Limit and shakedown analysis of kirchhoff-love plates under uncertainty of strength

Tran Ngoc Trinh

Abstract

A new formulation to calculate the shakedown limit load of Kirchhoff plates under stochastic conditions of strength is developed. Direct structural reliability design by chance constrained programming is based on the prescribed failure probabilities, which is an effective approach of stochastic programming if it can be formulated as an equivalent deterministic optimization problem.

We restrict uncertainty to strength, the loading is still deterministic. A new formulation is derived in case of random strength with lognormal distribution. Upper bound and lower bound shakedown load factors are calculated simultaneously by a dual algorithm.

Key words: *Kirchhoff Plate, Limit Analysis, Shakedown Analysis, Primal Dual Programming, Stochastic Programming, Chance Constrained Programming*

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1. Introduction

Plates are very important structural elements, that are widely used in civil and mechanical engineering. The common examples of plates are slabs in civil engineering structures, bearing plate under columns, many parts of mechanical components. In this paper, we consider bending of such plates subjected to lateral loads. The bending stiffness of a plate depends on the cube of its thickness. The classical theory divides plates into following groups: thin plates with small deflection, thin plates with large deflections, and thick plates.

The thin plate assumptions on which A.E.H. Love based his plate theory were proposed by Gustav R. Kirchhoff [1]. Consequently, thin plates with small deflections theory are called Kirchhoff-Love plate or Kirchhoff plate for short. This theory is suitable for plates with length of span at least 10 times the thickness. Many engineering problems lie in the above category and satisfactory results are obtained by the classical thin plates theory.

Limit analysis of plates in bending has been studied analytically and numerically [8]–[16]. Due to limitations of analytical methods, alternative numerical approaches such as finite element methods (FEM), meshfree methods or isogeometric analysis (IGA) have been developed.

Limit and shakedown analysis state problems as a mathematical programming. If the strength of a plate is a random variable, we may consider the problems as a stochastic programming problem. Many models of stochastic programming have been proposed such as approximate polyhedral dynamic programming [16-18], measurement-based optimization [19], worst-case and distributional robustness analysis [20], Cost horizons and certainty equivalence [21] and chance constrained optimization (CCOPT) [22]. In this paper the CCOPT approach is used to treat the problem of shakedown analysis of plate under uncertainty condition of strength. If the thickness deterministic and the yield stress is distributed normally or lognormaly a deterministic equivalent formulation can be derived which allows a most effective numerical solution for prescribed reliability of the structure.

2. Static approach with chance constrained programming

Consider a convex polyhedral load domain D and a special loading path consisting of all load vertices $\hat{P}_k\,(k=1,...,m)$ of $\mathcal{L}$. The total moment $\mathbf{m}(\mathbf{x},t)$ at a point $\mathbf{x}\in\Omega$ of the considered plate at time t is decomposed into an elastic reference moment $\mathbf{m}^E(\mathbf{x},t)$ and a residual moment $\boldsymbol{\rho}(\mathrm{x},t)$. Here, $\mathbf{m}^E(\mathbf{x},t)$ denotes the fictitious moment that would appear in a purely elastic reference structure $\mathscr{P}^E$ under the same loading conditions as the original structure, and $\boldsymbol{\rho}(\mathrm{x},t)$ represents a residual moment field that is induced by the evolution of plastic strains

$$\mathbf{m}(\mathbf{x},t)=\mathbf{m}^E(\mathbf{x},t)+\boldsymbol{\rho}(\mathbf{x},t) \qquad (2.1)$$

According to Melan's static shakedown theorem the structure will shakedown, if there exists a time-independent residual moment field $\bar{\boldsymbol{\rho}}(\mathbf{x})$ such that the yield condition is satisfied for any loading path at any time t and in any point $\mathbf{x}$ of the plate. Based on this lower bound theorem, for a plate made up of elastic perfectly plastic material, the maximum enlarging of the load domain allowing still for shakedown, characterized by load factor α^- that can be obtained by solving the following optimization problem (in FEM form)

$$\alpha^- = \max \alpha$$
$$\text{s.t.:} \begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \overline{\boldsymbol{\rho}}_i = 0 \\ f(\alpha \mathbf{m}_{ik}^E + \overline{\boldsymbol{\rho}}_i) \le m_0 \quad \forall i = \overline{1,NG} \quad \forall k = \overline{1,m} \end{cases} \tag{2.2}$$

in which $\mathbf{B}_i$ is the deformation matrix, w_i is integration weight at Gauss point i and NG denotes the total number of Gauss points of the structure.

