FUZZY INITIAL VALUE PROBLEM FOR FRACTIONAL DIFFERENTIAL EQUATION ON LINEAR CORRELATED FUZZY FUNCTION SPACE

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Abstract: *The paper focuses on linear correlated fuzzy function space* $\mathbb{R}_{\mathcal{F}_{(A)}}$ *with its special properties . Firstly, we show some special properties of the norm operator on two spaces* $\mathbb{R}_{\mathcal{F}_{(A_1)}}, \mathbb{R}_{\mathcal{F}_{(A2)}}$ *with* $A_2 = kA_1$, A_1, A_2 *given fuzzy numbers. Besides, we study the fractional differential equation in the space* $\mathbb{R}_{\mathcal{F}_{(A)}}$. *By combining the special properties of space with fractional order calculations and the fixed point theorem, we have built assumptions to ensure that the problem has unique solution.*

Key words: *Fractional equation, Fréchet derivative, Caputo-Fréchet derivative*

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1. INTRODUCTION

In mathematics, fuzzy sets (uncertain sets) are somewhat like sets whose elements have degrees of membership. Fuzzy sets were introduced independently by L.A. Zadeh [38] in 1965 as an extension of the classical sets. But applications of fuzzy sets in inventory control problems were around 15-20 years back. Among these works, one can refer the works of Mandal and Maiti [24], Wee et al. [37]. In [38], Zadeh defined operations on set of fuzzy numbers with the following subtraction $u \ominus_H v = w \Leftrightarrow u = v + w$. It is well-known that this usual Hukuhara difference between two fuzzy numbers only exists under very restrictive conditions [12]. To fix it, L. Stefanini introduced the fuzzy gH-difference in [34,35]. The gH-difference of two fuzzy numbers exists under much less restrictive conditions, however it does not always exist [35]. This makes the space of fuzzy numbers not a Banach space. To overcome this difficulty, Estevão Esmi [4] presented a practical way to introduce the space of linear correlated fuzzy numbers $\mathbb{R}_{\mathcal{F}(A)} = \{\psi_A(q,r) / (q,r) \in \mathbb{R}^2\}$ where ψ_A is the operator that associates each vector $(q,r) \in \mathbb{R}^2$ with a fuzzy number $q[A]^\alpha + r$ where $[A]^\alpha$ stands for the α-level of a fuzzy number A for all $\alpha \in [0,1]$. The set $\mathbb{R}_{\mathcal{F}(A)}$ equipped with addition and and scalar product defined in naturally 3.7 in [4] is a Banach space.

Hukuhara differentiability of fuzzy-valued functions is generalization of Hukuhara differentiability of set-valued functions. This differentiability is based on Hukuhara difference. Hukuhara introduced this difference (subtraction) of two sets in [10]. He introduced the notions of integral and derivative for set-valued mappings and considered the relationship between them. This derivative is widely studied and analysed by researchers for set-valued as well as fuzzy-valued functions. A wide range of applications of Hukuhara derivatives are studied in fuzzy differential equations and fuzzy optimization problems. Unfortunately, the derivative is very restrictive. Its existence is based on certain conditions. Estevão Esmi [4] gave the concept of the derivative Fréchet. Accordingly, the derivative of function $f(t) = \psi_A(q(t), r(t))$ is calculated through the derivative of $q(t), r(t)$.

The concepts of fractional derivatives for a fuzzy valued function are either based on the notion of Hukuhara derivative (H-derivative) or on the notion of strongly generalized derivative (G-derivative). The concept of Hukuhara derivative is old and well known [10], Puri and Ralescu [31] and the concept of G-derivative was recently introduced by Bede and Gal [2]. The Fréchet derivative was introduced the first time by D. Behmardi and E. Dehghan, Nayeri in [8]. Since then, many research works about Fréchet derivative have been published in [36].

Fractional calculus and fractional differential equations arise naturally in a variety of fields such as rheology, viscoelasticity, electrochemistry, diffusion process etc [6,7,8,9]They are usually applied to replace the derivative time in a given evolution equation by a derivative of fractional order. One can find applications of fractional differential equations in signal processing and in the complex dynamic in biological tissues [23,24,25]. For a general overview, we refer the reader to the monographs of Samko et al. [33], Podlubny [32], Kilbas et.al [13,14] and the papers [15,16, 26, 33, 34]. Some new research results for the fractional equation can be mentioned as [18,19,20,21,23,25,26]

This paper has 3 main results: Firstly, based on Fréchet derivative given by E. Esmi [4] , we introduce the definition of Fréchet-Caputo fractional derivative. The paper focuses on exploiting the application of the Fréchet derivative to the system of fuzzy frational equations. Applying the Lipschitz fixed point theorem, we make assumptions about the system of equations to have solutions. Finally, we present an example to illustrate the result.

