STABILITY OF STRONG SOLUTIONS TO SET OPTIMIZATION PROBLEMS

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ABSTRACT

In this paper, we study stability for parametric set optimization problems. We first mention the concepts of solutions to such problems based on set less order relation between sets. Then, we introduce the converse property of set-valued mappings. Under suitable assumptions, the upper and lower semicontinuity of strong solution mappings are established. Our results are new and different from the existing ones in the literature.

Keywords: Parametric set optimization problem, set less order relation, converse property, upper semicontinuity, lower semicontinuity.

1. INTRODUCTION

In recent years, set-valued optimization problems have been investigated by many authors (see [1-3] and the references therein). These problems with set-valued objective functions are extensions of vector optimization problems [4, 5] and arise from many fields such as industrial transportation robots, economics or finance [6, 7]. In generally, there are two approaches of set-valued optimization problems: vector criterion and set criterion. In the first one, Gaydu *et al*. [3] looked for minimal elements of the union of all image sets, while in the second one, Gutierrez *et al*. [2] and Mao *et al.* [8] considered set optimization problems with efficient sets based on set less order relations between them. These order relations have been independently introduced by Young in [9], Nishnianidze in [10] and Kuroiwa in [11]. The solution concepts for set optimization problems are suitable for studying the robust vector optimization problems.

One of the most important issues in optimization theory is stability analysis. There are two main approaches of stability. Some authors study stability by using the concepts of solution convergences in the sense of Painlevé-Kuratowski or Hausdorff [2, 3] while another approach is investigating (semi)continuity of solution mappings. In [12], Anh *et al*. considered sufficient/necessary conditions of the semicontinuity/continuity for the solution mappings to quasi-equilibrium problems with variable cones. In [13], Xu and Li discussed the upper, lower semicontiniuity of u-minimal and weak u-minimal solution mappings to parametric set optimization problems. Recently, Mao *et al*. [8] studied semicontinuity of solution mappings to set optimization problems with parametric feasible sets by using improvement sets.

Motivated by the above observations, in this paper we study stability of the strong solutions to parametric set optimization problems. Under suitable assumptions, the upper and lower semicontinuity of strong solution mappings to these problems are established. Our results are new and different from the existing ones in the literature.

The rest of the paper is organized as follows. In Section 2, we introduce the concepts of strong solution to set optimization problems and recall some necessary results needed in the sequel concerning. The upper and lower stability results are presented in Section 3. Some concluding remarks are included in the last section, Section 4.

2. PRELIMINARIES

Throughout this paper, unless otherwise specified, we use the following notations. Let Λ, X, Y be normed spaces, C be a proper pointed solid convex closed cone in Y. Let $K : \Lambda \rightrightarrows X$ and $F : X \times \Lambda \rightrightarrows Y$ be set-valued mappings.

For any nonempty subsets A, B in Y, we define the set less order relation as follows:

$$A \leq B \Leftrightarrow A \subset B - C.$$

For any given $\lambda \in \Lambda$, we consider the following set optimization problem:

$$\text{(P) Minimize } F(x,\lambda)$$
$$\text{subject to } x \in K(\lambda)$$

Definition 2.1. For each $\lambda \in \Lambda$, an element $x_0 \in K(\lambda)$ is said to be a strong solution to (P) if and only if $F(x_0,\lambda) \leq F(x,\lambda)$ for all $x \in K(\lambda)$.

For each $\lambda \in \Lambda$, we denote the strong solution set of (P) by $S(\lambda)$.

Definition 2.2. Let Z, T be normed spaces. A set-valued mapping $G : Z \rightrightarrows T$ is said to be

(i) upper semicontinuous (usc, shortly) at x_0 if for any open superset W of $G(x_0)$, there exists a neighborhood V of x_0 such that $G(V) \subset W$;

(ii) lower semicontinuous (lsc, shortly) at x_0 if for any open subset W of T with $G(x_0) \cap W \neq \emptyset$, there exists a neighborhood V of x_0 such that $G(x) \cap W \neq \emptyset, \forall x \in V$;

(iii) continuous at x_0 if it is both upper semicontinuous and lower semicontinuous at x_0.

Definition 2.3. Let Z, T and G be as in Definition 2.2. G is said to be

(i) Hausdorff upper semicontinuous (H-usc, shortly) at x_0 if for any neighborhood B of the origin in T, there exists a neighborhood N of x_0 such that $G(x) \subset G(x_0) + B$ for every $x \in N$;

(ii) Hausdorff lower semicontinuous (H-lsc, shortly) at x_0 if for any neighborhood B of the origin in T, there exists a neighborhood N of x_0 such that $G(x_0) \subset G(x) + B$ for every $x \in N$;

(iii) Hausdorff continuous at x_0 if it is both Hausdorff upper semicontinuous and Hausdorff lower semicontinuous at x_0.

