CONES GENERATED BY SEMI-INFINITE SYSTEMS AND THEIR APPLICATIONS ON OPTIMIZATION

Huynh Thi Kim Loan[1*] and Vo Duc Thinh[2]

[1]*ThuyLoi University - Southern Campus*

[2]*Department of Mathemetics Teacher Education, Dong Thap University*

[*]*Corresponding author: loanhtk@tlu.edu.vn*

Article history

Received: 05/08/2020; Received in revised form: 25/09/2020; Accepted: 28/09/2020

Abstract

In this paper, we first introduce the cones generated by semi-infinite systems. Then we use approaches of the semi-infinite programming to obtain formulas of normal cones and tangent cones to those cones. Thereby, we use obtained results in providing optimality conditions for conic optimization problems. The obtained results in the paper are new and they are generalized from some existing ones in the literature.

Keywords: *Cone generated by semi-infinite system, normal cone, optimality condition, tangent cone.*

VỀ NHỮNG NÓN SINH BỞI CÁC HỆ NỬA VÔ HẠN VÀ NHỮNG ÁP DỤNG CỦA CHÚNG VÀO TỐI ƯU

Huỳnh Thị Kim Loan[1*] và Võ Đức Thịnh[2]

[1]*Trường Đại học Thuỷ Lợi – Phân hiệu*

[2]*Khoa Sư phạm Toán học, Trường Đại học Đồng Tháp*

[*]*Tác giả liên hệ: loanhtk@tlu.edu.vn*

Lịch sử bài báo

Ngày nhận: 05/08/2020; Ngày nhận chỉnh sửa: 25/09/2020; Ngày duyệt đăng: 28/09/2020

Tóm tắt

Trong bài báo này, trước tiên chúng tôi giới thiệu những nón được sinh bởi những hệ nửa vô hạn. Sau đó, chúng tôi sử dụng cách tiếp cận cho hệ nửa vô hạn để thiết lập công thức nón pháp tuyến và nón tiếp tuyến cho nón mới này. Cuối cùng, chúng tôi sử dụng các kết quả đạt được để xây dựng điều kiện tối ưu cho bài toán tối ưu với ràng buộc nón. Các kết quả của trong bài báo này là mới và là sự tổng quát của các kết quả đã có trong các tài liệu tham khảo.

Từ khoá: *Điều kiện tối ưu, nón pháp tuyến, nón sinh bởi hệ nửa vô hạn, nón tiếp tuyến.*

1. Introduction

The second-order cone programming (SOCP) plays an important role in the optimization theory and has interested many authors (Alizadeh, F. and Goldfarb, D., 2003; Bonnans, J.F. and Ramírez, H.C., 2005; Hang, N.T.V. *et al.*, 2020; Liu, Y.J. and Zhang, L.W., 2008; Mordukhovich, B.S. *et al.*, 2016; Outrata, J.V. and Ramírez, H., 2011). The second order cone in $\mathbb{R}^{n+1}$ is defined by

$$\mathcal{K} := \{(x_0, x_r) \geq \| x_r \|\}, \tag{1.1}$$

where $\| x \|$ is Eucliean norm in $\mathbb{R}^n$.

The results on optimality conditions for (SOCP) and stability analysis of the solution set to (SOCP) were studied (see Alizadeh, F. and Goldfarb, D., 2003; Bonnans, J.F. and Ramírez, H.C., 2005; Hang, N.T.V. *et al.*, 2020; Mordukhovich, B.S. *et al.*, 2016; Outrata, J.V. and Ramírez, H., 2011; and references therein). Specifically, some results on first/second order tangent cones and optimality conditions for (SOCP) were studied in detail by Bonnans, J.F. and Ramírez, H.C. (2005). An important application of (SOCP) is to solve the problem of finding a maximum likelihood (ML) estimate of the parameter vector x can be expressed as (cf. Boyd, S. and Vandenberghe, L., 2004).

$$\max \quad l(x) \tag{1.2}$$
$$\text{subject to } x \in C,$$

where $x \in C$ gives the prior information or other constraints on the parameter vector x. In this optimization problem, the vector $x \in \mathbb{R}^n$ (which is the parameter in the probability density) is the variable, and the vector $y \in \mathbb{R}^m$ (which is the observed sample) is a problem parameter. Common cases of the problem (1.2) are given in the following forms:

(i) ML estimation for Gaussian noise densities:

$$\max\left(-\left(\frac{m}{2}\right)\log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \| Ax - y \|_2^2\right), \tag{1.3}$$

where m, σ, y are given and A is the matrix with rows $a_1^{\top}, \ldots, a_m^{\top}$. Therefore the ML estimate of x is $\bar{x} \in \operatorname{argmin}_x \| Ax - y \|_2^2$, the solution of a least-squares approximation problem.

(ii) ML estimation for Laplacian noise densities:

$$\max\left(-\left(\frac{m}{2}\right)\log\left(\frac{1}{2a}\right) - \frac{1}{a} \| Ax - y \|_1\right), \tag{1.4}$$

where $a > 0$ is given. In this case, the ML estimate is $\hat{x} \in \operatorname{argmin}_x \| Ax - y \|_1$, the solution of the l_1-norm approximation problem.