2. CONTENT

2.1. Preliminaries

2.1.1. The space of linear correlated fuzzy numbers

Denote by $\mathbb{R}_{\mathcal{F}}$ the space of all fuzzy numbers on the real line. According to [1], the characteristic properties of a fuzzy number u are presented via its α-cuts or level sets, which are defined by

$$[\boldsymbol{u}]^{\boldsymbol{\alpha}} = \begin{cases} \{\boldsymbol{x} \in \mathbb{R}: \boldsymbol{u}(\boldsymbol{x}) \geq \boldsymbol{\alpha}\}\ \boldsymbol{if}\ \boldsymbol{\alpha} \in (\boldsymbol{0}, \boldsymbol{1}] \\ \overline{\{\boldsymbol{x} \in \mathbb{R}: \boldsymbol{u}(\boldsymbol{x}) > \boldsymbol{0}\}}\ \boldsymbol{if}\ \boldsymbol{\alpha} = \boldsymbol{0}. \end{cases}$$

In addition, it is well-known that the level sets of u can be rewritten in the parametric form $[\boldsymbol{u}]^{\boldsymbol{\alpha}} = [\boldsymbol{u}_{\boldsymbol{\alpha}}^{-}, \boldsymbol{u}_{\boldsymbol{\alpha}}^{+}]$and the diameter of $[\boldsymbol{u}]^{\boldsymbol{\alpha}}$is given by $\boldsymbol{len}[\boldsymbol{u}]^{\boldsymbol{\alpha}} = \boldsymbol{u}_{\boldsymbol{\alpha}}^{+} - \boldsymbol{u}_{\boldsymbol{\alpha}}^{-}$for each $\boldsymbol{\alpha} \in [\boldsymbol{0}, \boldsymbol{1}]$

The space $(\mathbb{R}_{\mathcal{F}}, d_{\infty})$endowed with the supremum metric

$d_{\infty}(u, v) = \sup_{0 \leq \alpha \leq 1} d_H([u]^{\alpha}, [v]^{\alpha})$ for all $u, v \in \mathbb{R}_{\mathcal{F}}$, is a complete metric space (see [1]).

Definition 2.1.1 [4]

For each$\boldsymbol{A} \in \mathbb{R}_{\boldsymbol{\mathcal{F}}}$, define a mapping $\boldsymbol{\psi_A}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{\boldsymbol{\mathcal{F}}}$by $(\boldsymbol{q}, \boldsymbol{r}) \mapsto \boldsymbol{\psi_A}(\boldsymbol{q}, \boldsymbol{r})$where the level sets of $\boldsymbol{\psi_A}(\boldsymbol{q}, \boldsymbol{r})$are $[\boldsymbol{\psi_A}(\boldsymbol{q}, \boldsymbol{r})]^{\boldsymbol{\alpha}} = \{\boldsymbol{qa} + \boldsymbol{r}: \boldsymbol{a} \in [\boldsymbol{A}]^{\boldsymbol{\alpha}}\}$. For convenience, denote the fuzzy number $\boldsymbol{\psi_A}(\boldsymbol{q}, \boldsymbol{r})$by $\boldsymbol{qA} + \boldsymbol{r}$and the range of $\boldsymbol{\psi_A}(\boldsymbol{q}, \boldsymbol{r})$ by$\mathbb{R}_{\boldsymbol{\mathcal{F}(A)}}$.

Definition 2.1.2

A fuzzy number u said to be symmetric w.r.t. $x \in \mathbb{R}$ if $u(x-y) = u(x+y)$ for all $y \in \mathbb{R}$. The fuzzy number u is non-symmetric if there doesn't exist x such that u is symmetric.

From the results in [4], if $\boldsymbol{A} \in \mathbb{R}_{\boldsymbol{\mathcal{F}}}$is non-symmetric fuzzy number then the arithmetic operations on the space $\mathbb{R}_{\mathcal{F}(A)}$ such as addition, subtraction and scalar product are well-defined. Indeed, let us recall that for each $\boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}_{\boldsymbol{\mathcal{F}(A)}}$ and $\boldsymbol{\lambda} \in \mathbb{R}$it yields

(i) $B +_A C = \psi_A(\psi_A^{-1}(B) + \psi_A^{-1}(C))$

(ii) $\lambda B = \psi_A(\lambda \psi_A^{-1}(B))$

(iii) $B -_A C = B +_A (-1)C = \psi_A(\psi_A^{-1}(B) + (-1)\psi_A^{-1}(C))$.

In addition, the distance between elements of $\mathbb{R}_{\mathcal{F}(A)}$ can be measured by the metric d_A of the fuzzy number space. In particular, with $u = \psi_A(q_u, r_u), v = \psi_A(q_v, r_v), (q_u, r_u), (q_v, r_v) \in \mathbb{R}^2$, we have $d_A(u, v) = |q_u - q_v| + |r_u - r_v|$for all $u, v \in \mathbb{R}_{\mathcal{F}(A)}$.