We say that G satisfies a certain property on a subset $M \subset X$ if G satisfies it at every point of M.

Lemma 2.1. ([14]) Let $G : Z \rightrightarrows T$ be a set-valued mapping between two normed spaces. Then the following assertions hold.

(i) If $G(\bar{z})$ is compact, then G is usc at $\bar{z}$ if and only if for any sequence $\{z_n\} \subset Z$ converging to $\bar{z}$ and $t_n \in G(z_n)$, there is a subsequence $\{t_{n_k}\}$ that converges to some $\bar{t} \in G(\bar{z})$;

(ii) G is lsc at $\bar{z}$ if and only if, for any sequence $\{z_n\} \subset Z$ converging to $\bar{z}$ and $\bar{t} \in G(\bar{z})$, there exists a sequence $\{t_n\}$, $t_n \in G(z_n)$ such that $t_n \to \bar{t}$.

Lemma 2.2. ([15]) Let G be as in Lemma 2.1. Then the following assertions hold.

(i) If G is usc, then G is H-usc;

(ii) If G is H-usc with compact-valued, then G is usc;

(iii) If G is H-lsc, then G is lsc;

(iv) If G is lsc with compact-valued, then G is H-lsc.

In the following sections, we always assume that $S(\lambda)$ is nonempty for all λ in a neighborhood of the reference point.

3. MAIN RESULTS

Lemma 3.1. **S**uppose that for any given $\bar{\lambda} \in \Lambda$,

(i) $F(\cdot, \bar{\lambda})$ is Hausdorff lower semicontinuous with compact values on *X*;

(ii) $K(\bar{\lambda})$ is compact.

Then, $S(\bar{\lambda})$ is a compact set.

Proof. We first show that $S(\bar{\lambda})$ is a closed set. Let an arbitrary sequence $\{x_n\} \subset S(\bar{\lambda})$ such that $x_n \to x_0$. Since $x_n \in K(\bar{\lambda})$ and $K(\bar{\lambda})$ is closed, we obtain $x_0 \in K(\bar{\lambda})$. Let an arbitrary point $y \in K(\bar{\lambda})$, we have

$$F(x_n, \bar{\lambda}) \subset F(y, \bar{\lambda}) - C. \tag{3.1}$$

For any neighborhood B of the origin in Y, it follows from (3.1) and the Hausdorff lower semicontinuity of $F(\cdot, \bar{\lambda})$ that, for n large enough

$$F(x_0, \bar{\lambda}) \subset F(x_n, \bar{\lambda}) + B \subset F(y, \bar{\lambda}) - C + B. \tag{3.2}$$

Noting that $F(y, \bar{\lambda}) - C$ is closed, we conclude that $F(x_0, \bar{\lambda}) \subset F(y, \bar{\lambda}) - C$. The last inclusion implies that $x_0 \in S(\bar{\lambda})$. Thus, $S(\bar{\lambda})$ is a closed set. Moreover, since $K(\bar{\lambda})$ is compact and $S(\bar{\lambda}) \in K(\bar{\lambda})$, we have $S(\bar{\lambda})$ is a compact set.

□

Theorem 3.1. Suppose that the following assumptions hold:

(i) F is Hausdorff continuous with compact values on $X \times \Lambda$;

(ii) K is continuous with compact values on Λ.

Then, the solution mapping S is usc with compact-valued on Λ.

Proof. By Lemma 3.1, S is compact-valued on Λ. Suppose in the contrary that S is not usc at $\lambda_0 \in \Lambda$. Then there exist an open subset V of $S(\lambda_0)$, two sequences $\{x_n\} \subset X$ and $\{\lambda_n\} \subset \Lambda$ with $\lambda_n \to \lambda_0$ such that

$$x_n \in S(\lambda_n) \setminus V, \ \forall n \in \mathbb{N} . \tag{3.3}$$

Since K is usc with compact values, without loss of generality, we can assume that $x_n \to x_0 \in K(\lambda_0)$. We claim that $x_0 \in S(\lambda_0)$. Indeed, if it is not true, then there exists $y_0 \in K(\lambda_0)$ such that

$$F(x_0, \lambda_0) \not\subset F(y_0, \lambda_0) - C . \tag{3.4}$$

From the lower semicontinuity of K, there exists a sequence $\{y_n\}$ with $y_n \in K(\lambda_n)$ such that $y_n \to y_0$. For any neighborhood U of the origin θ_Y in Y, there is a balanced neighborhood U' of θ_Y such that $U' + U' \subset U$. It follows from the Hausdorff upper semicontinuity of F that for n sufficiently large,

$$F(y_n, \lambda_n) \subset F(y_0, \lambda_0) + U' . \tag{3.5}$$

Since $x_n \in S(\lambda_n)$, we obtain

$$F(x_n, \lambda_n) \subset F(y_n, \lambda_n) - C, \ \forall n \in \mathbb{N} . \tag{3.6}$$