Recently, many authors are interested in generalization of the second-order cones, namely, Glineur, F. and Terlaky, T. (2004); Gotoh, J. and Uryasey, S. (2015); Vinel, A. and Krokhmal, P. (2014a); Vinel, A. and Krokhmal, P. (2014b); Xue, G. and Ye, Y. (2000) considered p-order cones ($p \in [1, \infty]$), Ferrerira, O.P. and Németh S.Z. (2018); Németh, S.Z. and Xie, L. (2018); Németh, S.Z. *et al.* (2020); Sznajder, R. (2016) studied extended second order cones while Chang, Y.L. *et al.* (2013); Thinh, V.D. *et al.* (2020); Zhou, J. and Chen, J.S. (2013); Zhou, J. and Chen, J.S. (2015); Zhou, J. *et al.* (2015); Zhou, J. and Chen, J.S. (2017); Zhou, J. *et al.* (2017) are interested in circular cones. Some results on the projector onto those cones were presented. However, results on first/second order tangent cones and normal cone as well as first/second order optimality conditions for optimization problems under the second order cone/extended second order cones /circular cones constraints have not been provided in those papers.

In this paper, we first introduce new cones which are generated by semi-infinite systems. Then we will show that those cones are the generalizations of p-order cones, extended second order cones and circular cones. Thereby, we use approachs of semi-infinite programming to obtain exact formulas of normal and tangent cones to those new cones which will be applied on optimization problems. Our obtained results in the paper are new and they are generalized from some existing ones in the literature.

2. Preliminaries

Throughout the paper, we convention that $\frac{\mathbf{1}}{\infty} = \mathbf{0}$. The inner product in a finite-dimensional space $\mathbb{R}^{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N} := \{\mathbf{1}, \mathbf{2}, \ldots\}$, is defined by $\langle \boldsymbol{x}, \boldsymbol{y}\rangle := \boldsymbol{x}^\top \boldsymbol{y}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{\boldsymbol{n}}$. In $(\mathbb{R}^{\boldsymbol{n}}, \|\cdot\|_{\boldsymbol{p}})$, with $\boldsymbol{p} \in [\mathbf{1}, \infty]$, $\mathbb{B}_{\boldsymbol{r}}^{\boldsymbol{p}}(\boldsymbol{x})$ and $\mathbb{S}_{\boldsymbol{r}}^{\boldsymbol{p}}(\boldsymbol{x})$ are, respectively, the closed ball and the sphere centered at $\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$ with radius $\boldsymbol{r} > \mathbf{0}$, $\boldsymbol{e_i} = (\mathbf{0}, \ldots, \mathbf{1}, \ldots, \mathbf{0})^\top \in \mathbb{R}^{\boldsymbol{l}}$ $(\boldsymbol{i} = \mathbf{1}, \ldots, \boldsymbol{l})$ the $\boldsymbol{i}$-unit vector, $\boldsymbol{e} := (\mathbf{1}, \ldots, \mathbf{1})^\top$. In particular, we set $\mathbb{B}^{\boldsymbol{p}} := \mathbb{B}_{\mathbf{1}}^{\boldsymbol{p}}(\mathbf{0})$ and $\mathbb{S}^{\boldsymbol{p}} := \mathbb{S}_{\mathbf{1}}^{\boldsymbol{p}}(\mathbf{0})$ for simplicity. The symbols $\mathbf{cl}\,\boldsymbol{\Omega}$ and $\mathbf{bd}\,\boldsymbol{\Omega}$ stand for the closure and boundary of a set $\boldsymbol{\Omega} \subset \mathbb{R}^{\boldsymbol{n}}$, respectively. Let $\boldsymbol{x} = (\boldsymbol{x_0}, \boldsymbol{x_r}), \boldsymbol{y} = (\boldsymbol{y_0}, \boldsymbol{y_r}) \in \mathbb{R}^{\boldsymbol{m}+\mathbf{1}} := \mathbb{R} \times \mathbb{R}^{\boldsymbol{m}}$. The scalar product of $\boldsymbol{x}$ and $\boldsymbol{y}$ denote by $\boldsymbol{x}^\top \boldsymbol{y}$. The notation $\widehat{\boldsymbol{x}} := (\boldsymbol{x_0}, -\boldsymbol{x_r})$ and the notation $\widetilde{\boldsymbol{x}} := \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$ while $\boldsymbol{x}^\perp := \{\boldsymbol{y} \in \mathbb{R}^{\boldsymbol{m}+\mathbf{1}} \mid \boldsymbol{x}^\top \boldsymbol{y} = \mathbf{0}\}$. Let $\boldsymbol{C} \subset \mathbb{R}^{\boldsymbol{n}}$. We use the following notations:

$$\text{conv}\, C := \left\{\sum_{i=1}^{k} \lambda_i\, c_i \mid \lambda_i \in [0,1], c_i \in C, k \in \mathbb{N}\right\},$$

$$\text{cone}\, C := \{\sum_{i=1}^{k} \lambda_i\, c_i \mid \lambda_i \geq 0,$$

$$c_i \in \text{conv}C, k \in \mathbb{N}\} \ (\text{cone}\emptyset := \{0\}).$$

The notation C° (respectively, $C^\star$) the polar cone (respectively, the positive dual cone) of C which is defined by

$$C^\circ := \{y \in \mathbb{R}^n \mid y^\top x \leq 0 \ \ \forall x \in C\}$$

(respectively, $C^\star$: $\{y \in \mathbb{R}^n \mid y^\top x \geq 0 \ \ \forall x \in C\}$).