It is well-known that $(\mathbb{R}_{\mathcal{F}_{(A)}}, d_A)$is a complete metric space. Moreover, for each $u = \psi_A(q_u, r_u)$due to the fact that the space $\mathbb{R}_{\mathcal{F}(A)}$is isometric to the space $\mathbb{R}^2$ it implies that the space $(\mathbb{R}_{\mathcal{F}_{(A)}}, +_A, \cdot_A, \|.\|_{\psi_A})$ with the induced norm is a Banach space since $\mathbb{R}_{\mathcal{F}_{(A)}}$ is isometric to $\mathbb{R}^2$ with the norm $\|u\|_{\psi_A} = \|\psi_A^{-1}(u)\|_{\mathbb{R}^2} = |q_u| + |r_u|$.

Lemma 2.1.1

If A_1, A_2 be two non-symmetric fuzzy numbers such that $A_2 = kA_1$ with $k \in \mathbb{R}\setminus\{0\}$then two norms $\|\cdot\|_{A_1}$and $\|\cdot\|_{A_2}$ are equivalent.

Proof

Because $u \in \mathbb{R}_{\mathcal{F}(A_1)}$ then u also belongs to the space $\mathbb{R}_{\mathcal{F}(A_2)}$, there exist two pairs $({}^1q_u, {}^1r_u), ({}^2q_u, {}^2r_u) \in \mathbb{R}^2$ such that $u = \psi_{A_1}({}^1q_u, {}^1r_u) = \psi_{A_2}({}^2q_u, {}^2r_u)$, which follows that$\psi_{A_1}({}^1q_u, {}^1r_u) = \psi_{A_1}(k\,{}^2q_u, {}^2r_u)$.

Our aim is to prove that two norms $\|\cdot\|_{A_1}$and $\|\cdot\|_{A_2}$are equivalent, which means that there exist $m, M > 0$ such that $m\|u\|_{A_1} \leq \|u\|_{A_2} \leq M\|u\|_{A_1}$for all $u \in \mathbb{R}_{\mathcal{F}(A_1)}$.

Indeed, since A_1 is a non-symmetric fuzzy number then ψ_{A_1} is an injection and hence, we directly obtain$\begin{cases} {}^1q_u = {}^2q_u k \\ {}^1r_u = {}^2r_u \end{cases}$.

By the definition of norm, we have

$$\begin{cases} \|u\|_{A_1} = |{}^1q_u| + |{}^1r_u|, \\ \|u\|_{A_2} = |{}^2q_u| + |{}^2r_u| = \left|\dfrac{{}^1q_u}{k}\right| + |{}^1r_u|. \end{cases}$$

Now, we consider some following cases:

Case 1: If $|k| > 1$ then it implies that

$\frac{1}{|k|}(|{}^1q_u| + |{}^1r_u|) \leq \|u\|_{A_2} \leq |{}^1q_u| + |{}^1r_u|$ or equivalently $\frac{1}{|k|}\|u\|_{A_1} \leq \|u\|_{A_2} \leq \|u\|_{A_1}$which proves that $\|\cdot\|_{A_1} \sim \|\cdot\|_{A_2}$.

Case 2: If $|k| > 1$then it implies that

$$|{}^1q_u| + |{}^1r_u| \leq \|u\|_{A_2} \leq \frac{1}{|k|}(|{}^1q_u| + |{}^1r_u|)$$

or equivalently, $\|u\|_{A_1} \leq \|u\|_{A_2} \leq \frac{1}{|k|}\|u\|_{A_1}$which proves that $\|\cdot\|_{A_1} \sim \|\cdot\|_{A_2}$.

Case 3: If $|k| = 1$then it is obvious that $\|u\|_{A_1} = \|u\|_{A_2}$for all $u \in \mathbb{R}_{\mathcal{F}(A)}$

Hence, the proof is completed.

Remark 2.1.1

However, if A_1, A_2 are symmetry fuzzy numbers then the similar conclusion as in Lemma 2. 1 can't be obtained. Indeed, since $A_2 = kA_1$ if A_1 is a symmetric fuzzy number w.r.t. $x \in \mathbb{R}$ then so is A_2. Next, by using Theorem 3.15 in \cite{este}, our proof is divided into following cases:

Case 1: If ${}^1q_u = {}^2q_u k$ and ${}^1r_u = {}^2r_u$ then by similar arguments as in Lemma 2.1.1 , we have $\|\cdot\|_{A_1} \sim \|\cdot\|_{A_2}$.

