# THE SPACE OF LINEARLY CORRELATED FUZZY NUMBER $\mathbb{R}_{\mathcal{F}(A)}$

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***Abstract:*** *This article introduces the space of linearly correlated fuzzy number $\mathbb{R}_{\mathcal{F}(A)}$. It is a subspace of the space of fuzzy numbers. we first review the algebraic operations on $\mathbb{R}_{\mathcal{F}(A)}$ defined mean of linear isomorphism between $\mathbb{R}^2$ and $\mathbb{R}_{\mathcal{F}(A)}$ provided that $A$ is a non-symmetric fuzzy number. Second, we present a quotient set $\mathbb{R}^2/_{\equiv_A}$ by defining an appropriate equivalence relation on $\mathbb{R}^2$ when $A$ is a symmetric fuzzy number. After that, we will introduce some types of Fréchet derivative defined on the class of linear correlated fuzzy-valued functions namely Fréchet derivative and LC derivative . That allows us to introduce three types of Fréchet fractional derivatives, which are Fréchet Caputo derivative, Fréchet Riemann-Liouville derivative and Fréchet Caputo-Fabrizio derivative.*

***Keywords****: Linearly correlated fuzzy number, Fréchet derivative, LC derivative, Fréchet Caputo derivative, Fréchet Riemann-Liouville derivative and Fréchet Caputo-Fabrizio derivative.*

Received 5 July 2021

Revised and accepted for publication 23 August 2021

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## 1. INTRODUCTION

In [5], the H-difference of two fuzzy numbers has been introduced. It is well-known that the usual Hukuhara difference between two fuzzy numbers exists only under very restrictive conditions [11, 12, 16]. The gH-difference of two fuzzy numbers exists under much less restrictive conditions, however it does not always exist [31, 32]. The g-difference proposed in [38] overcomes these shortcomings of the above discussed concepts and the g-difference of two fuzzy numbers always exists. The same remark is valid if we regard differentiability concepts in fuzzy settings. Based on these types of differences, different types of derivatives on the space of fuzzy functions are also studied in turn.

In 2009 [30], generalized Hukuhara difference (gH difference) of two interval numbers which was more general than H-difference was introduced, based on which the generalized Hukuhara derivative (gH-derivative) was defined for interval-valued functions. Based on the g-difference, a very general fuzzy differentiability concept is defined and studied, the so-called g-derivative [5, 30]. Based on the gr-difference, Granular derivative (gr-derivative) is introduced [22]. In [13], Esmi et al has introduced the concept of difference of two fuzzy numbers in the linear correlated fuzzy-valued $\mathbb{R}_{\mathcal{F}(A)}$ space. In fact, the structure of the space of linearly correlated fuzzy numbers depends on the symmetry of the basic fuzzy number. Specifically, this space is a linear one if the basic fuzzy number is a non-symmetric fuzzy number, whereas if the basic fuzzy number is symmetric, then the space is not a linear space. Therefore, the calculus they established are mainly for the case where the basic fuzzy number is non-symmetric . In some special cases, subtraction in space $\mathbb{R}_{\mathcal{F}(A)}$ is always possible. Furthermore we will also have $B \ominus_A B = \hat{0}$ for all $B \in \mathbb{R}_{\mathcal{F}(A)}$. This is because the space $\mathbb{R}_{\mathcal{F}(A)}$ has some special analytical properties. The $\mathbb{R}_{\mathcal{F}(A)}$ space is a special space because it can be embedded into $\mathbb{R}_{\mathcal{F}}$ via $\psi_A$ function as a complete linear subspace if is $A$ a non-symmetric fuzzy number. Consequently, $(\mathbb{R}_{\mathcal{F}(A)}, \oplus_A, \odot_A)$ becomes a Banach space $\mathbb{R}_{\mathcal{F}}$. However, when the basic fuzzy number is symmetric, it is impossible to directly propose a suitable difference through the addition and the scalar multiplication mentioned above, because the operator is no longer a linear isomorphism and the space of linearly correlated fuzzy number spaces is also not linear. To deal with this problem, the author [26] introduced the LC-difference in the space of linearly correlated fuzzy numbers. Coincidentally, the LC-difference and the gH difference introduced in [26]are equal for interval numbers. It is worth mentioning that LC-difference is adaptable regardless of whether the basic fuzzy number is symmetric or non-symmetric, and this difference always exists in the space of linearly correlated fuzzy numbers.

