FINITE DIMENSIONALITY OF GLOBAL ATTRACTORS FOR A NONCLASSICAL DIFFUSION EQUATION LACKING INSTANTANEOUS DAMPING WITH MEMORY

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ABSTRACT: We consider a periodic boundary value problem for a nonclassical diffusion equation lacking instantaneous damping with hereditary memory $u_t - \Delta u_t - \int_0^{\infty} \kappa(s)\Delta u(t-s)ds + k_0 |u|^{p-1} u = g.$

The main feature of the model is that the equation does not contain a term of the form $-\Delta u$, which contributes to an instantaneous damping. Setting the problem in the history framework, we prove that the global attractor generated by the above-mentioned nonclassical diffusion equation is finite dimensional.

Keywords: Non-classical diffusion equation; lacking instantaneous damping; memory; global attractor; fractal dimension.

SỐ CHIỀU HỮU HẠN CỦA TẬP HÚT TOÀN CỤC CHO PHƯƠNG TRÌNH KHUẾCH TÁN KHÔNG CỔ ĐIỂN KHUYẾT SỐ HẠNG TẮT DẦN TỨC THỜI VÀ CHỨA NHỚ

TÓM TẮT: Trong bài báo này, chúng tôi xét bài toán giá trị biên tuần hoàn cho phương trình khuếch tán không cổ điển khuyết số hạng tắt dần và chứa nhớ $u_t - \Delta u_t - \int_0^{\infty} \kappa(s)\Delta u(t-s)ds + k_0 |u|^{p-1} u = g.$ Đặc điểm đáng chú ý của mô hình là phương trình không chứa số hạng có dạng $-\Delta u$, góp phần tạo ra sự tắt dần tức thời. Xét bài toán trong trường hợp mô hình chịu sự ảnh hưởng của các yếu tố trong quá khứ, chúng tôi chứng minh rằng tập hút toàn cục sinh ra bởi phương trình khuếch tán không cổ điển nêu trên là hữu hạn chiều fractal.

Từ khóa: Phương trình khuếch tán không cổ điển; số hạng tắt dần tức thời; nhớ; tập hút toàn cục; số chiều fractal.

1. Introduction

The study of the asymptotic behavior of dynamical systems arising from mechanics and physics is a capital issue, as it is essential, for practical applications, to be able to understand and even predict the long-time behavior of the solutions of such systems. One way to attack the problem for a dissipative dynamical system is to consider its global attractor. This is a compact invariant set, which contains much information about the long-time behavior of solutions.

The main goal of this paper is to discuss the asymptotic behavior of the solutions for the following autonomous equation

$$u_t - \Delta u_t - \int_0^\infty \kappa(s)\Delta u(t-s)ds + k_0 |u|^{p-1} u = g(x) \qquad (1)$$

with the damping coefficient $k_0 > 0$, $2 \le p \le 5$, and the initial data

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, t \le 0,$$
$$u(x,t) = u(x + Le_i, t), \quad x \in \mathbb{R}^n, t \in \mathbb{R}, i = 1, 2 \ldots, n,$$

where $e_i = \{0,\ldots,0,1,0,\ldots,0\}$ and L is a positive constant.

2. Literature review, theoretical framework and methodology

The nonclassical diffusion equation was introduced by E.C. Aifantis [1] as a model to describe physical phenomena, such as non-Newtonian flows, soil mechanics and heat conduction theory. In the case of the nonclassical diffusion equation containing viscoelasticity of the conductive medium, that is to say, we add a fading memory term to this equation,

$$u_t - \Delta u_t - \Delta u - \int_0^\infty \kappa(s)\Delta u(t-s)ds + f(u) = g.$$

The conduction of energy is not only affected by present external forces but also by historic external forces. The existence and asymptotic behavior of solutions to nonclassical diffusion equations with memory has been studied for both autonomous case (see [2, 3, 5, 6]) and non-autonomous case (see [3, 8]) where the memory term satisfies the conditions

$$\kappa(s) = \int_s^\infty \mu(r)dr, \quad \mu'(s) + \delta\mu(s) \le 0; \qquad (2)$$

or only requires conditions much weaker than (2) $\kappa(s)\le\theta\mu(s)$, which can be equivalently expressed in the form

$$\mu(\sigma+s)\le Me^{-\delta\sigma}\mu(s). \tag{3}$$

In particular, V. Pata et al [6] considered the nonclassical diffusion equation with memory lacking instantaneous damping

$$u_t-\Delta u_t-\int_0^\infty \kappa(s)\Delta u(t-s)ds+f(u)=g.$$

Under assumption on the memory kernel as in (2), and the nonlinearity f fulfills the growth restriction