Case 2: If ${}^1q_u = -{}^2q_u k$ and ${}^1r_u = 2{}^2q_u x + {}^2r_u$ then it implies that $\begin{cases} {}^2q_u = -\frac{{}^1q_u}{k} \\ {}^2r_u = 2\frac{{}^1q_u x}{k} + {}^1r_u. \end{cases}$

Therefore, we have

$$\|u\|_{A_2} = |{}^2q_u| + |{}^2r_u| \leq \left|-\frac{{}^1q_u}{k}\right| + \left|\frac{2x}{k}\right| |{}^1q_u| + |{}^1r_u| \leq \left(\frac{1}{|k|} + \left|\frac{2x}{k}\right|\right) |{}^1q_u| + |{}^1r_u|.$$

Then, by denoting $M = \max\left\{\frac{1}{|k|} + \left|\frac{2x}{k}\right|; 1\right\}$, we directly get that $\| u \|_{A_2} \leq M \| u \|_{A_1}$. For each $u \in \mathbb{R}_{\mathcal{F}(A_1)}$, we have

$$\begin{aligned} \mathrm{F}(\|u\|) &:= \|u\|_{A_2}^2 - m^2\|u\|_{A_1}^2 \\ &= \left(\left|-\frac{{}^1q_u}{k}\right| + \left|\frac{2x^1}{k}q_u + {}^1r_u\right|\right)^2 - m^2(|{}^1q_u| + |{}^1r_u|)^2. \end{aligned} \tag{1}$$

Subcase 2.1: If ${}^1q_u{}^1r_u \geq 0$ and $\frac{{}^1q_u}{k}\left(\frac{2{}^1q_u x}{k} + r\right) \geq 0$ then the expression (1) becomes

$$\begin{aligned} &\|u\|_{A_2}^2 - m^2\|u\|_{A_1}^2 \\ &= \left[\left(\frac{2x+1}{k}\right)^2 - m^2\right]^1 q_u^2 + \left(\frac{4x^1r_u}{k} + \frac{2^1r_u}{k} - 2^1r_u m^2\right)^1 q_u + {}^1r_u^2(1 \\ &- m^2) \end{aligned}$$

If the above expression is known as a quadratic function of variable ${}^1q_u$ then the delta discriminant Δ' of the quadratic equation $\mathrm{F}(\| u \|) = 0$ is given by

$$\Delta' = {}^1r^2m^2\left(\frac{2x+1}{k} - 1\right)^2 \geq 0,$$

which means that the solution set of the quadratic equation $\mathrm{F}(\| u \|) = 0$ is nonempty.

Subcase 2.2: If ${}^1q_u{}^1r_u \geq 0$ and $\frac{{}^1q_u}{k}\left(\frac{2{}^1q_u x}{k} + r\right) \leq 0$ then the expression (1) becomes

$$\| u \|_{A_2}^2 - m^2 \| u \|_{A_1}^2 = \left[\left(\frac{2x-1}{k} \right)^2 - m^2 \right]^1 q_u^2 + \left(\frac{4x^1 r_u}{k} - \frac{2^1 r_u}{k} - 2m^{21} r_u \right)^1 q_u + ^1 r_u^2 (1 - m^2)$$

If the above expression is known as a quadratic function of variable $^1 q_u$ then the delta discriminant Δ' of the quadratic equation $\mathrm{F}(\| u \|) = 0$ is given by $\Delta' = ^1 r^2 m^2 \left(\frac{2x-1}{k} - 1 \right)^2 \geq 0$

which means that the solution set of the quadratic equation $\mathrm{F}(\| u \|) = 0$ is nonempty.

Subcase 2.3 If $^1 q_u {}^1 r_u \leq 0$ and $\frac{^1 q_u}{k} \left(\frac{2^1 q_u x}{k} + r \right) \geq 0$ then the expression (1) becomes

$$\| u \|_{A_2}^2 - m^2 \| u \|_{A_1}^2 = \left[\left(\frac{2x+1}{k} \right)^2 - m^2 \right]^1 q_u^2 + \left(\frac{4x^1 r_u}{k} + \frac{2^1 r_u}{k} + 2m^{21} r_u \right)^1 q_u + ^1 r_u^2 (1 - m^2)$$

If the above expression is known as a quadratic function of $^1 q_u$ then the delta discriminant Δ' of the quadratic equation $\mathrm{F}(\| u \|) = 0$ is given by $\Delta' = ^1 r^2 \left(\frac{2x+1}{k} + 1 \right)^2 \geq 0$

which means that the solution set of the quadratic equation $\mathrm{F}(\| u \|) = 0$ is nonempty.

We can conclude that there always exists an element $u \in \mathbb{R}_{\mathcal{F}(A_1)}$ or equivalently, a pair $(^1 q_u, ^1 r_u) \in \mathbb{R}^2$ such that the inequality $\|u\|_{A_2}^2 \leq m^2 \|u\|_{A_1}^2$ doesn't hold. Therefore, it implies that two norms $\| \cdot \|_{A_1} and \| \cdot \|_{A_2}$ are not equivalent.

2.2.2. Fréchet-Caputo fractional derivative

Let E, F be normed spaces and denote $\mathcal{L}(E, F)$ by the space of all continuous mappings.

Definition 2.2.1

Let $A \in \mathbb{R}_{\mathcal{F}}$

and

$$f \in \mathcal{L}(J, \mathbb{R}_{\mathcal{F}(A)}), f(t) = \psi_A(q(t), r(t))$$

with

$$q(t), r(t) \in L^1(J, \mathbb{R}) \cap C(J, \mathbb{R})$$

.