Recently, the derivative concepts built in the space $\mathbb{R}_{\mathcal{F}(A)}$ are being studied. In [23], authors develop a theory of calculus for linearly correlated fuzzy processes via the Frechet derivative and the Riemann integral. The author [26] reconsidered the calculus of linearly correlated fuzzy number-valued functions with the help of their representation functions. In details, the differentiability of a linearly correlated fuzzy number-valued function can be characterized by its representation functions. If the basic fuzzy num-ber is non-symmetric, the differentiability is equivalent to the Fréchet differentiability proposed by Esmi et al. [13], and it is also equivalent to the differentiability of its representation functions. In addition, if the basic fuzzy number is symmetric, then the differentiability can be described by the representation functions of the canonical form of a linearly correlated fuzzy number-valued function. Along with the study of derivatives in the space $\mathbb{R}_{\mathcal{F}(A)}$, fractional derivatives are also gradually receiving much attention. In [29], we introduce the definition of Frechet Caputo fractional derivative, Frechet Riemann-Liouville fractional derivative and Frechet

Caputo-Fabrizio fractional derivative and the relationship between them. In this paper, we present recent studies on space $\mathbb{R}_{\mathcal{F}(A)}$, specifically, types of derivatives as well as some dynamical systems.

## 2. PRELIMINARY

Firstly, we recall from [5] some fundamental arithmetic operations on the fuzzy number space $\mathbb{R}_{\mathcal{F}}$

(i) The addition between two fuzzy numbers $u$ and $v$ is defined via the Minkowski sum of their $\alpha$ - level sets

$$[u \oplus v]^{\alpha} = [u]^{\alpha} + [v]^{\alpha} = \{a + b : a \in [u]^{\alpha}, b \in [v]^{\alpha}\}, \alpha \in [0,1].$$

(ii) The scalar multiplication of a fuzzy number $u$ with a scalar $\lambda$ is given by

$$[\lambda u]^{\alpha} = \lambda [u]^{\alpha} = \{\lambda a : a \in [u]^{\alpha}\}, \alpha \in [0,1].$$

(i) The H-difference of $u$ and $v$ is known as an element $w \in \mathbb{R}_{\mathcal{F}}$ such that $u = v \oplus w$ and is denoted by $u \ominus v$. Note that the H-difference $u \ominus v$ (if exists) is unique and its $\alpha-$level sets are

$$[u \ominus v]^{\alpha} = [u_{\alpha}^{-} - v_{\alpha}^{-}, u_{\alpha}^{+} - v_{\alpha}^{+}] \quad \text{for all } \alpha \in [0,1].$$

(ii) The gH-difference of $u$ and $v$, denoted by $u \ominus_{gH} v$, is known as an element $w \in \mathbb{R}_{\mathcal{F}}$ such that $u = v \oplus w$ or $v = u \oplus (-1)w$.

Here, it should be noted that if $u \ominus v$ exists then $u \ominus_{gH} v = u \ominus v$.

**Definition 2.1.** [13] A fuzzy number $A \in \mathbb{R}_{\mathcal{F}}$ is symmetric with respect to $x \in \mathbb{R}$ if $A(x - y) = A(x + y)$ for all $y \in \mathbb{R}.$. We say that $A$ is non-symmetric if there exists no $x$ such that $A$ is symmetric.

**Example 2.1.** Consider a fuzzy number u whose membership function is given by

$$u(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 < x \le 2 \\ 3 - x & \text{if } 2 < x \le 3 \\ 0 & \text{if } x > 3 \end{cases}$$

We can see that $u$ is a symmetric fuzzy number with respect to x=3/2 , see Figure 1 for illustration.

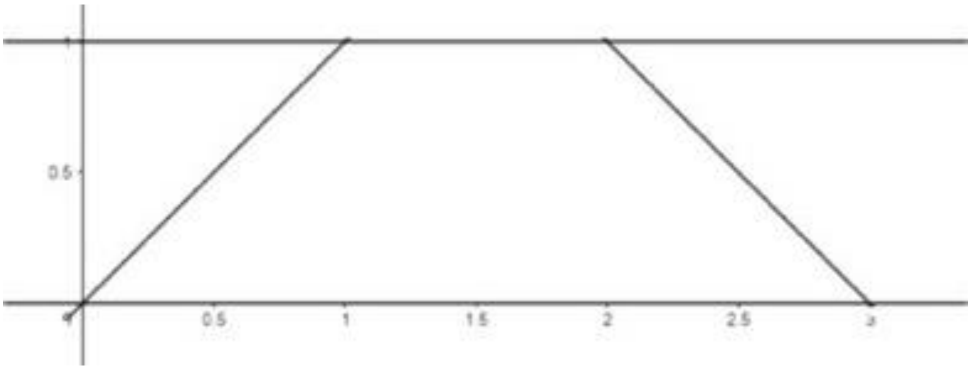


**Figure 1.** The symmetric fuzzy number $u$

**Example 2.2.** The fuzzy set $v:\mathbb{R}\to[0,1]$, given by is a symmetric fuzzy number with respect to $x$ =5/4, see Figure. 2.

$$v(x)=\begin{cases} 0 & \text{if } x<0 \\ x^3 & \text{if } 0\le x<1 \\ 1 & \text{if } 1\le x<1,5 \\ (2,5-x)^3 & \text{if } 1,5<x\le 2,5 \\ 0 & \text{if } x>2,5, \end{cases}$$

