χ – AUTOMORPHISM INVARIANT MODULES SATISFY C3-CONDITION

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ABSTRACT

In this article, we state the condition for an χ −endomorphism invariant module to be a $C2$ module and an χ − automorphism invariant module to be a $C3$ module. Finally, we discuss when an χ − automorphism invariant module is an χ − endomorphism invariant module.

Keywords: Automorphism invariant, endomorphism invariant, injective envelope, general envelope.

1. INTRODUCTION

In [1], the authors defined the notions of $\chi-$automorphism invariant modules and endomorphism-invariant modules where χ is any class of modules closed under isomorphisms. Clearly, an $\chi-$endomorphism invariant module is $\chi-$automorphism invariant, $\chi-$automorphism invariant modules need not be $\chi-$endomorphism invariant in general. Many interesting properties of $\chi-$automorphism invariant modules investigated. Namely, that if $u: M \to X$ is a monomorphic $\chi-$envelope of a module M such that M is $\chi-$automorphism invariant, $End(X)/J(End(X))$ is a von Neumann regular right self-injective ring and idempotents lift modulo $J(End(X))$, then $End(M)/J(End(M))$ is also von Neumann regular and idempotents lift modulo $J(End(M))$ and consequently, M satisfies the finite exchange property. Moreover, if every direct summand of M has an $\chi-$envelope, then $\chi-$automorphism-invariant M has a decomposition $M = A \oplus B$ where A is square-free, B is $\chi-$endomorphism-invariant and M is clean. In [2], the authors defined the notions of $\chi-$strongly purely closed modules. As a consequence, in this paper we study $\chi-$automorphism invariant $\chi-$strongly purely closed modules. We obtain that $\chi-$strongly purely closed, $\chi-$endomorphism invariant modules satisfy $C2$ condition ([**Theorem 2.10**]) and $\chi-$strongly purely closed, $\chi-$automorphism invariant modules satisfy $C3$ condition ([**Theorem 2.13**]). Throughout this article all rings are associative rings with identity and all modules are right unital. A submodule N of a module M is called essential in M (denoted as $N \leq^e M$) if $N \cap K \neq 0$ for any proper submodule K of M. Let χ be a class of right $R-$modules, we say that χ is closed under isomorphisms, if $M \in \chi$ and $N \cong M$ then $N \in \chi$. Let χ be a class of right R-modules which is closed under isomorphisms, a homomorphism $u: M \to X$ of right $R-$modules is an $\chi-$

envelope of a module M provided that: (1) $X \in \chi$ and, for every homomorphism $u' : M \to X'$ with $X' \in \chi$, there exists a homomorphism $f : X \to X'$ such that $u' = fu$;

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "u'"] & X \arrow[dl, "f"] \\ X' & \end{tikzcd}$$

(2) $u = fu$ implies that f is an automorphism for every endomorphism $f : X \to X$.

If (1) holds, then $u : M \to X$ is called an $\chi -$ preenvelope.

2. RESULTS

It is easy to see that the $\chi -$ envelope is unique up to isomorphisms. It has the following proposition.

Proposition 2.1 [3, Proposition 1.2.1] *If $u : M \to X$ and $u' : M \to X'$ are two different $\chi -$ envelopes of a right $R -$ module M, then $X' \cong X$.*

Proposition 2.2 [3, Theorem 1.2.5] *Let $M = M_1 \oplus M_2$, and $u_i : M_i \to X_i$ are $\chi -$ envelope of M_i. Then, $u_1 \oplus u_2 : M \to X_1 \oplus X_2$ is an $\chi -$ envelope of M.*

Let M, N be $R -$ modules. We say that N is $\chi - M -$ injective if there exist

$\chi -$ envelopes $u_N : N \to X_N$, $u_M : M \to X_M$ satisfying that for any homomorphism $g : X_N \to X_M$, there is a homomorphism $f : N \to M$ such that $gu_N = u_M f$:

$$\begin{tikzcd} X_N \arrow[r, "g"] & X_M \\ N \arrow[u, "u_N"] \arrow[r, "f"] & M \arrow[u, "u_M"'] \end{tikzcd}$$

If M is $\chi - M -$ injective, then M is said to be an $\chi -$ endomorphism invariant module. A class χ of right modules over a ring R, closed under isomorphisms is called an enveloping class if any right $R -$ module M has an $\chi -$ envelope.