Then the Riemann-Liouville (RL) fractional integral of order $p \in (0,1]$ of function f is defined by

$${}_{F}^{RL}\mathcal{J}_{0^+}^{p} f(t) = \psi_A\left(I_{0^+}^{p} q(t), I_{0^+}^{p} r(t)\right),$$

where $I_{0^+}^{p} q(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} q(s) ds, I_{0^+}^{p} r(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} r(s) ds.$

Definition 2.2.2

Let $A \in \mathbb{R}_{\mathcal{F}}$

And $f \in \mathcal{L}(J, \mathbb{R}_{\mathcal{F}(A)}), f(t) = \psi_A(q(t), r(t))$

Let $A \in \mathbb{R}_{\mathcal{F}}$ and $f \in \mathcal{L}(J, \mathbb{R}_{\mathcal{F}(A)}), f(t) = \psi_A(q(t), r(t))$ with. The Fréchet-Caputo fractional derivative of order of the function $p \in (0,1]$ of function f is defined by

$${}_{F}^{C}\mathcal{D}_{0^+}^{p} f(t) = {}_{F}^{RL}\mathcal{I}_{0^+}^{1-p} f_{\mathcal{F}'}(t) = \psi_A\left(I_{0^+}^{1-p} q'(t), I_{0^+}^{1-p} r'(t)\right).$$

Definition 2.2.3

Let $A \in \mathbb{R}_{\mathcal{F}}$ be non-symetric fuzzy number and $f \in \mathcal{L}(J, \mathbb{R}_{\mathcal{F}(A)}), f(t) = \psi_A(q(t), r(t))$ with $q(t), r(t) \in L^1(J, \mathbb{R}) \cap C(J, \mathbb{R})$. The Fréchet-Riemann-Liouville fractional derivative of order $p \in (0,1]$ of the function f is defined by

$${}_{F}^{RL}\mathcal{D}_{0^+}^{p} f(t) = ({}_{F}^{RL}\mathcal{I}_{0^+}^{1-p} f)_{\mathcal{F}'}(t) = \psi_A\left((I_{0^+}^{1-p} q(t))', (I_{0^+}^{1-p} r(t))'\right).$$

Definition 2.2.4

Let $A \in \mathbb{R}_{\mathcal{F}}$ and $f \in \mathcal{L}(J, \mathbb{R}_{\mathcal{F}(A)}), f(t) = \psi_A(q(t), r(t))$ with $q(t), r(t) \in L^1(J, \mathbb{R}) \cap C(J, \mathbb{R})$. The Fréchet-Riemann-Liouville fractional derivative of order $p \in (0,1]$ of the function f is defined by

$${}_{F}^{RL}\mathcal{D}_{0^+}^{p} f(t) = ({}_{F}^{RL}\mathcal{I}_{0^+}^{1-p} f)_{\mathcal{F}'}(t) = \psi_A\left((I_{0^+}^{1-p} q(t))', (I_{0^+}^{1-p} r(t))'\right).$$

2.2.3. Application of the Fréchet -Caputo derivative

Consider following fuzzy intitial value problem (FIVP) for fractional differential equation

$$\begin{cases} {}^{C}_{F}\mathcal{D}^{p}_{0^+}\mu(t) = f(t,\mu(t)) \\ \mu(0) = \mu_0, \end{cases} \tag{1}$$

where ${}^{C}_{F}\mathcal{D}^{p}_{0^+}\mu(.)$ is the Fréchet-Caputo derivative of order $p \in (0,1]$ of the continuous function $\mu(.), \mu_0 \in \mathbb{R}_{\mathcal{F}(A)}$ with A is a fuzzy number and the function $f: J \times C(J, \mathbb{R}_{\mathcal{F}(A)}) \to \mathbb{R}_{\mathcal{F}(A)}$ is continous function with $J = [0;b]$.

2.3. Existence of solution

If A is a fuzzy number and $\mu \in \mathbb{R}_{\mathcal{F}(A)}$ is a solution to the problem (2) then it satisfies the following integral equation

$$\mu(t) = \mu_0 +_A {}^{RL}_{\mathcal{F}}\mathcal{J}^{p}_{0^+} f(t,\mu(t)), \quad t \in J. \tag{2}$$

Proof

Because $f(t,\mu(t)) \in \mathbb{R}_{\mathcal{F}(A)}$, there exists $q_{f\mu}, r_{f\mu}: J \to \mathbb{R}$ such that

$$f(t,\mu(t)) = \psi_A(q_{f\mu}(t), r_{\mu t}(t)).$$

Assume that is a solution of (2).

From ${}^{C}_{\mathcal{F}}\mathcal{D}^{p}_{0^+}\mu(t) = f(t,\mu(t))$, we have ${}^{RL}_{\mathcal{F}}\mathcal{J}^{1-p}_{0^+}\mu'(t) = f(t,\mu(t))$ wher $\mu'(t) = \psi_A(q'_{f\mu}(t), r'_{f\mu}(t))$.