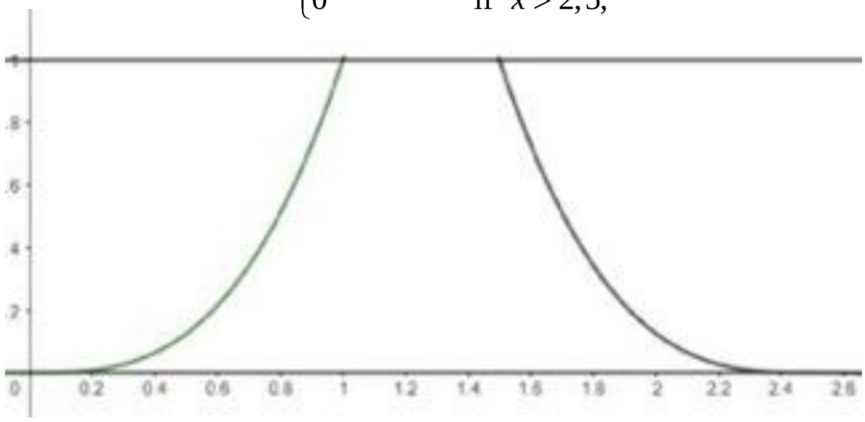


**Figure 2.** The symmetric fuzzy number v

**Example 2.3.** The fuzzy set $w:\mathbb{R}\to[0,1]$

$$w(x)=\begin{cases} 0 & \text{if } x<-1 \\ \dfrac{x+1}{4} & \text{if } -1\le x<3 \\ \dfrac{5-x}{2} & \text{if } 3\le x\le 5 \\ 0 & \text{if } x>5, \end{cases}$$

is a non-symmetric fuzzy number since there is no $x\in\mathbb{R}$ such that the equality

$w(x+y)=w(x-y)$ is satisfied for all $y\in\mathbb{R}$.

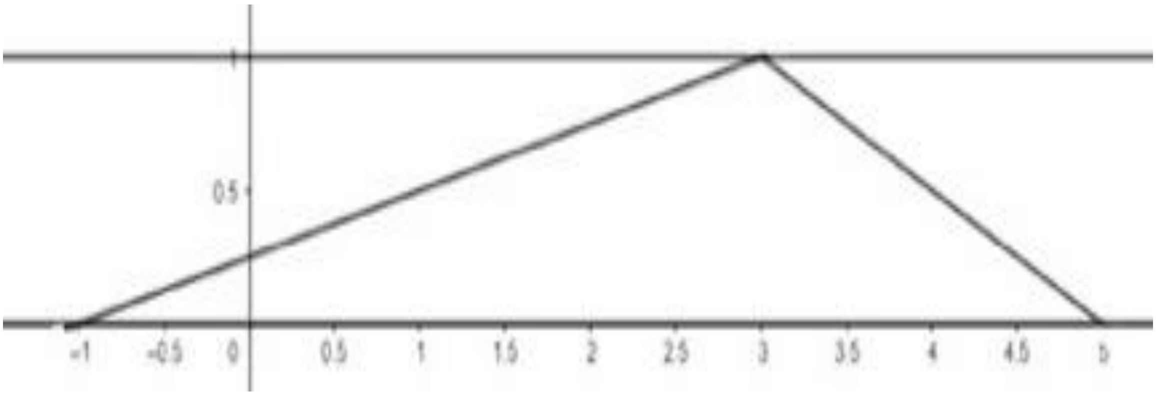


**Figure 3.** The symmetric fuzzy number w

**2.1. The sapce of linearly correlated fuzzy number** $\mathbb{R}_{\mathcal{F}(A)}$.

***a.*** $A$ ***is non-symmetric***

According to [13], for each non-symmetric fuzzy number $A\in\mathbb{R}_{\mathcal{F}}$, there is a linear isomorphism

$$\psi_A:\mathbb{R}\times\mathbb{R}\to\mathbb{R}_{\mathcal{F}}$$
$$(q,r)\mapsto\psi_A(q,r)$$

where $\psi_A(q,r)$ is a fuzzy number whose $\alpha-$level sets are given by

$$[\psi_A(q,r)]^\alpha=\{qa+r:a\in[A]^\alpha\},\ \forall\alpha\in[0,1],\ q,r\in\mathbb{R}.$$

Additionally, we denote the fuzzy number $\psi_A(q,r)$ by $q\odot_A A+r$ and the range of the isomorphism $\psi_A$ by $\mathbb{R}_{\mathcal{F}(A)}$. An important fact that is worth emphasizing is that $\mathbb{R}$ can be embedded in $\mathbb{R}_{\mathcal{F}(A)}$ since every real number $r$ can be identified with the fuzzy number $\psi_A(0,r)\in\mathbb{R}_{\mathcal{F}(A)}$ that is $\mathbb{R}\subseteq\mathbb{R}_{\mathcal{F}(A)}$.