Let us now consider the situation that the plastic moment of the plate is not given but must be modelled $m_0 = m_0(\omega)$ a random variable on a certain probability space. Under uncertainty, the inequalities in (2.2) are not always satisfied, the probability of the i^{th} yield condition is required to be satisfied is greater than some reliability level ψ_i. Problem (2.2) becomes a chance constraint stochastic program:

$$\alpha^- = \max \alpha$$
$$s.t.: \begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \overline{\boldsymbol{\rho}}_i = \mathbf{0} \quad \text{in } \Omega \\ \text{Prob}\left[f(\alpha \mathbf{m}_{ik}^E + \overline{\boldsymbol{\rho}}_i) - m_{0i}(\omega) \le 0 \right] \ge \psi_i \\ \qquad \forall i = \overline{1,NG} \quad \forall k = \overline{1,m} \end{cases} \tag{2.3}$$

Let the plastic moment $m_i(\omega)$ be distributed normally with mean μ_i and standard deviation σ_i, in short $m_i(\omega) \sim \mathcal{N}(\mu_i, \sigma_i^2)$. Based on the methodology of chance constrained programming, problem (2.3) can be converted into a equivalent deterministic program as shown in [10,11]:

$$\alpha^- = \max \alpha$$
$$\text{s.t.:} \begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \overline{\boldsymbol{\rho}}_i = 0 \quad \text{in } \Omega \\ f(\alpha \mathbf{m}_{ik}^E + \overline{\boldsymbol{\rho}}_i) \le \mu_i - \kappa \sigma_i \quad \forall i = \overline{1,NG} \quad \forall k = \overline{1,m} \end{cases} \tag{2.4}$$

where $\kappa = \Phi^{-1}(\psi_i)$ is the inverse normal cumulative distribution function (normal quantile function) of the plastic moment at Gauss point i.

Let the plastic moment $m_i(\omega)$ be distributed lognormally. This means that $\ln[m_i(\omega)]$ is distributed normally with mean μ_i and standard deviation σ_i, in short $\ln[m_i(\omega)] \sim \mathcal{N}(\mu_i, \sigma_i^2)$. The stochastic program (2.3) can be relaxed into an equivalent deterministic optimization problem after some transformations [2,3]:

$$\alpha^- = \max \alpha$$
$$\text{s.t.:} \begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \overline{\boldsymbol{\rho}}_i = 0 \quad \text{in } \Omega \\ f(\alpha \mathbf{m}_{ik}^E + \overline{\boldsymbol{\rho}}_i) \le e^{\mu_i - \kappa \sigma_i} \quad \forall i = \overline{1,NG} \quad \forall k = \overline{1,m} \end{cases} \tag{2.5}$$

3. Kinematic approach with chance constrained programming

An upper bound to the shakedown limit of plates can be obtained using the kinematic shakedown theorem. In this investigation, we use von Mises yield criterion.The plastic dissipation power of the plate domain Ω can be written in form of curvature vector χ

$$\dot{D}_{\text{int}}(\dot{\chi}) = m_0 \int_{\Omega} \sqrt{\dot{\chi}^T \mathbf{Q} \dot{\chi}}\, d\Omega \tag{3.1}$$

In which

$$\mathbf{Q} = \frac{1}{3} \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.2}$$

m_0 is the plastic limit moment per unit length of a plate section.

We introduce here an admissible cycle of a plastic curvature field $\Delta\boldsymbol{\chi}^p$. At each load vertex, the plastic curvature rate may not necessarily be compatible at each instant during the time cycle, but the plastic curvature accumulation over the cycle is required to be kinematically compatible such that

$$\Delta\boldsymbol{\chi}^p = \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}^p = \nabla^2 \dot{w} \tag{3.3}$$

Based on the above statements and the mathematical programming theory, an upper bound of the shakedown load factor can be found by solving the following convex nonlinear programming:

$$\alpha^+ = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i m_0 \sqrt{\dot{\boldsymbol{\chi}}_{ik}^T \mathbf{Q} \dot{\boldsymbol{\chi}}_{ik}}$$
$$\text{s.t.:} \begin{cases} \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} \quad \forall i = \overline{1,NG} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\chi}}_{ik}^T \mathbf{m}_{ik}^E = 1 \end{cases} \tag{3.5}$$

If the yield stress of the material is random, then the plastic moment is an uncertain quantity and the objective function of (3.5) is a stochastic variable. Firstly, we must properly define the minimum of a random function. This can be done in such a way that one looks for a minimum lower bound of the objective function under the constraint that the probability of violation of that bound is prescribed in [22]

$$\min \eta$$
$$\text{s.t.:} \begin{cases} \text{Prob}\left(\sum_{k=1}^{m} \sum_{i=1}^{NG} w_i m_0(\omega) \sqrt{\dot{\boldsymbol{\chi}}_{ik}^T \mathbf{Q} \dot{\boldsymbol{\chi}}_{ik}} \ge \eta \right) = \psi \\ \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} \quad \forall i = \overline{1,NG} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\chi}}_{ik}^T \mathbf{m}_{ik}^E = 1 \end{cases} \tag{3.6}$$