By taking fractional integration of order p of two sides, we obtain

$${}^{RL}_{\mathcal{F}}\mathcal{I}^{p}_{0^+}\,{}^{RL}_{\mathcal{F}}\mathcal{I}^{1-p}_{0^+}\mu'(t) = {}^{RL}_{\mathcal{F}}\mathcal{I}^{1-p}_{0^+} f(t,\mu(t)).$$

This imlies $\int_0^t \mu'(s)ds = {}^{RL}_{\mathcal{F}}\mathcal{I}^{p}_{0^+} f(t,\mu(t))$. Therefore $\mu(t) = \mu_0 + {}^{RL}_{A\mathcal{F}}\mathcal{I}^{p}_{0^+} f(t,\mu(t)), t \in J$.

Definition 3.1.1

$\mu(t) \in \mathbb{R}_{\mathcal{F}(A)}$ is called an intergral solution of the problem (2) if it satifies integral equation $\mu(t) = \mu_0 + {}^{RL}_{A\mathcal{F}}\mathcal{I}^{p}_{0^+} f(t,\mu(t))$.

On space $C(J, \mathbb{R}_{\mathcal{F}(A)})$ we defined the supremum metric $\mathcal{H}$ and weighted metric ${}^{r}d$ as follows $\mathcal{H}(\mu,\Upsilon) = \sup_{t\in J} d_A(\mu(t),\Upsilon(t))$ and ${}^{r}d = \sup_{t\in J}\{t^r d_A(\mu(t),\Upsilon(t))\}$.

Theorem 3.1.1

Assume that A is a fuzzy number, the fuzzy-valued function f is jointly continuous and satisfies the Lipschitz condition w.r.t. the last argument, i.e., there exists a constant $L > 0$ so that

$$d_A(f(t,\mu(t)), f(t,\Upsilon(t))) \le L d_A(\mu(t),\Upsilon(t)).$$

for $all\, t \in J, \mu, \Upsilon \in C\left(J, \mathbb{R}_{\mathcal{F}(A)}\right)$.

Then, the fuzzy initial value problem (2) has a unique integral solution defined on J.

Proof: Let us define an operator $\mathcal{G} : C\left(J, \mathbb{R}_{\mathcal{F}(A)}\right) \to C\left(J, \mathbb{R}_{\mathcal{F}(A)}\right)$ by

$\mathcal{G}[\mu](t) = \mu_0 + {}^{RL}_{AF}\mathcal{I}^p f(t,\mu(t))$.

Assume that $(q_{f\mu}(t), r_{f\mu}(t)) \in \mathbb{R}\times\mathbb{R}, (q_{f\Upsilon}(t), r_{f\Upsilon}(t)) \in \mathbb{R}\times\mathbb{R}$ such that

$$f(t,\mu(t)) = \psi_A(q_{f\mu}(t), r_{f\mu}(t)), f(t,\Upsilon(t)) = \psi_A(q_{f\Upsilon}(t), r_{f\Upsilon}(t)).$$

For each $t \in J$ and $\mu, \Upsilon \in C\left(J, \mathbb{R}_{\mathcal{F}(A)}\right)$, we have

$d_A\left(\mathcal{G}[\mu](t), \mathcal{G}[\Upsilon](t)\right)$	$= d_A({}^{RL}_{F}\mathcal{I}^p f(t,\mu(t)), {}^{RL}_{F}\mathcal{I}^p f(t,\Upsilon(t)))$
	$= \lvert I^p_{0^+}(q_{fu}(t) - q_{fv}(t))\rvert + \lvert I^p_{0^+}(r_{fu}(t) - r_{fv}(t))\rvert$
	$= \lvert I^p_{0^+}(q_{f\mu}(t) - q_{f\Upsilon}(t))\rvert + \lvert I^p_{0^+}(r_{f\mu}(t) - r_{f\Upsilon}(t))\rvert$
	$\le \dfrac{1}{\Gamma(p)}\displaystyle\int_0^t (t-s)^{p-1}\left(\lvert q_{fu}(s) - q_{fv}(s)\rvert + \lvert r_{fu}(s) - r_{fv}(s)\rvert\right)ds$
	$\le \dfrac{1}{\Gamma(p)}\displaystyle\int_0^t (t-\gamma)^{p-1} d_A(f(\gamma,\mu(\gamma)), f(\gamma,\Upsilon(\gamma)))d\gamma$
	$\le \dfrac{1}{\Gamma(p)}\displaystyle\int_0^t (t-\gamma)^{p-1}\left(\lvert q_{f\mu}(\gamma) - q_{f\Upsilon}(\gamma)\rvert + \lvert r_{f\mu}(s) - r_{f\Upsilon}(\gamma)\rvert\right)d\gamma$
	$\le \frac{L\Gamma(p)}{\Gamma(2p)} t^{2p-1\ 1-p} d_A(\mu,\Upsilon)$.