From the results in [13], if $A\in\mathbb{R}_{\mathcal{F}}$ is a non-symmetric fuzzy number then the arithmetic operations on the space $\mathbb{R}_{\mathcal{F}(A)}$. such as addition, subtraction and scalar product are well-defined as follows:

(i) $u\oplus_A v=\psi_A\left(\psi_A^{-1}(u)+\psi_A^{-1}(v)\right)$

(ii) $\lambda\odot_A u=\psi_A\left(\lambda\psi_A^{-1}(u)\right)$;

(iii) $u-_A v=u+_A(-1)v=\psi_A\left(\psi_A^{-1}(u)+(-1)\psi_A^{-1}(v)\right)$,

where $u,v\in\mathbb{R}_{\mathcal{F}(A)}$ and $\lambda\in\mathbb{R}$.

The metric on the space $\mathbb{R}_{\mathcal{F}(A)}$ is defined by

$$d_A(u,v) = |q_u - q_v| + |r_u - r_v| \quad \text{for all } u,v \in \mathbb{R}_{\mathcal{F}(A)}.$$

Moreover, the space $\mathbb{R}_{\mathcal{F}(A)}$ endowed with metric $d_A$ is a complete metric space.

In addition, if the fuzzy number $A$ is non-symmetry then the space $\mathbb{R}_{\mathcal{F}(A)}$ is isometric to the Banach space $\mathbb{R}^2$ and hence, it implies that the space $(\mathbb{R}_{\mathcal{F}(A)}, +_A, \cdot_A, \|\cdot\|_{\psi_A})$ is a Banach space with the induced norm $\|u\|_{\psi_A} = \|\psi_A^{-1}(u)\|_{\mathbb{R}^2} = |q_u| + |r_u| = d_A(u,\hat{0})$ where $u = \psi_A(q_u, r_u) \in \mathbb{R}_{\mathcal{F}(A)}$ and $\hat{0} = \psi_A(0,0)$ is the neutral element of the space $\mathbb{R}_{\mathcal{F}(A)}$.

***b.* $A$ *is non-symmetric***

When $A$ is a symmetric fuzzy number, a linearly correlated fuzzy number-valued function fmay have many infinitely representation functions. Considering the structure of the space $\mathbb{R}_{\mathcal{F}(A)}$, from [26], author introduce the canonical form of a linearly correlated fuzzy number-valued function $f(t) = q(t)A + r(t)$ provided that $A$ is a symmetric fuzzy number *namely* an equivalence realtion $\equiv_A$ is defined in $\mathbb{R}^2$ by $(q,r) \equiv_A (p,s)$ if and only if $(q,r) = (p,s)$ or $(q,r) = (-p, 2px^* + s)$ for $(q,r),(p,s) \in \mathbb{R}^2$. Define the equivalence class

$$[q,r]_{\equiv_A} \overset{\text{def}}{=} \{(q,r),(-q,2qx^* + r)\}.$$

Using the equivalence relation $\equiv_A$, the quotient set of $\mathbb{R}^2$ is defined by

$$\mathbb{R}^2/_{\equiv_A} \overset{\text{def}}{=} \{[q,r]_{\equiv_A} \mid (q,r) \in \mathbb{R}^2\}.$$

Note that we choose $(q,r)$ with $q \geq 0$ as the representative element. The function $\hat{\psi} : \mathbb{R}^2/\equiv_A \to \mathbb{R}_{\mathcal{F}(A)}$ is defined by

$$\hat{\psi}_A([q,r]_{\equiv_A}) = q \odot_A A + r$$

where $[q,r]_{\equiv_A} \in \mathbb{R}^2/\equiv_A$. Clearly, $\hat{\psi}_A$ is a bijection.

For $u = \hat{\psi}_A([q_u, r_u]_{\equiv_A}), v = \hat{\psi}_A([q_v, r_v]_{\equiv_A})$, with $\lambda \in \mathbb{R}, q_u, q_v \geq 0.$ The addition $\hat{\oplus}_A$ and the

$(i)$ $\quad u \,\hat{\oplus}_A\, v = \hat{\psi}_A([q_u + q_v, r_u + r_v]_{\equiv_A}),$

$(ii)$

$$\lambda \,\hat{\odot}\, u = \begin{cases} \hat{\psi}_A([\lambda q_u, \lambda r_v]_{\equiv_A}), & \lambda \geq 0, \\ \hat{\psi}_A([-\lambda q_u, 2\lambda q_u x^* + \lambda r_u]_{\equiv_A}), & \lambda < 0, \end{cases}$$

scalar multiplication $\hat{\odot}_A$ in $\mathbb{R}^2/\equiv A$ are defined

**Definition 3.2.** [26] For $u=\psi_A(q_u,r_u), v=\psi_A(q_v,r_v)$ are non symmetric fuzzy number. The LC-difference of $u$ and $v$ is defined by

**Example 3.4.** Let $A=(1;3;4)$ be a non-symmetric fuzzy number, $u=\psi_A(t^2,2t)$ and $v=\psi_A(t,t^2)$ for all $t\in[0,\infty)$. Then we have

$u\boxminus_A v=\psi_A(t^2-t,2t-t^2)=(t^2-t)\odot_A A+(2t-t^2).$

If $t\geq 1$ then $t^2-t\geq 0$, we have $u\boxminus_A v=(t;2t^2-t;3t^2-2t)$.