Let C be a convex subset in $\mathbb{R}^n$. The normal cone and the tangent cone to C at $\bar{x} \in C$ are respectively defined by (cf. Mordukhovich, B.S., 2005; Mordukhovich, B.S. and Nam N.M., 2014)

$$N_C(\bar{x}) := \left\{x^* \in \mathbb{R}^n \,\middle|\, \begin{array}{c}\langle x^*, x - \bar{x}\rangle \leq 0 \\ \forall x \in C\end{array}\right\},$$

$$T_C(x) := \left\{u \in \mathbb{R}^n \,\middle|\, \begin{array}{c}\exists t_k \to 0^+, u_k \to u \\ \text{with } x + t_k u_k \in C, \forall k\end{array}\right\}.$$

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a convex function and $\bar{x} \in \mathbb{R}^n$. The (convex) *subdifferential* of f at $\bar{x}$ is defined by

$$\partial f(\bar{x}) :=$$

$$\{x^* \in \mathbb{R}^n \mid \langle x^*, x - \bar{x}\rangle \leq \mathrm{f(x)} - \mathrm{f(\bar{x})}\ \forall x \in \mathbb{R}^n\}.$$

Let $\|\cdot\|$ be a norm of $\mathbb{R}^n$. The dual norm $\|\cdot\|_*$ of $\|\cdot\|$ is defined by

$$\| x \|_* := \sup_{y \in \mathbb{S}} \langle x, y\rangle,$$

where $\mathbb{S} := \{x \in \mathbb{R}^n \mid \| x \| = 1\}$.

For any $p \in [1, \infty]$, it is well known that $\|\cdot\|_{p_*} = \|\cdot\|_q$ with $\frac{1}{p} + \frac{1}{q} = 1$.

We now recall some generalizations of the *second order cone* as follows (cf. Sznajder, R., 2016; Vinel, A. and Krokhmal, P., 2014; Zhou, J. *et al.*, 2015).

Given $p \geq 1$, the *p-order cone* in $\mathbb{R}^{n+1}$ is defined by

$$\mathcal{K} := \{x = (x_0, x_r) \in \mathbb{R} \times \mathbb{R}^m \mid x_0 \geq \| x_r \|_2\}, \quad (2.3)$$

where $\| x \|_p$ is p-norm in $\mathbb{R}^n$. *The circular cone* in $\mathbb{R}^{n+1}$ is given by

$$\mathcal{K}_\theta := \{x = (x_0, x_r) \in \mathbb{R} \times \mathbb{R}^m \mid x_0 \tan\theta \geq \| x_r \|_2\}, (2.2)$$

with an angle $\theta \in \left(0, \frac{\pi}{2}\right)$. The *extended second order cone* in $\mathbb{R}^l \times \mathbb{R}^s$ is defined by

$$\mathcal{K}(l, s) := \{(x, u) \in \mathbb{R}^l \times \mathbb{R}^s | x \succcurlyeq \| u \| e\}, \quad (2.3)$$

where $x \succcurlyeq y$ means that $x - y \in \mathbb{R}_+^l$ for all $x, y \in \mathbb{R}^l$ and $e := (1,1, \ldots, 1) \in \mathbb{R}^l$.

It is obviously that $\mathcal{K}_p, \mathcal{K}_\theta$ and $\mathcal{K}(l, s)$ are, respectively, the solution set of the following semi-infinite systems

$$x = (x_0, x_r) \in \mathbb{R} \times \mathbb{R}^n$$

$$(-1, a)^\top (x_0, x_r) \leq 0 \quad \forall a \in \mathbb{B}^p$$

with

$$\frac{1}{p} + \frac{1}{q} = 1,$$

$$x = (x_0, x_r) \in \mathbb{R} \times \mathbb{R}^n$$

$$(-1, a)^\top (x_0, x_r) \leq 0 \quad \forall a \in \mathbb{B}_r^2(0)$$

with $r = \frac{1}{\tan\theta}$ and

$$z = (x, u) \in \mathbb{R}^l \times \mathbb{R}^s,$$

$$(-e_i, a)^\top (x, u) \leq 0 \quad \forall a \in \mathbb{B}^2, i = 1, \ldots, l.$$

Inspired by above representations, we consider the following semi-infinite system

$$(x, u) \in \mathbb{R}^l \times \mathbb{R}^s,$$

$$(-e_i, a)^\top (x, u) \leq 0 \ \forall a \in \mathbb{B}(r, \|\cdot\|),$$

$$\text{with } i = 1, \ldots, l, \tag{2.4}$$

where $r > 0$, $\mathbb{B}(r, \|\cdot\|) := \{x \mid \| x \| \leq r\}$ and $\|\cdot\|$ is a norm of $\mathbb{R}^s$.