This implies

$$t^{1-p} d_A(\mathcal{G}[\mu], \mathcal{G}[\Upsilon]) \le \frac{Lt^p\Gamma(p)}{\Gamma(2p)}^{1-p} d_A(\mu,\Upsilon) \tag{3}$$

The operator $\mathcal{G}^n$ is defined by

$\mathcal{G}^n[\mu](t) = \mathcal{G}(\mathcal{G}^{n-1}[\mu](t))$ for all $n \in \mathbb{N}, t \in J$.

Then, we will show that $\mathcal{G}^n$ is a contraction mapping for a big enough $n \in \mathbb{N}$. To this end, by induction principle, we will show that the following estimation holds for all $\mu \in C\left(J, \mathbb{R}_{\mathcal{F}(A)}\right)$. We have

$$d_A(\mathcal{G}^n[\mu](t), \mathcal{G}^n[\Upsilon](t)) \leq \frac{L^n t^{np+p-1}\Gamma(p)}{\Gamma((n+1)p)}(\quad {}^{1-p}d_A(\mu,\Upsilon)). \tag{4}$$

Indeed, if $n=1$ then we gain the estimation (3) from the inequality (4). Moreover, let us assume that the estimation (3) is true for $n=k$ and then, we will prove that (3) holds for $n=k+1$. For each $\mu,\Upsilon \in C\left(J, \mathbb{R}_{\mathcal{F}(A)}\right)$ and $t \in J$, we have

$d_A(\mathcal{G}^{k+1}[u](t), \mathcal{G}^{k+1}[\Upsilon](t))$	$= d_A(\mathcal{G}(\mathcal{G}^k[u](t)), \mathcal{G}(\mathcal{G}^k[u](t)))$
	$\leq d_A\left({}^{RL}_{F}\mathcal{J}^p_{0^+} f(t, \mathcal{G}^k[\mu](t)), {}^{RL}_{F}\mathcal{J}^p_{0^+} f(t, \mathcal{G}^k[\Upsilon](t))\right)$
	$\leq \frac{1}{\Gamma(p)} \int_0^t (t-\gamma)^{p-1}\, d_A(f(s, \mathcal{G}^k[\mu](\gamma), f(s,[\mu](\gamma))d\gamma$
	$\leq \frac{L}{\Gamma(p)} \int_0^t (t-\gamma)^{p-1}\, d_A(\mathcal{G}^k[\mu](\gamma), \mathcal{G}^k[\Upsilon](\gamma))d\gamma.$

By employing the induction hypothesis, we have

$$\begin{aligned} d_A(\mathcal{G}^{k+1}[\mu](t), \mathcal{G}^{k+1}[\Upsilon](t)) &\leq \frac{L^{k+1} \quad {}^{1-p}d_A(\mu,\Upsilon)}{\Gamma((k+1)p)} \int_0^t (t-\gamma)^{p-1}\, \gamma^{kp+p-1} d\gamma \\ &\leq \frac{L^{k+1} t^{(k+2)p-1}}{\Gamma((k+2)p)} B(p,(k+1)p)(\quad {}^{1-p}d_A(\mu,\Upsilon)) \\ &\leq \frac{L^{k+1} t^{(k+2)p-1}\Gamma(p)}{\Gamma((k+2)p)}(\quad {}^{1-p}d_A(\mu,\Upsilon)), \end{aligned}$$

where $B(p,q)$ is Beta function [14]. Therefore, the inequality (3) holds for $n=k+1$. From the inequality (3), we have

$$t^{1-p}d_A(\mathcal{G}^n[\mu](t), \mathcal{G}^n[\Upsilon](t)) \leq \frac{L^n t^{np}\Gamma(p)}{\Gamma((n+1)p)}(\quad {}^{1-p}d_A(\mu,\Upsilon)).$$

Then, by taking supremum both sides, we obtain $\quad {}^{1-p}d_A(\mathcal{G}^n[\mu], \mathcal{G}^n[\Upsilon]) \leq \frac{Lb^{np}\Gamma(p)}{\Gamma((n+1)p)}(\quad {}^{1-p}d_A(\mu,\Upsilon)) \to 0$ as $n \to \infty$.

It implies that the operator $\mathcal{G}^n$ is a contraction when $n \in \mathbb{N}$ is big enough. Applying contraction principle, we guarantee the unique existence of fixed point μ of the operator $\mathcal{G}^n$, that is the unique integral solution of the problem (2). The proof is complete.