If $t\in(0,1)$ then $t^2-t<0$,we have $u\boxminus_A v=(3t^2-2t;2t^2-t;t)$.

**Definition 3.3.** [26] For $u=\hat{\psi}_A([q_u,r_u]_{\equiv_A}), v=\hat{\psi}_A([q_v,r_v]_{\equiv_A})$ are symmetric fuzzy numbers with symmetric point $x^*$ . The LC-difference of $u$ and $v$ is defined by

$$u\,\hat{\boxminus}\,v=\begin{cases}\hat{\psi}_A([(q_u-q_v),r_u-r_v]_{\equiv_A}), & q_u\geq q_v,\\ \hat{\psi}_A([(q_v-q_u),2(q_u-q_v)x^*+r_u-r_v]_{\equiv_A}), & q_u<q_v.\end{cases}$$

In [27], it is proved that the LC-difference and the gH-difference are equivalent for interval numbers, since each interval number can be regarded as a symmetric fuzzy number. Therefore, the LC-difference can be seen as a gener-alization of the gH-difference of interval numbers in the space of linearly correlated fuzzy numbers $\mathbb{R}_{\mathcal{F}(A)}$.

**Example 3.5.** Let $A=(1;3;5)$ is a symmetric fuzzy number with the symmetry point $x^*=2$ , $u=\psi_A(t^2,2t)$ and $v=\psi_A(t,t^2)$ for all $t\in[0,\infty)$. Then we have

- if $t\geq 1$ then $t^2\geq t$, we get

  $B\,\hat{\boxminus}\,C=\hat{\psi}_A([t^2-t,2t-t^2]_{\equiv_A}),$

- if $t\in(0,1)$ then $t^2-t<0$, we get

$B\,\hat{\boxminus}\,C=\hat{\psi}_A([t-t^2,2(t^2-t)2+2t-t^2]_{\equiv_A})=\hat{\psi}_A([t-t^2,3t^2-2t]_{\equiv_A})$.

From [26], $u=\psi_A(q_u,r_u)$ is non-symmetric fuzzy number, the norm $\|.\|_A$ in $\mathbb{R}_{\mathcal{F}(A)}$ is introduced as follows

$\|u\|_A=\max\{|q_u|,|r_u|\}.$

Based on the LC-difference [26], for $u=\psi_A(q_u,r_u), v=\psi_A(q_v,r_v)$ are non-symmetric fuzzy number with the symmetric point $x^*$. The metric $d_{\psi_A}$ is given by

$d_{\psi_A}(u,v) = \| u \boxminus_A v \|_A = \max\{|q_u - q_v|, |r_u - r_v|\}$.

If $B = \hat{\psi}_A([q_u, r_u]_{\equiv_A})$ is a symmetric fuzzy number with the symmetric point $x^*$ then the norm $\|.\|_{\hat{A}}$ in $\mathbb{R}_{\mathcal{F}(A)}$

$\| u \|_{\hat{A}} = \max\{|q_u|, |r_u|, |2q_u x^* + r_u|\}$,

and for $u = \hat{\psi}_A([q_u, r_u]_{\equiv_A}), v = \hat{\psi}_A([q_v, r_v]_{\equiv_A}) \in \mathbb{R}_{\mathcal{F}(A)}$ the metric $d_{\hat{\psi}_A}$ is given by

$$d_{\hat{\psi}_A}(u,v) = \| u \hat{\boxminus}_A v \|_{\hat{A}} = \max\{|q_u - q_v|, |2(q_u - q_v)x^* + r_u - r_v|, |r_u - r_v|\}$$

**2.2. Derivative of linear correlated fuzzy-valued function**

**Remark 4.1.** [13] Let $g : J \to \mathbb{R}_{\mathcal{F}(A)}$ and the functions $q, r : J \to \mathbb{R}$ such $g(t) = \psi_A(q(t), r(t))$, $t \in J$. We have

1. when $A$ is a non-symmetric fuzzy number, then g is Fréchet differentiable at $t \in J$ if and only if $q'(t), r'(t) \in \mathbb{R}$ exist. Additionally, $g'_{\mathcal{F}}(t) = \psi_A(q'(t), r'(t))$, $t \in J$;

2. when $A$ is a symmetric fuzzy number then we define $g'_{\mathcal{F}}(t) = \psi_A(q'(t), r'(t)), t \in J$; provided there exist $q'(t), r'(t) \in \mathbb{R}, t \in J$.