The solution set to the system (2.4) denoted by $\mathcal{S}_l^r(\|\cdot\|)$ is a convex cone (see Lemma 2.1) and thus it is called the *cone generated by the semi-infinite system* (2.4). We get

$$\mathcal{S}_l^r(\|\cdot\|) := \{(x, u) \in \mathbb{R}^l \times \mathbb{R}^s \mid (-e_i, a)^\top (x, u) \leq 0, \ \forall a \in \mathbb{B}(r, \|\cdot\|), \ i = 1, \ldots, l\}. \tag{2.5}$$

The semi-infinite system (2.4) is a special case of the semi-infinite system (Chuong, T.D. and Jeyakumar V., 2016). In that paper, the authors used the approaches of semi-infinite propgramming to obtain some results on the error bound of semi-infinite systems. We will see in the next section that the system (2.4) admits a global error bound at any boundary point. This result is important to obtain the formulas of normal cone and tangent cone to $\mathcal{S}_l^r(\|\cdot\|)$.

The following lemma shows that the cone generated by the semi-infinite system (2.4) is a closed convex cone.

Lemma 2.1. *Let $r > 0, l, s \in \mathbb{N}$ and $\|\cdot\|$ is a norm in $\mathbb{R}^s$. Then the set $\mathcal{S}_l^r(\|\cdot\|)$ given by* (2.5) *is a closed convex cone in $\mathbb{R}^l \times \mathbb{R}^s$.*

Proof. We first show that $\mathcal{S}_l^r(\|\cdot\|)$ is a cone. Indeed, with $x_0 = 0 \in \mathbb{R}^l, u_0 = 0 \in \mathbb{R}^s$, we get $(x_0, u_0) \in \mathcal{S}_l^r(\|\cdot\|)$. Take $(x, u) \in \mathcal{S}_l^r(\|\cdot\|)$ and $\lambda > 0$. We put

$$\mathbb{B}(r, \|\cdot\|) := \{x \in \mathbb{R}^s \mid \| x \| \leq r\}.$$

For any $a \in \mathbb{B}(r, \|\cdot\|)$, we gain

$$(-e_i, a)^\top (\lambda(x, u)) = \lambda(-e_i, a)^\top (x, u) \leq 0,$$
$$i = 1, \ldots, l.$$

It implies that $\lambda(x, u) \in \mathcal{S}_l^r(\|\cdot\|)$ and thus $\mathcal{S}_l^r(\|\cdot\|)$ is a cone.

Moreover, since $\mathbb{B}(r, \|\cdot\|)$ is closed and convex, the mapping $\varphi: \mathbb{R}^l \times \mathbb{R}^s \to \mathbb{R}^n$ defined by $\varphi(x, u) = (-e_i, a)^\top (x, u)$ is continuous and convex. So, $\mathcal{S}_l^r(\|\cdot\|)$ is a closed convex set. $\square$

From (1.1), (2.1), (2.2), (2.3) and (2.5), we have the following remark.

Remark 2.2. (i) $\mathcal{S}_1^1(\|\cdot\|_2)$ is the second order cone in $\mathbb{R}^{s+1}$.

(ii) $\mathcal{S}_1^r(\|\cdot\|_2)$ is the circular cone in $\mathbb{R}^{s+1}$ with the angle $\theta := \arctan\frac{1}{r} \in \left(0, \frac{\pi}{2}\right)$.

(iii) $\mathcal{S}_1^1(\|\cdot\|_p)$ is the p-order cone in $\mathbb{R}^{s+1}$.

(iv) $\mathcal{S}_l^1(\|\cdot\|_2)$ is the extended second order cone in $\mathbb{R}^l \times \mathbb{R}^s$.

Example 2.3. Some special cases of the cone generated by semi-infinite system (2.4).

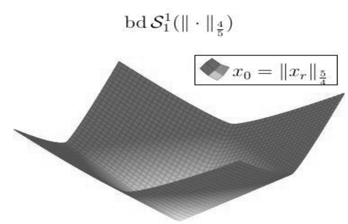

Figure 1. The boundary of $\frac{4}{5}$-order cone

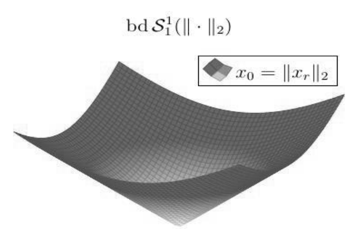

Figure 2. The boundary of second order cone

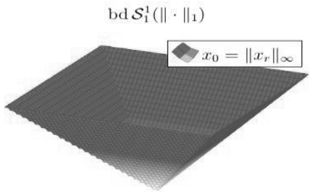

Figure 3. The boundary of ∞-order cone

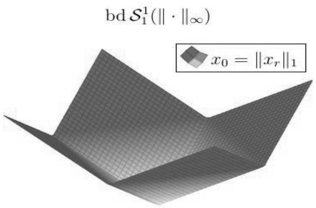

Figure 4. The boundary of 1-order cone

3. Global error bound of cones generated by semi-infinite systems

Let $\|\cdot\|$ be a norm of $\mathbb{R}^s$. Consider the system (2.4). For each $(x,u) \in \mathbb{R}^l \times \mathbb{R}^s$, we put $\varphi_i(x,u) = \max_{a\in\mathbb{B}(r,\|\cdot\|)} \{-e_i^\top x + a^\top u\}$ for each $i = 1,\ldots,l$ and $\varphi(x,u) := \max_{i=1,\ldots,l} \varphi_i(x,u)$. The *index active set* and *the active set* of (x,u) are, respectively, denoted by

$$I(x,u) := \{i \in \{1,\ldots,l\} |\ \varphi_i(x,u) = \varphi(x,u)\},$$

$$U(x,u) := \{(-e_i, a) \in \mathbb{R}^s \times \mathbb{B}(r,\|\cdot\|) \mid i \in I(x,u), -e_i^\top x + a^\top = \varphi(x,u)\}.$$

