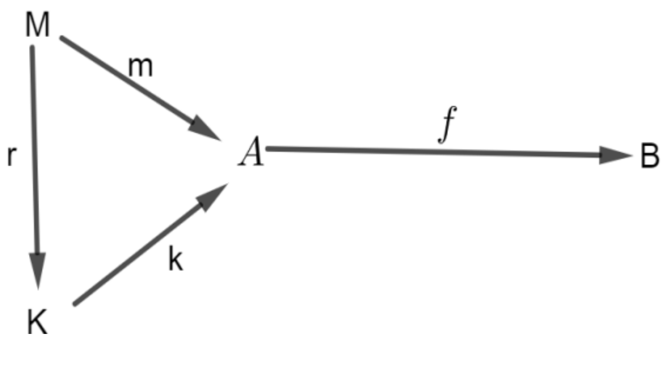
Remark 9. Let $f: A \to B$ be an arrow in category $\mathcal{C}$. We consider a category $\mathcal{A}^o$ with: Objects of $\mathcal{A}^o$ are epimorphisms $v: A \to I$ such that there is exist an arrow $u: I \to B$ satisfy $f = uv$. For two objects $v: A \to I$ and $t: A \to J$ correspond to arrows $u: I \to B, s: J \to B$ satisfying $f = uv = st$, an arrow from object v to object t is an arrow $n: J \to I$ in the category $\mathcal{C}$ such that $v = nt$ and $s = un$.



The composition of arrows in $\mathcal{A}^o$ is the composition of arrows in $\mathcal{C}$.

According to definition 8, we see that the coimage of f is the terminal of category $\mathcal{A}^o$. The category $\mathcal{A}^o$ is the dual category of $\mathcal{A}$.

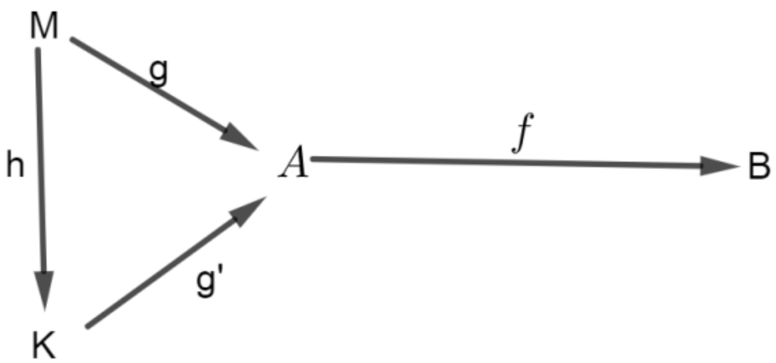
2.3. Kernel and cokernel



Let $\mathcal{C}$ be a category whose a zero object.

Definition 10. Let $f: A \to B$ be an arrow in the category $\mathcal{C}$. We will call a morphism $k: K \to A$ the *kernel* of f if $fk = 0$, and if for every morphism $m: M \to A$ such that $fm = 0$ we have a unique morphism $r: M \to K$ such that $m = kr$.

Remark 11. Let category $\mathcal{C}$ whose zero object, $f: A \to B$ is an arrow in $\mathcal{C}$. Let $\mathcal{C}_f$ be a category that defined as the following: Objects of $\mathcal{C}_f$ are arrows $g: M \to A$ such that $fg = 0_{M \to B}$. An arrow from object $g: M \to A$ and object $g': K \to A$ in $\mathcal{C}_f$ is an arrow $h: M \to K$ in the category $\mathcal{C}$ such that $g = g'h$.



The composition of arrows in $\mathcal{C}_f$ is the composition of arrows in $\mathcal{C}$.

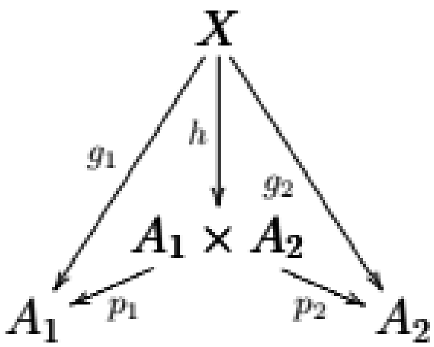
According to definition 10, if $\mathcal{C}_f$ has a terminal it is the kernel of f. The kernel of f will be denoted by $Kerf$ or (K, k) or simplify K.

Definition 12. A morphism $B \to Coker(f)$ is called the *cokernel* of f if it is the kernel of f in the dual category.

Remark 13. It is easily seen that cokernel of $f: A \to B$ is initial in the dual category $\mathcal{C}_f^o$ of $\mathcal{C}_f$.

2.4. Products and coproducts

Definition 14. Let A_1, A_2 be two objects in an arbitrary category $\mathcal{C}$. A product for A_1, A_2 is an object A along with two morphisms $p_1: A \to A_1, p_2: A \to A_2$ such that for any object X along with two morphisms $g_1: X \to A_1, g_2: X \to A_2$ there is a unique morphism $h: X \to A$ such that $g_1 = p_1 h$, $g_2 = p_2 h$.