**Definition 4.1.** [26] Let $A$ is a symmetric fuzzy number with the symmetry point $x^*$ and Let $f : J \to \mathbb{R}_{\mathcal{F}(A)}$ is a linearly correlated fuzzy number value function with $f(t) = q(t)A + r(t)$. Then the canonical form of f is defined by

$f(t) = \hat{\psi}_A([q(t), \tilde{r}(t)]_{\equiv_A}) = q(t) \hat{\odot}_A A + \tilde{r}(t)$,

Wherre

$$q(t) = \begin{cases} q(t), & q(t) \ge 0 \\ -q(t), & q(t) < 0 \end{cases} \qquad \tilde{r}(t) = \begin{cases} r(t), & q(t) \ge 0, \\ 2q(t)x^* + r(t), & q(t) < 0. \end{cases}$$

**Example 4.1.**

- Let $A = (-2;0;2)$ be a symmetric triangular fuzzy number with the symmetry point $x^* = 0$. Assume that $f(t) = (3-t) \hat{\odot} A + 2t + 1$ for all $t \in [0, \infty)$. Then the canonical form of $f(t)$ is $f(t) = |3-t| \hat{\odot}_A A + 2t + 1$.
- Let $A = (1;2;3)$ be a symmetric triangular fuzzy number with the symmetry point $x^* = 1$. Assume that $f(t) = -t^2 \hat{\odot}_A A + t^2$ for all $t \in [0; +\infty)$. We can see that $-t^2 \le 0$ for all $t \ge 0$. Then the canonical form of $f(t)$ is $f(t) = \hat{\psi}_A(t^2, -2t^2.1 + t^2) = \hat{\psi}_A(t^2, -t^2)$.

**Definition 4.2.** [26] Let $A$ is a non-symmetric fuzzy number and let : $f: J \to \mathbb{R}_{\mathcal{F}(A)}$ be a linearly correlated fuzzy number-valued function with $f(t) = q(t)A + r(t)$ . For $t_0 \in int(J)$, we say that

- (i) $f$ is left LC-differentiable at $t_0$ provided that the following limit

$$\lim_{t \to t_0^-} \frac{1}{t - t_0} \odot_A (f(t) \boxminus_A f(t_0))$$

exists in the sense of the metric $d_{\psi_A}$:

- (i) $f$ is right LC-differentiable at $t_0$ provided that the following limit

$$\lim_{t \to t_0^+} \frac{1}{t - t_0} \odot_A (f(t) \boxminus_A f(t_0))$$

exists in the sense of the metric $d_{\psi_A}$.

Meantime, the left LC-derivative and the right LC-derivative of $f$ at $t_0$ are denote by $f'_-(t_0)$ and $f'_+(t_0)$, respectively.

**Definition 4.3.** [26] Let $A$ is a non-symmetric fuzzy number and let $f:(a,b) \to \mathbb{R}_{\mathcal{F}(A)}$ be a linearly correlated fuzzy number-valued function with with $f(t) = q(t)A + r(t)$. For $t_0 \in (a,b)$, we say that $f$ is LC-differentiable at $t_0$ provided that $f$ is both left and right LC-differentiable and $f'_-(t_0) = f'_+(t_0)$. Furthermore, the LC-derivative is denoted by $f'(t_0)$.

**Definition 4.4.** [26] Let $A$ is a symmetric fuzzy number with the symmetry point $x^*$ and let $f: J \to \mathbb{R}_{\mathcal{F}(A)}$ be a linearly correlated fuzzy-valued function with the canonical form $f(t) = q(t)A + \tilde{r}(t)$. For $t_0 \in int(J)$, we say that

- (i) $f$ is left LC-differentiable at $t_0$ provided that the following limit

$$\lim_{t \to t_0^-} \frac{1}{t - t_0} \hat{\odot}_A (f(t) \hat{\boxminus}_A f(t_0))$$

exists in the sense of the metric $d_{\hat{\psi}_A}$,

- (i) $f$ is right LC-differentiable at $t_0$ provided that the following limit

$$\lim_{t \to t_0^+} \frac{1}{t - t_0} \hat{\odot}_A (f(t) \hat{\boxminus}_A f(t_0))$$

exists in the sense of the metric $d_{\hat{\psi}_A}$.

**Remark 4.2.** [27] We define

$$q_1(t_0) = \begin{cases} q'(t_0), & q'_-(t_0) \geq 0 \\ -q'(t_0), & q'_-(t_0) < 0 \end{cases} \qquad and \qquad r_1(t_0) = \begin{cases} \tilde{r}'(t_0), & q'_-(t_0) \geq 0 \\ -q'(t_0), & q'_-(t_0) < 0 \end{cases}$$

Then we can obtain $f^{'}(t_0)=q_1A+r_1(t_0)$. More generally, for any $t\in(a;b)$ the derivative function of $f$ can be represented by

$f^{'}(t)=\hat{\psi}_A[q_1(t),r_1(t)]_{\equiv_A}$.