It is obvious that

$$\varphi_i(x,u) = -x_i + r \parallel u \parallel, \qquad \forall i = 1,\ldots,l$$

and

$$\varphi(x,u) = -\min_{i=1,\ldots,l} x_i + r \parallel u \parallel,$$

$$I(x,u) = \begin{cases} \operatorname{argmin}_i\{x_i |\ i \in \{1,\ldots,l\}\} & \text{if } (x,u) \neq 0, \\ \{1,\ldots,l\} & \text{if } (x,u) = 0, \end{cases} \quad (3.1)$$

$$U(x,u) = \begin{cases} \left\{\left(-e_i, \left(\frac{u_1}{\|u\|},\ldots,\frac{u_s}{\|u\|}\right)\right) \mid i \in I(x,u)\right\} & \text{if } (x,u) \neq 0, \\ \{(-e_i, a) \mid i \in I(x,u), a \in \mathbb{B}(r,\|\cdot\|)\} & \text{if } (x,u) = 0. \end{cases} \quad (3.2)$$

for each $(x,u) \in \mathbb{R}^l \times \mathbb{R}^s$ with $x = (x_1,\ldots,x_l)^\top$.

Definition 3.1. Let $\|\cdot\|$ be a norm of $\mathbb{R}^s$. Consider the system (2.4). Then

(i) We said that the system (2.4) admits a *global error bound* at $(\bar{x},\bar{u}) \in \mathcal{S}_l^r(\|\cdot\|)$ if there exists $\tau > 0$ such that

$$d((x,u), \mathcal{S}_l^r(\|\cdot\|) \leq \tau[\varphi(x,u)]_+$$

$\forall (x,u) \in \mathbb{R}^l \times \mathbb{R}^s$.

(ii) We said that the system (2.4) admits a *local error bound* at $(\bar{x},\bar{u}) \in \mathcal{S}_l^r(\|\cdot\|)$ if there exist $\tau > 0$ and $\delta > 0$ such that

$$d((x,u), \mathcal{S}_l^r(\|\cdot\|) \leq \tau[\varphi(x,u)]_+$$

$\forall (x,u) \in \mathbb{B}_\delta(\bar{x},\bar{u})$.

Note that, in this definition

$$[a]_+ := \max\{a; 0\}, \forall a \in \mathbb{R}.$$

It is clear that if the system (2.4) admits a global error bound at $(\bar{x},\bar{u}) \in \mathcal{S}_l^r(\|\cdot\|)$, then it also (2.4) admits a local error bound at $(\bar{x},\bar{u})$. The following theorem gives us that the system (2.4) admits a global error bound at any $(\bar{x},\bar{u}) \in \operatorname{bd} \mathcal{S}_l^r(\|\cdot\|)$.

Therem 3.2. Let $(\bar{x},\bar{u}) \in \mathcal{S}_l^r(\|\cdot\|)$. The system (2.4) always admits a global error bound at any $(\bar{x},\bar{u}) \in \operatorname{bd} \mathcal{S}_l^r(\|\cdot\|)$.

Proof. We can assume that $\bar{x} = (\bar{x}_0,\ldots,\bar{x}_l)$ and $\bar{u} = (\bar{u}_0,\ldots,\bar{u}_l)$. If $(x,u) \in \mathcal{S}_l^r(\|\cdot\|)$ then (3.1) trivially holds. Thus we only need to consider the case of $(x,u) \notin \mathcal{S}_l^r(\|\cdot\|)$. Without any loss of generality, we can assume that $x_1,\ldots,x_{l_1} < r \parallel u \parallel$ and $x_{l_1+1},\ldots,x_l \geq r \parallel u \parallel$. We put $\tilde{x} = (\tilde{x}_1,\ldots,\tilde{x}_l)$ with $\tilde{x}_i = r \parallel u \parallel$ for all $i = 1,\ldots,l_1$ and $\tilde{x}_i = x_i$ for all $i = l_{l_1+1},\ldots,l$. Then $\tilde{x} \in \mathcal{S}_l^r(\|\cdot\|)$ and

$$\begin{aligned} & d((x,u), \mathcal{S}_l^1(\|\cdot\|) \\ & \leq \parallel x - \tilde{x} \parallel_2 \\ & \leq \sqrt{\sum_{i=1}^{l_1} \varphi_i^2(x,u)} \leq \sqrt{l_1}\varphi(x,u) \\ & = \sqrt{l_1}[\varphi(x,u)]_+. \end{aligned}$$

Hence, the system (2.4) admits a global error bound at $(\bar{x},\bar{u})$. □

By combining Theorem 3.2 and Corollary 3.9 (Chuong, T.D. and Jeyakumar V., 2016), we obtain formulas of normal cone and tangent cone to $\mathcal{S}_l^r(\|\cdot\|)$ as follows.