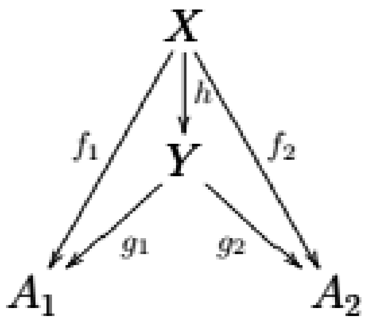


The object A will be denoted by $A_1 \times A_2$.

Remark 15. Let category $\mathcal{C}$ and A_1, A_2 be two objects in $\mathcal{C}$. Let $\mathcal{T}$ be a category that defined as the following: Objects of $\mathcal{T}$ are triples (X, f_1, f_2) where X be an object of $\mathcal{C}$, and $f_1: X \to A_1, f_2: X \to A_2$ are arrows in $\mathcal{C}$. An arrow from object (X, f_1, f_2) to object (Y, g_1, g_2) in $\mathcal{T}$ is an arrow $h: X \to Y$ in the category $\mathcal{C}$ such that $f_1 = g_1 h, f_2 = g_2 h$.

The composition of arrows in $\mathcal{T}$ is the composition of arrows in $\mathcal{C}$.

According to definition 14, If the product $(A_1 \times A_2, p_1, p_2)$ of A_1, A_2 exists, it will be a terminal of the category $\mathcal{T}$.



Definition 16. The coproduct of A_1, A_2 is defined dually to the product. Thus the coproduct is is an object A along with two morphisms $i_1: A_1 \to A, p_2: i_2 \to A$ such that for any object X along with two morphisms $g_1: A_1 \to X, g_2: A_2 \to X$ there is a unique morphism $h: A \to X$ such that $g_1 = hi_1,\ g_2 = hi_2$. The object A will be denoted by $A_1 \oplus \quad A_2$.

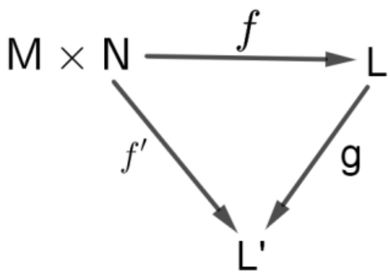
Remark 17. It is easily seen that if the coproduct $(A_1 \oplus A_2, i_1, i_2)$ of A_1, A_2 exists, it will be initial in the dual category $\mathcal{T}^o$ of $\mathcal{T}$.

2.5. Tensor product of modules

Let A be a commutative ring.

Definition 18. Let M, N be $A-$modules. Tensor product of M, N is a pair (T, g) consisting of an $A-$module T and an $A-$bilinear mapping $g: M \times N \to T$, with the following property: Given any $A-$module P and any $A-$bilinear mapping $f: M \times N \to P$, there exists a unique an $A-$linear mapping $f': T \to P$ such that $f = f'g$. Moreover, if (T, g) and (T', g') are two pairs with this property, then there exists a unique isomorphism $j: T \to T'$ such that $jg = g'$. Tensor product of M, N will be denoted by $M \otimes N$.

Remark 19. Let M, N be $A-$modules. Consider $\mathcal{C}(tx)$ be a category that defined as the following: Objects of $\mathcal{C}(tx)$ are pairs (L, f) where L be $A-$module and f be $A-$bilinear mapping $f: M \times N \to L$. An arrow from object (L, f) and object (L', f') in $\mathcal{C}(tx)$ is a $A-$linear mapping $g: L \to L'$ such that $f' = gf$.



The composition of arrows in $\mathcal{C}(tx)$ is the composition of arrows in $\mathcal{C}$.

According to definition 18, if the tensor product $M\otimes N$ of M, N exists, it will be initial of the category $\mathcal{C}(tx)$.

2.6. Localisation

Let A be a commutative ring with a unit and $S \subset A$ a multiplicative set.

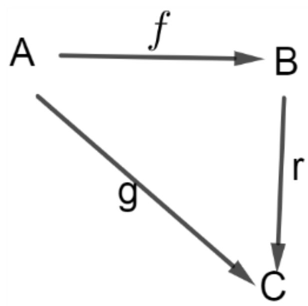
Definition 20. Suppose that $f: A \to B$ is a ring homomorphism satisfying the two conditions

(i)$f(x)$ is a unit of B for all $x \in S$;

(ii)If $g: A \to C$ is a homomorphism of rings taking every element of S to a unit of C then there exists a unique homomorphism $h: B \to C$ such that $g = hf$;

then B is uniquely determined up to isomorphism, and B is called the *localization* or the *ring of fractions* of A with respect to S. We write $B = S^{-1}A$ or A_S, and call $f: A \to A_S$ the canonical map.