Problem (3.6) is a stochastic program, it can be converted into an equivalent deterministic program by using a chance constrained program technique [10,11].

$$\alpha^+ = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \left(\mu_i - \kappa \sigma_i \right) \sqrt{\dot{\boldsymbol{\chi}}_{ik}^T \mathbf{Q} \dot{\boldsymbol{\chi}}_{ik}}$$
$$\text{s.t.:} \begin{cases} \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} \quad \forall i = \overline{1,NG} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\chi}}_{ik}^T \mathbf{m}_{ik}^E = 1 \end{cases} \tag{3.7}$$

In case of a lognormal distribution of strength, the stochastic problem (3.6) can be converted into the equivalent deterministic program (3.8) by using the duality property:

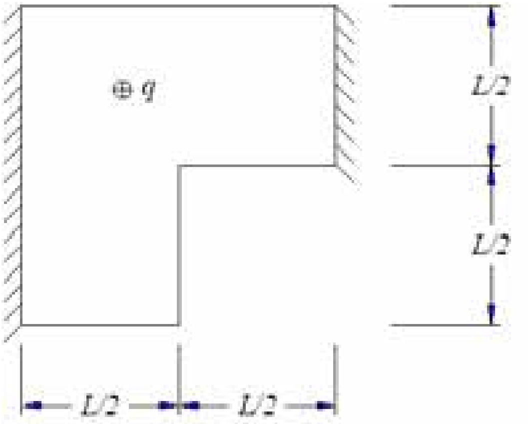

Figure 1: L-shape plate loaded by a uniform pressure

$$\alpha^{+} = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i e^{(\mu_i - \kappa\sigma_i)} \sqrt{\dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{Q} \dot{\boldsymbol{\chi}}_{ik}}$$

$$\text{s.t.}: \begin{cases} \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} \qquad \forall i = \overline{1, NG} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{m}_{ik}^{E} = 1 \end{cases} \tag{3.8}$$

4. A dual algorithm for shakedown analysis of Kirchhoff plate

For the sake of simplicity, we set some new notations:

$$\dot{\mathbf{k}}_{ik} = w_i \mathbf{Q}^{1/2} \dot{\boldsymbol{\chi}}_{ik}, \ \mathbf{t}_{ik} = \left(\mathbf{Q}^{-1/2}\right)^{T} \mathbf{m}_{ik}^{E}, \ \hat{\mathbf{B}}_i = w_i \mathbf{Q}^{1/2} \mathbf{B}_i, \tag{4.1}$$

where

$$\mathbf{Q}^{1/2} \mathbf{Q}^{-1/2} = \mathbf{I}, \ \mathbf{Q} = \left(\mathbf{Q}^{1/2}\right)^{T} \mathbf{Q}^{1/2}. \tag{4.2}$$

By substituting (4.1) into (3.8) one obtains a simplified version for the upper bound of the shakedown limit load (primal problem)

$$\alpha^{+} = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} e^{(\mu_i - \kappa\sigma_i)} \sqrt{\dot{\mathbf{k}}_{ik}^{T} \dot{\mathbf{k}}_{ik}}$$

$$\text{s.t.}: \begin{cases} \sum_{k=1}^{m} \dot{\mathbf{k}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} \qquad \forall i = \overline{1, NG} \\ \sum_{i=1}^{NG} \sum_{k=1}^{m} \dot{\mathbf{k}}_{ik}^{T} \mathbf{t}_{ik} - 1 = 0 \end{cases} \tag{4.3}$$

Using a penalty method to eliminate the first constraint in (4.3) leads to the penalty function

Table 1: Limit load factor in comparison for case of simple supported plate $\left(\dfrac{m_0}{qL^2}\right)$

Author	Lower bound	Upper bound	
Le et al. [7]	--	6.219	
Tran et al.[15]	6.044	6.173	deterministic
	6.022	6.190	
Present	3.785	3.882	normal
	4.135	4.242	lognormal