**Example 4.2**. Let $A=(1;2;3)$ be a symmetric triangular fuzzy number with the symmetry point $x* = 1$ and let $f(t)=(t^2+1)\hat{\odot}_A A+t^2, t\in(-1;1)$. We can get $q(t) = t^2 + 1$, $r(t) = t^2$. For each $t\in(-1;1)$, we can obtain $q_-^{'}(t)=q_+^{'}(t)=2t, \tilde{r}_-^{'}(t)=\tilde{r}_+^{'}(t)=2t,$. Furthermore, we have

$$q_1(t)=\begin{cases}2t & t\geq 0\\ -2t & t<0\end{cases} \quad \text{and } r_1(t)=\begin{cases}2t & t\geq 0\\ -2t & t<0.\end{cases}$$

Hence, we conclude that $f(t)$ is differentiable in $(-1;1)$ and $f'(t)=q_1\hat{\odot}_A A+r_1(t)$ for each $t\in(-1;1)$.

**Corollary 4.2** [27] Let $A\in\mathbb{R}_{\mathcal{F}}\setminus\mathbb{R}$ be a symmetric fuzzy number with the symmetric point $x^*$ and let be the linearly correlated fuzzy number function in $(a;b)$. Suppose that $q(t),r(t)$ are differentiable with $q(t)$ has the same sign and $q'(t)\geq 0$ in $(a;b)$. Then $f(t)$ is differentiable in $(a;b)$ and $f'(t)=q'(t)\hat{\odot}_A+r'(t)$.

**Corollary 4.3**. [27] Let $A\in\mathbb{R}_{\mathcal{F}}\setminus\mathbb{R}$ be a symmetric fuzzy number with the symmetric point $x^*$ and let $f(t)=q(t)\odot_A+r(t)$ be the linearly correlated fuzzy number function in $(a;b)$. Suppose that $q(t),r(t)$ are differentiable with $q(t)$ has the same sign and $q'(t)<0$ in $(a;b)$. Then f (t) is differentiable in $(a;b)$ and $f'(t)=-q'(t)\hat{\odot}_A+2q'(t)x^*+r'(t)$.

**Example 4.3.** Let $A\in\mathbb{R}_{\mathcal{F}}\setminus\mathbb{R}$ be a symmetric fuzzy number with the symmetric point $x^*$ and let $f(t)=t^3\hat{\odot}_A+1$ for all $t\in\mathbb{R}$. By Corollary 3.1, we know that $f(t)$ is differentiable in $\mathbb{R}$ and $f'(t)=t^2\hat{\odot}_A$.

**Example 4.4**. Let $A\in\mathbb{R}_{\mathcal{F}}\setminus\mathbb{R}$ be a symmetric fuzzy number with the symmetric point $x*$ and let $f(t)=-t^3\hat{\odot}_A+1$ for all $t\in\mathbb{R}$. By Corollary 3.2, we know that $f(t)$ is differentiable in R and and $f'(t)=t^2\hat{\odot}_A-2t^2x^*$.

**2.3. Fractional derivatives of linear correlated fuzzy-valued function**

**Definition 5.1.** [29] Let $A\in\mathbb{R}_{\mathcal{F}}$ be non-symmetric fuzzy number, $f\in\mathcal{L}(J,\mathbb{R}_{\mathcal{F}(A)})$ and $f(t)=q(t)\odot_A A+r(t)$ with $q,r\in L^1(J,\mathbb{R})\cap C(J,\mathbb{R})$. Then, the Riemann-Liouville fractional integral of order $p \in (0;1]$ of the function f is defined

$${}^{RL}_{\mathcal{F}}\mathcal{I}^{p}_{0^+} f(t) = \psi_A\left(I^{p}_{0^+}q(t), I^{p}_{0^+}r(t)\right), t \in J$$

**Example 5.1.** Let $A = (0;1;4)$ be non-symmetric fuzzy number and $f(t) = (t^2 - t) \odot_A + t$ for all $t \in [0,\infty)$. We have $q(t) = t^2 - t, r(t) = t$. Then

$$I^{1/2}_{0^+}q(t) = \frac{16}{3\sqrt{\pi}}\sqrt{t^5} - \frac{8}{3\sqrt{\pi}}\sqrt{t^3}, \; I^{1/2}_{0^+}r(t) = \frac{8}{3\sqrt{\pi}}\sqrt{t^3}$$

This implies $${}^{RL}_{\mathcal{F}}\mathcal{I}^{1/2}_{0^+} f(t) = \psi_A\left(\frac{16}{3\sqrt{\pi}}\sqrt{t^5} - \frac{8}{3\sqrt{\pi}}\sqrt{t^3}, \frac{8}{3\sqrt{\pi}}\sqrt{t^3}\right).$$