Taking into account (3.5), (3.6) with the fact that F is H-lsc,

$$F(x_0, \lambda_0) \subset F(x_n, \lambda_n) + U' \subset F(y_0, \lambda_0) + U' - C + U' \subset F(y_0, \lambda_0) - C + U .$$

Noting that $F(y_0, \lambda_0) - C$ is closed and U is arbitrary, we conclude that $F(x_0, \lambda_0) \subset F(y_0, \lambda_0) - C$, which contradicts (3.4). Thus, $x_0 \in S(\lambda_0) \subset V$. Therefore, $x_n \in V$ for n large enough, which is a contradiction with (3.3). This brings the proof to its end. □

The following example illustrates the essentialness of the Hausdorff continuity of F.

Example 3.1. Let $X = \Lambda \equiv \mathbb{R}, Y \equiv \mathbb{R}^2, C = \mathbb{R}^2_+$, $K(\lambda) = [1, 1 + \lambda^2]$ and

$$F(x, \lambda) = \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - 1)^2 + (y_2 - 1)^2 \le 1\}, x < 2; \\ \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 \le 1\}, x \ge 2. \end{cases}$$

It is easy to check that all assumptions in Theorem 3.1 are satisfied, excepted the Hausdorff continuity of F. From the direct calculation, we have

$$S(\lambda) = \begin{cases} [1, 1 + \lambda^2], \lambda \in (-1, 1); \\ [2, 1 + \lambda^2], \lambda \notin (-1, 1). \end{cases}$$

Clearly, S is not usc. The reason is that F is not Hausdorff continuous.

Remark 3.1. Although the imposed assumption of Hausdorff continuity of F in Theorem 3.1 is essential, we can also replace it with the continuous condition of F. Indeed, since F is lower semicontinuous with compact-valued, by Lemma 2.2, F is Hausdorff lower semicontinuous. Similarly, the upper semicontinuity of F implies the Hausdorff upper semicontinuity of F.

Picking up the idea of Xu and Li [13], we introduce the definition of the converse property of set-valued mappings as follows:

Definition 3.1. A set-valued mapping $F: X \times \Lambda \rightrightarrows Y$ is said to have converse property at (x_0, λ_0) with respect to $y_0 \in X$ iff, either $F(y_0, \lambda_0) \not\le F(x_0, \lambda_0)$ or for any sequences $\{x_n\}$, $\{y_n\}$ and $\{\lambda_n\}$ with $x_n \to x_0$, $y_n \to y_0$, $\lambda_n \to \lambda_0$, there exists $m \in \mathbb{N}$ such that $F(y_m, \lambda_m) \le F(x_m, \lambda_m)$.

Example 3.2. Let $X = \Lambda \equiv \mathbb{R}, Y \equiv \mathbb{R}^2, C = \mathbb{R}^2_+$ and

$$F(x, \lambda) = \left\{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - x)^2 + (y_2 - x)^2 = 1\right\}$$

It is easy to check that *F* has converse property at every $(x, \lambda) \in X \times \Lambda$ with respect to each $y \in X, y \ne x$.

The following Theorem illustrates the lower semicontinuity of the solution mapping.

Theorem 3.2. Suppose that the following assumptions hold:

(i) *K* is continuous with compact values on Λ;

(ii) *F* has converse property on $X \times \Lambda$ with respect to each $y \in X$.

Then, the solution mapping *S* is lsc on Λ.

Proof. Suppose that *S* is not lsc at λ_0, then there exist an open subset *W* with $S(\lambda_0) \cap W \ne \emptyset$, a sequence $\{\lambda_n\}$ satisfying $\lambda_n \to \lambda_0$ such that

$$S(\lambda_n) \cap W = \emptyset, \ \forall n \in \mathbb{N}. \tag{3.7}$$

Taking any fixed point $x_0 \in S(\lambda_0) \cap W$, since $x_0 \in K(\lambda_0)$ and *K* is lsc at λ_0, there exists a sequence $\{x_n\}$ with $x_n \in K(\lambda_n)$ such that $x_n \to x_0$. For each $n \in \mathbb{N}$, let the arbitrary point $y_n \in K(\lambda_n)$. Since *K* is usc with compact values, without loss of generality, we can assume that $y_n \to y_0 \in K(\lambda_0)$. Because $x_0 \in S(\lambda_0)$, we obtain