Lemma 2.3 *Let χ be an enveloping class. If N is $\chi - M -$ injective, then N' is $\chi - M' -$ injective for any N' is a direct summand of N and any M' is a direct summand of M.*

Proof. Let $N = N' \oplus K, M = M' \oplus L$ for some submodules K of N and L of M. Let $u_{N'} : N' \to X_{N'}$, $u_K : K \to X_K$, $u_{M'} : M' \to X_{M'}$, $u_L : L \to X_L$ be $\chi -$ envelopes of N', K, M', L, respectively. We have $u_{N'} \oplus u_K : N \to X_{N'} \oplus X_K$ is an $\chi -$ envelope of N and $u_{M'} \oplus u_L : M \to X_{M'} \oplus X_L$ is an $\chi -$ envelope of M. Let $\alpha : X_{N'} \to X_{M'}$ be any homomorphism, and $\pi : X_{N'} \oplus X_K \to X_{N'}$ be the canonical projection, $i : X_{M'} \to X_{M'} \oplus X_L$ be the inclusion map. Let $g = i\alpha\pi : X_{N'} \oplus X_K \to X_{M'} \oplus X_L$. Since

N is $\chi - M -$injective, there exists an homomorphism $f : N \to M$ such that $g(u_{N'} \oplus u_K) = (u_{M'} \oplus u_L) f$.

$$\begin{tikzcd} X_N \arrow[r, "g"] & X_M \\ N \arrow[u, "u_{N'} \oplus u_K"] \arrow[r, "f"] & M \arrow[u, "u_{M'} \oplus u_L"'] \end{tikzcd}$$

It follows that $gu_{N'} = u_{M'} f$. Now, take $\pi_{M'} : M' \oplus L \to M'$ the canonical projection and $i_{N'} : N' \to N' \oplus K$ the inclusion map. Let $g' = \pi_{M'} f i_{N'} : N' \to M'$, then we can check that $\alpha u_{N'} = u_{M'} g'$.

$$\begin{tikzcd} X_{N'} \arrow[r, "\alpha"] & X_{M'} \\ N' \arrow[u, "u_{N'}"] \arrow[r, "g'"] & M' \arrow[u, "u_{M'}"'] \end{tikzcd}$$

Therefore, N' is $\chi - M' -$injective. □

The following corollaries are straightforward and we can omit their proofs.

Corollary 2.4 *If N is an $\chi - M -$injective module and L is a direct summand of M, then N is an $\chi - L -$injective module.*

Corollary 2.5 *Every direct summand of an $\chi - M -$injective module is also an $\chi - M -$ injective module.*

Corollary 2.6 *Any direct summand of an $\chi -$endomorphism invariant module is $\chi -$ endomorphism invariant.*

Corollary 2.7 *Assume that $M = M_1 \oplus M_2$. If M is $\chi -$endomorphism invariant, then M_1 is an $\chi - M_2 -$injective module and M_2 is an $\chi - M_1 -$injective module.*

Definition 2.8 An $R-$module M is called $\chi -$strongly purely closed if every submodule A of M and any homomorphism $f : A \to X$, with $X \in \chi$, extends to a homomorphism $g : M \to X$ such that $gi = f$ in which $i : A \to M$ is the inclusion map

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] & M \arrow[dl, "g"] \\ X \in \chi & \end{tikzcd}$$

A module M is called satisfying $C2$ condition if every submodule A of M such that A is isomorphic to a direct summand of M, then A is a direct summand of M.

Theorem 2.10 *Let χ be an enveloping class and M is an $\chi -$strongly purely closed module. If M is an $\chi -$endomorphism invariant module then M satisfies $C2$ condition.*

Proof. Let A be a submodule of M, and B is a direct summand of M such that $A \cong B$. Let $\varphi : B \to A$ be an isomorphism. Let $u_B : B \to X_B$ be an $\chi-$envelope of B, $u_M : M \to X_M$ be an $\chi-$envelope of M. Since M is an $\chi-$strongly purely closed module, the homomorphism $u_B\varphi^{-1} : A \to X_B$ extends to a homomorphism $\beta : M \to X_B$ such that $u_B\varphi^{-1} = \beta i$. Since $u_M : M \to X_M$ is an $\chi-$preenvelope of M, there exists $k : X_M \to X_B$ such that $\beta = ku_M$.