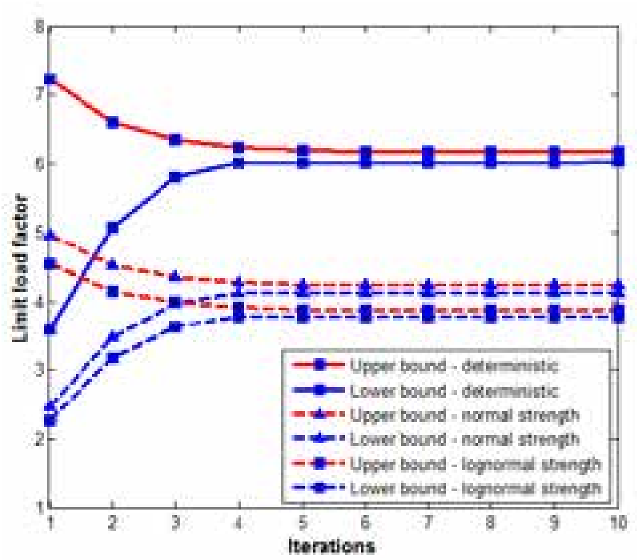

Figure 2: Convergence of limit load factors

$$F_p = \sum_{i=1}^{NG} \left\{ \begin{array}{l} \sum_{k=1}^{m} e^{(\mu_i - \kappa\sigma_i)} \sqrt{\dot{\mathbf{k}}_{ik}^{T} \dot{\mathbf{k}}_{ik} + \varepsilon_0^2} + \\ + \dfrac{c}{2} \left(\sum_{k=1}^{m} \dot{\mathbf{k}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} \right)^{T} \left(\sum_{k=1}^{m} \dot{\mathbf{k}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} \right) \end{array} \right\} \tag{4.4}$$

where $\underline{c}$ is a penalty parameter such that $\underline{c>>1}$. The corresponding Lagrange function of (4.4) is

$$L = F_p - \alpha \left(\sum_{i=1}^{NG} \sum_{k=1}^{m} \dot{\mathbf{k}}_{ik}^{T} \mathbf{t}_{ik} - 1 \right) \tag{4.5}$$

By employing Newton method to solve the Karush-Kuhn-Tucker (KKT) conditions of the Lagrange function (4.5), we have the incremental vectors of nodal variables $\dot{\mathbf{u}}$, curvature rate $\dot{\mathbf{k}}_{ik}$ and $\boldsymbol{\beta}_i$ as follows :

$$\begin{aligned} & d\dot{\mathbf{u}} = d\dot{\mathbf{u}}_1 + d\dot{\mathbf{u}}_2 \left(\alpha + d\alpha\right) \\ & d\dot{\mathbf{k}}_{ik} = (d\dot{\mathbf{k}}_{ik})_1 + (d\dot{\mathbf{k}}_{ik})_2 \left(\alpha + d\alpha\right) \\ & d\boldsymbol{\beta}_i = (d\boldsymbol{\beta}_i)_1 + (d\boldsymbol{\beta}_i)_2 \left(\alpha + d\alpha\right) \end{aligned} \tag{4.6}$$

$$\left(\alpha + d\alpha\right) = \frac{1 - \sum_{i=1}^{NG} \sum_{k=1}^{m} \mathbf{t}_{ik}^{T} \left[\dot{\mathbf{k}}_{ik} + \left(d\dot{\mathbf{k}}_{ik}\right)_1 \right]}{\sum_{i=1}^{NG} \sum_{k=1}^{m} \mathbf{t}_{ik}^{T} \left(d\dot{\mathbf{k}}_{ik}\right)_2} \tag{4.7}$$

The vectors $\mathrm{d}\dot{\mathbf{q}}, \mathrm{d}\dot{\mathbf{k}}_{ik}, \mathrm{d}\boldsymbol{\beta}_i$. and $\mathrm{d}\alpha$ are actually Newton directions, which assure that a suitable step along them will lead to a decrease of the objective function of the primal problem (3.8) and to an increase of the objective function of the objective function of the dual problem (2.5). Based on (4.6 - 4.7) we can update the vectors of $\dot{\mathbf{q}}, \dot{\mathbf{k}}_{ik}$, $\boldsymbol{\beta}_i$ and α. The dual algorithm for limit and shakedown analysis is presented in detail in [2-6].

5. Numerical examples

We investigate a L-shape plate subjected to uniform pressure. Length $L=10m$, plate thickness $t=0{,}1m$, the mean value of yield stress $E(\sigma_i)=250\mathrm{MPa}$ and the standard deviation $\sigma = 0.1E(\sigma_0)$ .The reliability level is assumed $\psi = 0.9999$. Let

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reduction at the age of 3 and 7 days is significantly lower than that of the 28 days.

Therefore, air-entraining admixture should be used at the minimum content, depending on the purpose of strength and bulk density as well as the ease of construction of fresh concrete. In this study, the most reasonable AD content is at 0.02% by weight of the binder.

And it is worth noting that the method as presented can be used to evaluate the relative stratification reduction effect of AD for lightweight concrete mixes./.

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us calculate limit load fators. This example is investigated in [5-6] for case of normal distribution of strength.

In this analysis, the plate is modelled by 768 DKQ (discrete kirchhoff quadrilateral) elements. Figure 2 shows the convergence of the upper bound and lower bounds for simple supported case. Table 1 shows the results in comparison with Le [7] and Tran [15]./.

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