If $A$ is a symmetric fuzzy number then the operator $\psi_A$ is not injective, we introduce the follows concept of Riemann-Liouville fractional integral of order $p \in (0,1]$ when $A$ is symmetric as

**Definition 5.2.** [29] Let $A \in \mathbb{R}_{\mathcal{F}}$ be a symmetric fuzzy number and $f : J \subset \mathbb{R} \to \mathbb{R}_{\mathcal{F}(A)}$ be a linear correlated fuzzy-valued function which is given by $f(t) = \psi_A(q(t), r(t))$. If there exist $q(t), r(t) \in L^1(J,\mathbb{R}) \cap C(J,\mathbb{R})$ and $q(t)$ does not change the sign in $J$ then the Riemann-Liouville (RL) fractional integral of order $p \in (0,1]$ of the function $f(t)$ is defined by

$${}^{RL}_{\mathcal{F}}\mathcal{I}^{p}_{0^+} f(t) = \psi_A\left(I^{p}_{0^+}q(t), I^{p}_{0^+}r(t)\right), t \in J.$$

**Example 5.2.** Let $A = (0;1;2)$ be a symmetric fuzzy number with the symmetry point $x^* = 1$

and $f(t) = t^2 \odot_A + t$ for all $t \in [0,\infty)$. We have $q(t) = t^2, r(t) = t$. This implies

$$I^{1/2}_{0^+}q(t) = \frac{16}{r\sqrt{\pi}}\sqrt{t^5} \qquad I^{1/2}_{0^+}r(t) = \frac{8}{3\sqrt{\pi}}\sqrt{t^3}$$

Therefore

$${}^{RL}_{\mathcal{F}}\mathcal{I}^{1/2}_{0^+} f(t) = \psi_A\left(\frac{16}{3\sqrt{\pi}}\sqrt{t^5}, \frac{8}{3\sqrt{\pi}}\sqrt{t^3}\right).$$

**Definition 5.3.** Let $A$ be a non-symmetric fuzzy number and $f : J \subset \mathbb{R} \to \mathbb{R}_{\mathcal{F}(A)}$, $f(t) = q(t)A \oplus r(t)$ is Frechet differentiable for each $t \in J$. The Frechet Caputo-Fabrizio (FCF) fractional derivative of order $p \in (0;1]$ of fuzzy-valued function f is defined by

$$
\begin{aligned}
{}^{CF}_{\mathcal{F}}D^{p}_{0^{+}}f(t) &= \frac{1}{1-p}\int_0^t e^{\frac{-p(t-s)}{1-p}} f'_{\mathcal{F}}(s)ds \\
&= \psi_A\left({}^{CF}D^{p}_{0^{+}}q(t), {}^{CF}D^{p}_{0^{+}}r(t)\right).
\end{aligned}
\tag{2}
$$

In the case $A$ is a symmetric fuzzy number, assume that $f(t) = q(t)A \oplus r(t)$ is Frechet differentiable for each $t \in J$ and $q'(\cdot), r'(\cdot)$ do not change the sign in $J$, we define the FCF fractional derivative of order $p \in [0,1)$ of f $f$ by formula (2).

## 3. CONCLUSION

In this paper, we present knowledge around the space of linearly correlated fuzzy numbers $\mathbb{R}_{\mathcal{F}(A)}$. The construction of the two-element difference in space, especially the LC difference has opened a new way for approaching integral definitions on space $\mathbb{R}_{\mathcal{F}(A)}$. This leads to dynamical systems on $\mathbb{R}_{\mathcal{F}(A)}$ that can be converted to systems on the set of real numbers. This makes it easier to understand the solution posture as well as show the existence of solutions.

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## KHÔNG GIAN CÁC HÀM SỐ MỜ TƯƠNG QUAN TUYẾN TÍNH

***Abstract:*** *Trong bài báo này, tôi giới thiệu không gian không gian số mờ có giá trị tương quan tuyến tính $R_{F(A)}$. Nó là một không gian con của không gian các số mờ . Đầu tiên chúng tôi cung cấp một số thông tiên cơ bản về nguồn gốc hình thành, cách xác định các phép toán, các định nghĩa metric trên không gian $R_{F(A)}$. Sau đó, tôi đưa ra một số loại đạo hàm bậc một đã được nghiên cứu trong không gian này cụ thể ở đây là đạo hàm Fréchet. Cuối cùng, một số định nghĩa về đạo hàm bậc phân thứ, cụ thể là đạo hàm bậc phân thứ Fréchet Caputo, Fréchet Riemann-Liouville, Fréchet Caputo-Fabrizio cũng như một số tính chất của nó.*

***Từ khóa***: *Đạo hàm bậc phân thứ Fréchet Caputo, đạo hàm bậc phân thứ Fréchet Riemann-Liouvill, đạo hàm bậc phân thứ Fréchet Caputo-Fabrizio fractional derivative, không gian các hàm số mờ tương quan tuyến tính.*