Theorem 3.3. (Formulas of normal cone and tangent cone) *For any* $(\bar{x},\bar{u}) \in \mathcal{S}_l^r(\|\cdot\|)$, *we get*

$$N((\bar{x},\bar{u}),\mathcal{S}_l^r(\|\cdot\|)) = \begin{cases} \{0\} & \text{if } (\bar{x},\bar{u}) \in \operatorname{int} \mathcal{S}_l^r(\|\cdot\|), \\ \operatorname{cone}\left\{\left(-e_i,\left(\frac{u_1}{\|u\|},\dots,\frac{u_s}{\|u\|}\right)\right) \,\middle|\, i \in I(x,u)\right\} & \text{if } (\bar{x},\bar{u}) \in \operatorname{bd} \mathcal{S}_l^r(\|\cdot\|)\backslash\{0\}, \\ -\mathcal{S}_l^{1/r}(\|\cdot\|_*) & \text{if } (\bar{x},\bar{u}) = 0, \end{cases}$$

and

$$T((\bar{x},\bar{u}),\mathcal{S}_l^r(\|\cdot\|)) = \begin{cases} \mathbb{R}^l \times \mathbb{R}^s & \text{if } (\bar{x},\bar{u}) \in \operatorname{int} \mathcal{S}_l^r(\|\cdot\|), \\ \left\{\left(-e_i,\left(\frac{u_1}{\|u\|},\dots,\frac{u_s}{\|u\|}\right)\right) \mid i \in I(x,u)\right\}^0 & \text{if } (\bar{x},\bar{u}) \in \operatorname{bd} \mathcal{S}_l^r(\|\cdot\|)\backslash\{0\}, \\ \mathcal{S}_l^r(\|\cdot\|) & \text{if } (\bar{x},\bar{u}) = 0. \end{cases}$$

Proof. We will first prove the formula of normal cone. We consider the following three cases.

Case 1. $(\bar{x},\bar{u}) \in \operatorname{int} \mathcal{S}_l^r(\|\cdot\|)$. By the definition of normal cone to a convex set, we get

$$N((\bar{x},\bar{u}),\mathcal{S}_l^r(\|\cdot\|)) = \{0\}.$$

Case 2. $(\bar{x},\bar{u}) \in \operatorname{bd} \mathcal{S}_l^r(\|\cdot\|)\backslash\{0\}$. In this case, we get

$I(x,u) = \operatorname{argmin}_i\{x_i \mid i \in \{1,\dots,l\}\}$ and

$U(x,u) = \left\{\left(-e_i,\left(\frac{u_1}{\|u\|},\dots,\frac{u_s}{\|u\|}\right)\right) \mid i \in I(x,u)\right\}$.

By Corollary 3.9 (Chuong, T.D. and Jeyakumar V., 2016), we get

$$\begin{aligned} &N((\bar{x},\bar{u}),\mathcal{S}_l^r(\|\cdot\|)) \\ &= \operatorname{cone} U(x,u) \\ &= \operatorname{cone}\left\{\left(-e_i,\left(\frac{u_1}{\|u\|},\dots,\frac{u_s}{\|u\|}\right)\right) \,\middle|\, \text{i} \in I(x,u)\right\}. \end{aligned}$$

Case 3. $(\bar{x},\bar{u}) = 0$. In this case, we get

$$I(x,u) = \{1,\dots,l\}$$

and

$$U(x,u) = \{(-e_i, a) \mid i \in I(x,u), a \in \mathbb{B}(r,\|\cdot\|)\}.$$

It implies that

$$\begin{aligned} &N((\bar{x},\bar{u}),\mathcal{S}_l^r(\|\cdot\|)) \\ &= \operatorname{cone}\{(-e_i, a) \mid i \in I(x,u), a \in \mathbb{B}(r,\|\cdot\|)\}. \end{aligned}$$

Taking $(\mathrm{y},\mathrm{v}) \in N((\bar{x},\bar{u}),\mathcal{S}_l^r(\|\cdot\|))$, there exist $i \in \{1,\dots,l\}$, $a \in \mathbb{B}(r,\|\cdot\|)$ and $\alpha > 0$ such that $(\mathrm{y},\mathrm{v}) = \alpha(-e_i, a)$. For any $b \in \mathbb{B}(1/r,\|\cdot\|_*)$, we have $\|b\|_* = \max_{c\in\mathbb{S}} \langle b,c\rangle \leq \frac{1}{r}$ and thus $|b^T a| \leq \|a\|.\|b\|_* \leq 1$. Therefore,

$$\begin{aligned} (-e_i, b)^T(-y,-v) &= \alpha(-e_i, b)^T(e_i, -a) \\ &= \alpha(-1 - b^T a) \leq 0 \end{aligned}$$

which implies that $(-y,-v) \in \mathcal{S}_l^{1/r}(\|\cdot\|_*)$.

The formula of $T((\bar{x},\bar{u}),\mathcal{S}_l^r(\|\cdot\|))$ is implied from the formula of

$N((\bar{x},\bar{u}),\mathcal{S}_l^r(\|\cdot\|))$ and the fact that

$$T((\bar{x},\bar{u}),\mathcal{S}_l^r(\|\cdot\|)) = \left(N((\bar{x},\bar{u}),\mathcal{S}_l^r(\|\cdot\|))\right)^0. \square$$

Remark 3.4. If $\mathcal{S}_l^r(\|\cdot\|)$ is the second order cone or the circular cone then the results in Theorem 3.3 reduce to those ones in Bonnans, J.F. and Ramírez H.C. (2005); Thinh, V.D. *et al.* (2020).