$$F(x_0, \lambda_0) \subset F(y_0, \lambda_0) - C$$

It follows from (ii) that there exist three subsequences $\{x_{n_k}\}$ of $\{x_n\}$, $\{y_{n_k}\}$ of $\{y_n\}$ and $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ such that $F(x_{n_k}, \lambda_{n_k}) \subset F(y_{n_k}, \lambda_{n_k}) - C$ for all *k*. This means that $x_{n_k} \in S(\lambda_{n_k})$. Noting that $x_0 \in W$, we obtain $x_{n_k} \in W$ for *k* large enough. This gives a contradiction with (3.7). Therefore, *S* is lsc at λ_0 and the proof is completed. □

The following example shows that assumption (ii) in Theorem 3.2 is essential.

Example 3.3. Let $X = \Lambda \equiv \mathbb{R}, Y \equiv \mathbb{R}^2, C = \mathbb{R}^2_+$, $K(\lambda) = [0, \lambda^2 + 1]$ and

$$F(x, \lambda) = \begin{cases} \{(t_1, t_2) \in \mathbb{R}^2 : x \le t_1^2 + t_2^2 \le 1\} \cup \{(0,0)\}, & \lambda \ne 0; \\ \{(t_1, t_2) \in \mathbb{R}^2 : t_1^2 + t_2^2 \le 1\}, & \lambda = 0. \end{cases}$$

It is easy to check that the assumption (i) in Theorem 3.2 is satisfied. By direct calculating, we obtain

$$S(\lambda)=\begin{cases}[0,1], & \lambda=0;\\ (1,\lambda^2+1], & \lambda\neq 0.\end{cases}$$

Clearly, S is not lsc at $\lambda_0=0$. The reason is that the assumption (ii) is violated. Indeed, let $x_0=1, y_0=\lambda_0=0$, we have $F(y_0,\lambda_0)\le F(x_0,\lambda_0)$. Let $x_n=1+\dfrac{1}{n}, y_n=\dfrac{1}{n}$ for all $n\ge 1$ and a sequence $\{\lambda_n\}$ with $\lambda_n\to\lambda_o, \lambda_n\neq\lambda_o$. Then, $F(x_n,\lambda_n)=\{(0,0)\}$ and $F(y_n,\lambda_n)=\{(t_1,t_2)\in\mathbb{R}^2 : \dfrac{1}{n}\le t_1^2+t_2^2\le 1\}\cup\{(0,0)\}$. Thus, we can not find any $m\in\mathbb{N}$ such that $F(y_m,\lambda_m)\le F(x_m,\lambda_m)$.

From Lemma 3.1, Theorem 3.1, Theorem 3.2 and Remark 3.1, we obtain the following Corollary.

Corollary 3.1. Suppose that the following assumptions hold:

(i) *F* is continuous with compact values on $X\times\Lambda$;

(ii) *K* is continuous with compact values on Λ;

(iii) *F* has converse property on $X\times\Lambda$ with respect to each $y\in X$.

Then, the solution mapping *S* is continuous with compact-valued on Λ.

Remark 3.2. Taking into account Lemma 2.2, we also conclude that if all assumptions in Corollary 3.1 are satisfied, then the solution mapping *S* is Hausdorff continuous with compact-valued on Λ.

4. CONCLUSION

In this paper, we focus our attention on the stability of strong solutions to parametric set optimization problems. This kind of solutions is different from the one that based on improvement sets. To the best of our knowledge, the (semi)continuity of strong solution mappings to such problems in the set criterion is not available in the literature. So, our results are new and complement some previously known ones.

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TÓM TẮT

TÍNH ỔN ĐỊNH CỦA NGHIỆM MẠNH CỦA BÀI TOÁN TỐI ƯU TẬP

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Bài báo này nghiên cứu tính ổn định của bài toán tối ưu tập chứa tham số. Trước hết, tác giả đề cập các ý tưởng về nghiệm của bài toán dựa trên quan hệ thứ tự giữa các tập hợp, sau đó giới thiệu tính chất ngược của ánh xạ đa trị. Dưới các giả thiết thích hợp, tính nửa liên tục trên và nửa liên tục dưới của ánh xạ nghiệm được thiết lập. Các kết quả trong nghiên cứu này là mới và khác với các kết quả đã có.

Từ khóa: Bài toán tối ưu tập chứa tham số, quan hệ thứ tự giữa các tập hợp, tính chất ngược, tính nửa liên tục trên, tính nửa liên tục dưới.