Remark 21. Consider $\mathcal{C}(S)$ is a category that defined as the following: Objects of $\mathcal{C}(S)$ are pairs (B, f) where B is a commutative ring with unit and homomorphism $f: A \to B$ such that $f(s)$ is a unit of B for all $x \in S$. An arrow from object (B, f) and object (C, g) in $\mathcal{C}(S)$ is a homomorphism $r: B \to C$ such that $g = rf$.

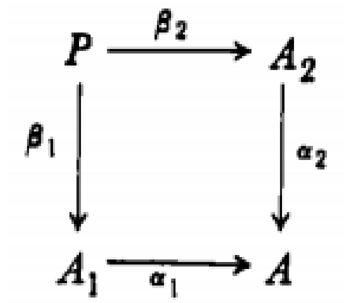


The composition of arrows in $\mathcal{C}(S)$ is the composition of homomorphisms.

According to definition 20, we see that the ring of fractions of A with respect to S is the initial of category $\mathcal{C}(S)$.

2.7. Pullbacks and Pushouts

Definition 22. Giving two morphisms $\alpha_1: A_1 \to A$ and $\alpha_2: A_2 \to A$ with a common codomain, a commutative diagram is called a *pullback* for α_1 and α_2 if for every pair of morphisms $\beta_1': P' \to A_1$ and $\beta_2': P' \to A_2$ such that $\alpha_1\beta_1' = \alpha_2\beta_2'$, there exists a unique morphism $\gamma: P' \to P$ such that $\beta_1' = \beta_1\gamma$ and $\beta_2' = \beta_2\gamma$.

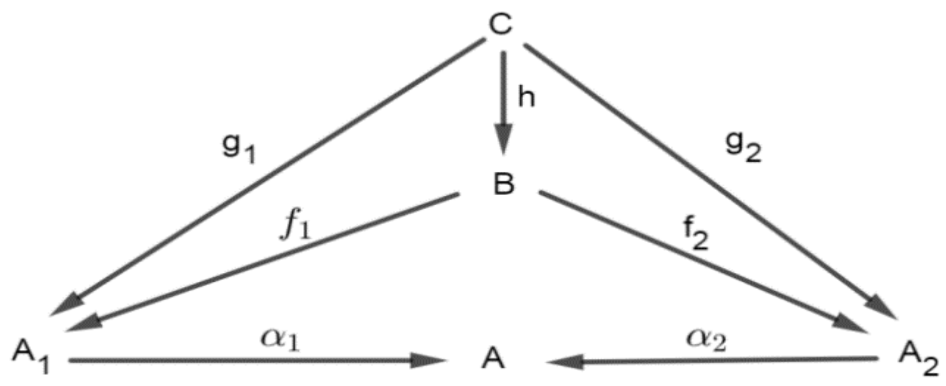


The pullback for α_1 and α_2 is uniquely determined up to isomorphism. In other words, if P' is also a pullback, there exists an isomorphism $\gamma': P \to P'$.

Remark 23. Giving two morphisms $\alpha_1: A_1 \to A$ and $\alpha_2: A_2 \to A$ with a common. Consider $\mathcal{C}(P)$ a category that defined as the following: Objects of $\mathcal{C}(P)$ are triples (B, f_1, f_2) where $f_1: B \to A_1, f_2: B \to A_2$ such that $\alpha_1 f_1 = \alpha_2 f_2$. An arrow from object (B, f_1, f_2) and object (C, g_1, g_2) in $\mathcal{C}(S)$ is a morphism $h: C \to B$ such that $g_1 = f_1 h$ and $g_2 = f_2 h$.

The composition of arrows in $\mathcal{C}(P)$ is the composition of morphisms.

According to definition 22, we see that *pullback* is the terminal of category $\mathcal{C}(P)$.



Definition 24. The dual of a pullback is called a *pushout*. Thus a pushout diagram is obtained by reversing the direction of all arrows in the diagram of pullback.

Remark 25. It is easily seen that if the pushout exists, it will be initial in the dual category $\mathcal{C}(P)^o$ of $\mathcal{C}(P)$.

3. CONCLUSION

In this article, I have reiterated the definition of some concepts in algebra and category theory. After that, I defined those concepts by showing that they are the initial or the terminal of a certain category. This shows a close relationship between category theory and different disciplines of mathematics such as algebra, topology; and also gives us a deeper understanding of some mathematical concepts.

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XEM XÉT MỘT SỐ KHÁI NIỆM TRONG ĐẠI SỐ VÀ PHẠM TRÙ THÔNG QUA VẬT KHỞI ĐẦU VÀ VẬT TẬN CÙNG TRONG LÝ THUYẾT PHẠM TRÙ

Tóm tắt: *Bài báo chỉ ra rằng một số khái niệm đại số là vật khởi đầu hoặc vật tận cùng của một phạm trù nào đó. Nói cách khác, một số khái niệm đại số có thể được định nghĩa thông qua vật khởi đầu hoặc vật tận cùng trong lý thuyết phạm trù.*

Từ khóa: *Lý thuyết phạm trù, vật khởi đầu, vật tận cùng, khái niệm đại số.*