4. Applications

Let $(\mathbb{R}^n, |||\,.\,|||)$ and $(\mathbb{R}^m,\|\cdot\|)$ be finite dimension spaces, and let $A: \mathbb{R}^n \to \mathbb{R}^m$ be a continuous linear mapping. We first consider the following problem which gives us the solution sets of problems (1.3) (if $\|\cdot\|$ is $\|\cdot\|_2$) and (1.4) (if $\|\cdot\|$ is $\|\cdot\|_1$).

$$\min_x \|Ax - y\|, \tag{4.1}$$

where $y \in \mathbb{R}^m$ is given.

Theorem 4.1. Consider the problem (4.1). Let $\bar{x}$ be an optimal solution of (4.1). Then $A^*(A\bar{x} - y) = 0$.

Proof. Denote by $\bar{z} := A\bar{x} - y$. By using Theorem 2.51 and Theorem 4.14 (Mordukhovich, B.S. and Nam N.M., 2014), we get

$$\begin{aligned} 0 \in \partial_{\mathrm{x}} \|\mathrm{A}\bar{\mathrm{x}} - \mathrm{y}\| &= \mathrm{A}^* \partial \|\bar{\mathrm{z}}\| \\ &= \{\mathrm{A}^*\mathrm{v} \mid \mathrm{v} \in \partial \|\bar{\mathrm{z}}\|\}. \end{aligned}$$

It implies from Example 2.38 (Mordukhovich, B.S. and Nam N.M., 2014) that

$$\partial \|\bar{\mathrm{z}}\| = \begin{cases} \mathbb{B} & \text{if } \bar{\mathrm{z}} = 0, \\ \left\{\frac{\bar{\mathrm{z}}}{\|\bar{\mathrm{z}}\|}\right\} & \text{if } \bar{\mathrm{z}} \neq 0. \end{cases}$$

Thus $\mathrm{A}^*(\mathrm{A}\bar{\mathrm{x}} - \mathrm{y}) = 0$.

Hence, the proof is completed. □

In what follows, we consider the following optimization problem.

$$\min \ f(x) \tag{4.2}$$

$$\text{subject to } \| g(x) \| \leq h_i(x), \quad \forall i = 1, \dots, l,$$

where $f: \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable, $h_i: \mathbb{R}^n \to \mathbb{R}, g: \mathbb{R}^n \to \mathbb{R}^s$ for all $i = 1, \dots, l$ are twice continuously differentiable.

We put

$$h(x) = \begin{pmatrix} h_1(x) \\ \vdots \\ h_l(x) \end{pmatrix}$$

and

$$G(x) = \begin{pmatrix} h(x) \\ g(x) \end{pmatrix}.$$

Then the problem (4.2) can be rewritten by

$$\min \ f(x) \tag{4.3}$$

$$\text{subject to } G(x) \in \mathcal{S}_l^1(\|\cdot\|).$$

To provide the optimality conditions for (4.3) (and thus (4.2)), we need the following qualification condition.

Definition 4.2. Let $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. We say that F is *metrically subregular* at $(\bar{x}, \bar{y}) \in \text{gph}F$ if there exist $r, \tau > 0$ such that

$$d\big(x, F^{-1}(\bar{y})\big) \leq \tau d\big(\bar{y}, F(x)\big),$$
$$\forall x \in \mathbb{B}_r(\bar{x}).$$

Theorem 4.3. Let $\bar{x}$ be the locally solution to the problem (4.3). Assume that the multifunction $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^l \times \mathbb{R}^s$ defined by $F(x) := G(x) - \mathcal{S}_l^1(\|\cdot\|)$ is metrically subregular at $(\bar{x}, 0)$. Then

$$-\nabla f(\bar{x}) \in \begin{pmatrix} \nabla h(x) \\ \nabla g(x) \end{pmatrix}^{\top} N((g(\bar{x}), h(\bar{x}))^{\top}, \mathcal{S}_l^1(\|\cdot\|)). \tag{4.4}$$

Proof. We denote by

$$\mathcal{C} := \{x \in \mathbb{R}^n | G(x) \in \mathcal{S}(l, 1, \|\cdot\|)\}.$$

By using Corollary 4.15 (Mordukhovich, B.S. and Nam N.M., 2014), we get

$$0 \in \nabla f(\bar{x}) + N(\bar{x}, \mathcal{C}).$$

Since $F(x)$ is metrically subregular at $(\bar{x}, 0)$, we get from Proposition 4.2 (Mohammadi, A. *et al.*, 2020), that

$$N(\bar{x}, \mathcal{C}) = \nabla G(\bar{x})^{\top} N(G(\bar{x}), \mathcal{S}_l^1(\|\cdot\|))$$

which gives us that

$$0 \in \nabla f(\bar{x}) + \nabla G(\bar{x})^{\top} N(G(\bar{x}), \mathcal{S}_l^1(\|\cdot\|)).$$

Thus, we obtain (4.4) and hence the proof is completed. □

5. Conclusion and discussion

In this paper, we have first presented the cone generated by semi-infinite systems which is a generalization of the second order cone, extended second order cones, and circular cones. Then, we have provided exact formula for the normal and tangent cones to this cone. Thereby, we have provided first order necessary conditions for local optimal solutions to mathematical programs.