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[drr, "u_B"'] & A \arrow[r, "i"] \arrow[dr, "u_B\varphi^{-1}"] & M \arrow[r, "u_M"] \arrow[d, "\beta"] & X_M \arrow[dl, "k"] \\ & & X_B & \end{tikzcd}$$

Since M is an $\chi-$endomorphism invariant module and B is a direct summand of M, M is $\chi-B-$ injective by Corollary 2.4. Therefore, there exists $f : M \to B$ such that $ku_M = u_B f$

$$\begin{tikzcd} X_M \arrow[r, "k"] & X_B \\ M \arrow[u, "u_M"] \arrow[r, "f"] & B \arrow[u, "u_B"'] \end{tikzcd}$$

Now, we have $\quad u_B\varphi^{-1} = \beta i = ku_M i = u_B fi$

As u_B is a monomorphism, so $\varphi^{-1} = fi$. It follows that $i\varphi$ is a split monomorphism. It means that $Im(i\varphi) = A$ is a direct summand of M. □

Definition 2.11 A right $R-$module M having an $\chi-$envelope $u : M \to X$ is said to be $\chi-$automorphism invariant if for any automorphism g of X, there exists an endomorphism f of M such that $uf = gu$.

$$\begin{tikzcd} X \arrow[r, "g"] & X \\ M \arrow[u, "u"] \arrow[r, "f"] & M \arrow[u, "u"] \end{tikzcd}$$

Lemma 2.12 *Let $M = M_1 \oplus M_2$ be an $\chi-$automorphism invariant module. Then M_1 is $\chi-M_2-$injective.*

Proof. Let $u_1 : M_1 \to X_1, u_2 : M_2 \to X_2$ be $\chi-$envelopes of M_1, M_2, respectively. Thus, $u = u_1 \oplus u_2 : M \to X = X_1 \oplus X_2$ is an $\chi-$envelope of M. For any homomorphism $g : X_1 \to X_2$, $\bar{g} : X \to X$ via $\bar{g}(x_1 + x_2) = x_1 + x_2 + g(x_1)$ is an isomorphism of X. Since X is an $\chi-$automorphism invariant module, there exists $h : M \to M$ such that $\bar{g}u = uh$. Let $f = \pi_2(h-1)i_1$, where $\pi_2 : M \to M_2$ is the canonical projection and $i_1 : M_1 \to M$ is the inclusion map, then we have $u_2 f = gu_1$

$$\begin{tikzcd} X_1 \arrow[r, "g"] & X_2 \\ M_1 \arrow[u, "u_1"] \arrow[r, "f"] & M_2 \arrow[u, "u_2"] \end{tikzcd}$$

Therefore, M_1 is $\chi - M_2 -$ injective. □

Theorem 2.13 *Let χ be an enveloping class and M is an $\chi-$automorphism invariant module. If M is an $\chi-$strongly purely closed module, then M satisfies $C3$ condition.*

Proof. Assume that A, B are direct summands of M with $A \cap B = 0$. Let A' be some submodule of M such that $M = A \oplus A'$. We claim that, there exists $M' \le M$ such that $M = A \oplus M'$ and $B \le M'$. Let $\pi : M \to A$, $\pi' : M \to A'$ be the projections. Since $A \cap B = 0$, $\pi'|_B : B \to A'$ is a monomorphism. Moreover, M is an $\chi-$strongly purely closed module, A' is too. It follows that $u\pi'|_B : B \to X_{A'}$ is a preenvelope, where $u : A \to X_A$ and $u' : A' \to X_{A'}$ are envelopes.

$$\begin{tikzcd} B \arrow[r, "\pi'|_B"] \arrow[d, "\pi|_B"'] & A' \arrow[r, "u'"] & X_{A'} \arrow[lldd, dashed, "h"'] \\ A \arrow[d, "u"'] & & \\ X_A & & \end{tikzcd}$$

By definition of preenvelope, there exists $h : X_{A'} \to X_A$ such that $hu'\pi'|_B = u\pi|_B$.

$$\begin{tikzcd} X_{A'} \arrow[r, "h"] & X_A \\ A' \arrow[u, "u'"] \arrow[r, "g"] & A \arrow[u, "u"] \end{tikzcd}$$

Since A' is $\chi - A-$injective by Lemma 2.12, there exists $g : A' \to A$ such that $hu' = ug$. Therefore $u\pi|_B = hu'\pi'|_B = ug\pi'|_B$. As u is a momomorphism, $\pi|_B = g\pi'|_B$. Let $M' = \{a' + g(a') | a' \in A'\}$. For every $b \in B$, we have $b = \pi'(b) + \pi(b) = \pi'(b) + g\pi'(b)$. It follows that $b \in M'$. Then $B \le M'$. It is easy to see that $A \cap M' = 0$ and for every $m \in M$,

$$m = a + a' = a - g(a') + (a' + g(a')) \in A + M, (a \in A, a' \in A').$$

Thus $M = A + M'$, and so $M = A \oplus M'$. On the other hand, we have $M = B \oplus B'$ for some $B' \le M$, then $M' = B \oplus (M' \cap B')$. We deduce that $M = A \oplus M' = A \oplus B \oplus (M' \cap B')$. It means that $A \oplus B$ is a direct summand of M .□

We will say that M is $\chi-$extending invariant (or $\chi-$extending) if there exists an $\chi-$envelope $u : M \to X$ such that for any idempotent $g \in End(X)$ there exists an idempotent $f : M \to M$ such that $g(X) \cap u(M) = uf(M)$ or $uf = guf$.