2.4. Example

In this example, we consider the equation of the mass-Spring-DamperSystem as

$$\frac{m}{\sigma^{2(1-\gamma)}}{}_{F}^{C}\mathcal{D}_{0^+}^{2\gamma}\mu(t)+\frac{\beta}{\sigma^{1-\gamma}}+_{A}k\mu(t)=F(t)\qquad 0<\gamma\leq 1$$

where A is a fuzzy number, $\mu(t), F(t) \in \mathbb{R}_{\mathcal{F}(\mathcal{A})}$ the mass is m, the damping coefficient is β the spring costant is k anf $F(t)$ represent the forcing function, an auxilary parameter σ is introduced into the fractional tenporal operator:

$$\frac{d}{dt}\to\frac{1}{\sigma^{1-\gamma}}.\frac{d^{\gamma}}{dt^{\gamma}},\qquad m-1\leq\gamma\leq m, m\in M=1,2,3,\ldots$$

$$\frac{d^2}{dt^2}\to\frac{1}{\sigma^{2(1-\gamma)}}.\frac{d^{2\gamma}}{dt^{2\gamma}},\qquad m-1\leq\gamma\leq m, m\in M=1,2,3,\ldots$$

Consider a constant source, $F(t)=f_0, \mu(0)=\mu_0, \frac{d\mu}{dt}(0)=0$, we have equation (2), may be writen as follow:

$${}_{F}^{C}\mathcal{D}_{0^+}^{2\gamma}\mu(t)=\frac{\eta^2}{k}f_0-_{A}\eta^2\mu(t)$$

where $\eta^2=\frac{k\sigma^{2(1-\gamma)}}{m}$.

We can see that $f(t,\mu(t))=\frac{\eta^2}{k}f_0-_{A}\eta^2\mu(t)$is jointly continous function and satisfies the Lipschitz condition. So, we obtain the solution of problem is

$$\mu(t)=\left(\mu_0-_{A}\frac{f_0}{k}\right).E_{2\gamma}\{-\eta^2t^2\gamma\}+_{A}\frac{f_0}{k}.$$

where the Mittag-Leffler function $E_a(t)$ is defined by a power as: $E_a(t)=\sum_{m=0}^{\infty}\frac{t^m}{\Gamma(am+1)}$.

3. CONCLUSION

Firstly, in this paper, we introduced a new concept of fractional differentiability for a class of linear correlated fuzzy valued function namely the Fréchet-Caputo fractional derivative.

After that, we studied fuzzy fractional PDEs under Fréchet- Caputo differentiability. By using the fixed point theorem, we have proved some new results on the existence and uniqueness of fuzzy solution for the fuzzy intitial value problem. Fréchet derivative makes this problems have only a unique solution instead of having two different types of solutions such as the usual gH derivative .

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BÀI TOÁN VỀ GIÁ TRỊ CỦA BIẾN MỜ CHO CÁC PHƯƠNG TRÌNH VI PHÂN PHÂN SỐ TRÊN KHÔNG GIAN TUYẾN TÍNH

Tóm tắt: *Bài báo tập trung vào không gian hàm mờ tương quan tuyến tính với các tính chất đặc biệt của nó $\mathbb{R}_{\mathcal{F}_{(A)}}$ Đầu tiên, chúng tôi hiển thị một số thuộc tính đặc biệt của toán tử chuẩn trên hai không gian với $\mathbb{R}_{\mathcal{F}_{(A_1)}}, \mathbb{R}_{\mathcal{F}_{(A2)}}$ with$A_2 = kA_1$, A_1, A_2 mang lại các giá trị mờ. Bên cạnh đó, chúng tôi nghiên cứu phương trình vi phân phân số trong không gian $\mathbb{R}_{\mathcal{F}_{(A)}}$bằng cách kết hợp các tính chất đặc biệt của không gian với các phép tính bậc phân số và định lý điểm cố định, chúng tôi đã xây dựng các giả thiết để đảm bảo rằng bài toán có nghiệm duy nhất.*

Từ khoá: *Phương trình vi phân, đạo hàm Fréchet, đạo hàm Caputo-Fréchet.*

SMARTPHONE BASED OPTICAL SENSOR FOR ENVIRONMENTAL APPLICATION

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Abstact: *In the recent years there has been a huge development on the detection devices using smartphone that are reliable, easy-to-use, and low cost. In this work, a smartphone based optical sensor is constructed by implementing an external light source, a collimating lens, a diffraction grating, and a CMOS chip of a smartphone as a detector. The construction allows the device to function with an optical bandwidth of 300 nm (from 400 to 700 nm) and its resolution of 0.26 nm/pixel. It can be used both for measuring the absorption, transmission emission spectrum. As a proof of concept, the optical sensor using smartphone is then applied to investigate concentration of methylene blue (MB), a reactive dye in wastewater from textile industry. Despite of its cost-effectiveness, the sensor exhibits reliable results, which can be considerably comparable with that of laboratory instrument.*

Keywords: *Optical sensor, smartphone, absorption, methylene blue.*

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1. INTRODUCTION

In recent years, there has been an increased interest in the development of simple, light-weight, low-cost, portable and rapid detection devices for applications related to clinical diagnosis, health care and environmental monitoring [1–3]. The combinations of portable mobile devices with internet connectivity, touch screen displays, high resolution cameras, and high-performance CPUs have facilitated the development of a new device generation. These kinds of devices are not only suited for scientific research but also for daily work which normally does not require dedicated instruments and laboratory conditions for sensing, detection and analysis. Since smart phones are ubiquitous, thus, integrating detecting smart phones into devices is a promising approach for the creation of a detection device for public health and environmental protection. As a result, many research groups