For possible developments, we are planning to employ the obtained results to calculate the second order tangent cone of the cone generated by semi-infinite systems. Moreover, inspired by (Thinh, V.D. *et al.*, 2020), necessary and sufficient conditions for optimal solutions of cone generated by semi-infinite systems complementarity programs would be established by using the current approach.

Acknowledgements: This work was partially supported by Dong Thap University and Vietnam Ministry of Education and Training under the grant number B2019.SPD.01./.

References

Alizadeh, F. and Goldfarb, D. (2003). Second-order cone programming. *Mathematical Programming*, (95), 3-51.

Bonnans, J.F. and Ramírez H.C. (2005). Perturbation analysis of second-order cone programmming problems. *Mathematical Programming*, (104), 205-227.

Boyd, S. and Vandenberghe, L. (2004). *Convex Optimization.* Cambridge University Press.

Chang, Y.L., Yang, C. Y., and Chen, J.S. (2013). Smooth and nonsmooth analyses of vector-valued functions associated with circular cones. *Nonlinear Analysis,* (8), 160-173.

Chuong, T.D. and Jeyakumar, V. (2016). Characterizing robust local error bounds for linear inequality systems under data uncertain. *Linear Algebra and its Application*, (489), 199-216.

Ferreira, O.P. and Németh, S.Z. (2018). How to project onto extended second order cones. *Journal of Global Optimization*, (70), 707-718.

Glineur, F. and Terlaky, T. (2004). Conic formulation for l_p-norm optimization. *Journal of Optimization Theory and Applications*, (122), 285-307.

Gotoh, J. and Uryasev, S. (2015). Two pairs of families of polyhedral norms versus l_p-norms: proximity and applications in optimization. *Mathematical Programming*, Series A, (96), 1-41.

Hang, N.T.V., Mordukhovich, B.S., and Sarabi, M.E. (2020). Second-order variational analysis in second-order cone programming. *Mathematical Programming,* (180), 75-116.

Liu, Y. J. and Zhang, L. W. (2008). Convergence of the augmented Lagrangian method for nonlinear optimization problems over second-order cones. *Journal of Optimization Theory and Applications*, (139), 557-575.

Németh, S. Z. and Xie, J. (2018). Linear complementarity problems on extended second order cones. *Journal of Optimization Theory and Applications,* (176), 269-288.

Németh, S.Z., Xie, J., and Zhang, G. (2020). Positive operators on extended second order cones. *Acta Mathematica Hungarica*, (160), 390-404.

Mordukhovich, B. S. (2005). *Variational analysis and generalized differentiation I.* Springer, Berlin.

Mohammadi, A., Mordukhovich, B.S., and Sarabi, M.E. (2020). Parabolic regularity in geometric variational analysis, to appear in *Transactions of the American Mathematical Society*, arxiv:1909.00241.

Mordukhovich, B.S. and Nam, N.M. (2014). *An easy path to convex analysis and applications.* Morgan & Claypool Publishers, Willistion.

Mordukhovich, B. S., Outrata, J., and Sarabi, M.E. (2014). Full stability of locally optimal solution in second-order cone programming. *SIAM Journal on Optimization*, (24), 1581-1613.

Outrata, J.V. and Ramírez, H.C. (2011). On the Aubin property of critical points to perturbed second-order cone programs. *SIAM Journal on Optimization,* (21), 798-823.

Sznajder, R. (2016). The Lyapunov rank of extended second order cones. *Journal of Global Optimization*, (66), 585-593.

Thinh, V.D., Chuong, T.D., and Anh, N.L.H. (2020). Optimality conditions for circular cone complementarity programs. *Optimization*, accepted.

Vinel, A. and Krokhmal, P. (2014a). On valid inequalities for mixed integer p-order cone programming. *Journal of Optimization Theory and Applications,* (160), 439-456.

Vinel, A. and Krokhmal, P. (2014b). Polyhedral approximations in p-order cone programming. *Optimization Methods Software*, (29), 1210-1237.

Xue, G. and Ye, Y. (2000). An efficient algorithm for minimizing a sum of p-norms. *SIAM Journal on Optimization*, (10), 551-579.

Zhou, J., Chang, J.L, and Chen, J.S. (2015). The H-differentiability and calmness of the circular cone functions. *Journal of Global Optimization*, (63), 811-833.

Zhou, J. and Chen, J. S. (2013). Properties of circular cone and spectral factorization associated with circular cone. *Journal of Nonlinear and Convex Analysis*, (14), 807-816.

Zhou, J. and Chen, J.S. (2017). Monotonicity and circular cone monotonicity associated with circular cones. *Set-Valued and Variational Analysis*, (25), 211-232.

Zhou, J., Tang, J. and Chen, J.S. (2017). Parabolic second-order directional differentiability in the Hadamard sense of the vector-valued functions associated with circular cones. *Journal of Optimization Theory and Applications*, (172), 802-823.