Theorem 2.14 *Let M be an $\chi -$ extending invariant modules and $u: M \to X$ is a monomorphic $\chi -$ envelope with $u(M)$ essential in X. Assume $End(X)/J(End(X))$ is a von Neumann regular, right self-injective ring and idempotents lift modulo $J(End(X))$. If M is $\chi -$ automorphism invariant, $\chi -$ strongly purely closed module then M is an $\chi -$ endomorphism invariant module.*

Proof. Let g be any endomorphism of X. By [1, Theorem 3.14], $End(X)$ is clean, so $g = e + f$ in which e is an idempotent endomorphism of X and f is an automorphism of X. Since M is an $\chi -$ automorphism coinvariant module, there exists a homomorphism $\alpha : M \to M$ such that $fu = u\alpha$. On the other hand, since M is an $\chi -$ extending invariant modules and e is an idempotent endomorphism of X, there exists an idempotent endomorphism e' of X such that $e(X) \cap u(M) = ue'(M)$. Therefore $A = u^{-1}(e(X)) \cap M = e'(M)$ is a direct summand of M. Since $(1-e)$ is also an idempotent endomorphism of X, $B = u^{-1}((1-e)(X)) \cap M$ is a direct summand of M. It is easy to see that $A \cap B = 0$. By Theorem 2.13, $A \oplus B$ is a direct summand of M. Let $M = A \oplus B \oplus C$ for some $C \le M$, then $M = A \oplus A'$ where $A' = B \oplus C \ge B$. Let $\pi : A \oplus A' \to A$ be the canonical projection. We show that $eu = u\pi$.

Assume that, there exists $0 \ne m \in M$ such that $(eu - u\pi)(m) \ne 0$. Since $u(M) \le^e X$, there exists $m_1 \in M$ such that $u(m_1) = (eu - u\pi)(m) \ne 0$. Hence $u(m_1 + \pi(m)) = eu(m) \in e(X)$, and so $m_1 + \pi(m) \in A$. Moreover, $eu(m_1 + \pi(m)) = e^2u(m) = eu(m)$, so $eu(m_1 + \pi(m) - m) = 0$. Now,

$$u(m_1 + \pi(m) - m) = (1-e)u(m_1 + \pi(m) - m) \in (1-e)(X),$$

so $m_1 + \pi(m) - m \in B.$

Let $m = a + a'$, where $a \in A, a' \in A'$, then $m_1 + \pi(m) - a - a' \in B \le A'$ and $\pi(m) = a$. Therefore $m_1 + \pi(m) - a \in A' \cap A = 0$. Thus $m_1 = 0$, a contradiction.

Let $h = \alpha + \pi \in End(M)$, it follows that

$$gu = (e + f)u = eu + fu = u\pi + u\alpha = u(\pi + \alpha) = uh.$$

That means M is an $\chi -$ endomorphism invariant module. □

3. CONCLUSION

The article has just provided some general results about $\chi -$ automorphism invariant modules satisfying the C-conditions. And stating the condition so that $\chi -$ automorphism invariant modules is $\chi -$ endomorphism invariant. In the case of class χ is a class of specific modules such as the class of injective modules, the class of all pure-injective modules, we will have the corresponding specific results as well known for injective modules, pure-injective modules. We guarantee that the results in the paper belong to us and are completely different from existing ones.

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TÓM TẮT

MÔ-ĐUN BẤT BIẾN DƯỚI CÁC TỰ ĐẲNG CẤU CỦA BAO TỔNG QUÁT

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Trong bài viết này, chúng tôi nêu điều kiện để một môđun $\chi -$ bất biến tự đồng cấu thỏa điều kiện C2 và môđun $\chi -$ bất biến tự đẳng cấu thỏa điều kiện C3. Cuối cùng, chúng tôi thảo luận khi nào một môđun $\chi -$ bất biến tự đẳng cấu là một môđun $\chi -$ bất biến tự đồng cấu.

Từ khóa: Bất biến tự đẳng cấu, bất biến tự đồng cấu, bao xạ ảnh, bao tổng quát.