$$|f(u)-f(v)|\le k_f|u-v|(1+|u|^4+|v|^4) \quad \text{for some } k_f>0,$$

they proved the existence of global attractors and their regularity. Besides, they pointed out that the whole dissipation is contributed by the convolution term only. Then, the weaker condition (3) no longer suffices to ensure energy decay. In this paper, by the fractal dimension theorem given by A.O. Celebi et al [4], we prove that the fractal dimension of the global attractor for the problem is finite. Now, denote $\Omega=\prod_{i=1}^{n}(0;L)$ and the spaces

$$\dot{L}^2_{\text{per}}(\Omega)=\left\{u\in L^2(\Omega):\int_\Omega udx=0, u \text{ is periodic in } x \text{ with period } L\right\},$$
$$\dot{H}^s_{\text{per}}(\Omega)=\left\{u\in H^s(\Omega):\int_\Omega udx=0, u \text{ is periodic in } x \text{ with period } L\right\},$$

$(\cdot,\cdot)$ and $\|\cdot\|$ are inner product and norm of $\dot{L}^2_{\text{per}}(\Omega)$ along with $(\cdot,\cdot)_s$ and $\|\cdot\|_s$ are inner product and norm of $\dot{H}^s_{\text{per}}(\Omega)$. To study the problem (1), we assume that the external force g and the memory kernel κ satisfy the following conditions **[H1]** The time-independent external force $g\in L^2(\Omega)$. **[H2]** The memory kernel κ is a nonnegative summable function (we take it of unitary mass) of the form $\kappa(s)=\int_s^\infty \mu(r)dr$, where $\mu\in L^1(\mathbb{R}^+)$ is a decreasing (hence nonnegative) piecewise absolutely continuous function such that $\mu'(s)+\delta\mu(s)\le 0$, for some $\delta>0$ and almost every $s>0$. In particular, these assumptions imply that $\kappa(0)=\int_0^\infty \mu(s)ds<\infty$, and $\mu(0)<\infty$, namely, μ can be continuously extended to the origin.

To avoid the presence of unnecessary constants, from now on we assume $\kappa(0)=1$ which can be always obtained by rescaling the memory kernel.

To this aim, following C.M. Dafermos [7], we consider a new variable reflecting the history of (1) which is introduced, that is to be, $\eta^t(x,s)=\eta(x,t,s)=\int_0^s u(x,t-r)dr, \quad s\geq 0,$ then we can check that $\partial_t\eta^t(x,s)=u(x,t)-\partial_s\eta^t(x,s), s\geq 0.$ Since $\mu(s)=-\kappa'(s)$, problem (1) can be transformed into the following system

$$\begin{cases} u_t-\Delta u_t-\int_0^\infty \mu(s)\Delta\eta^t(x,s)ds+k_0|u|^{p-1}u=g(x), x\in\mathbb{R}^n, t\in\mathbb{R}, \\ \partial_t\eta^t(x,s)=-\partial_s\eta^t(x,s)+u(x,t), x\in\mathbb{R}^n, t>\tau, s\geq 0, \\ u(x,t)=u(x+L,t), x\in\mathbb{R}^n, t\leq 0, \\ \eta^t(x,s)=\eta^t(x+L,s), x\in\mathbb{R}^n, s\in\mathbb{R}^+,\ t\leq 0, \\ u(x,0)=u_0(x), x\in\mathbb{R}^n, \\ \eta^0(x,s)=\eta_0(x,s)=\int_0^s u_0(x,-\tau)d\tau, x\in\mathbb{R}^n, s\in\mathbb{R}^+. \end{cases} \tag{4}$$

Denote $z(t)=(u(t),\eta^t)$, and $z_0=(u_0,\eta_0)$. We now define the history spaces $L^2_\mu(\mathbb{R}^+,\dot{H}^r_{\text{per}}(\Omega))$, which is the Hilbert space of functions $\varphi:\mathbb{R}^+\to H^r_{\text{per}}(\Omega)$ endowed with the inner product $\langle\varphi_1,\varphi_2\rangle_{r,\mu}=\int_0^\infty \mu(s)\langle\varphi_1(s),\varphi_2(s)\rangle_r\, ds,$ and let $\|\cdot\|_{r,\mu}$ denote the corresponding norm. We now introduce the following Hilbert spaces $\mathcal{H}_i=\dot{H}^i_{\text{per}}(\Omega)\times L^2_\mu(\mathbb{R}^+,\dot{H}^i_{\text{per}}(\Omega)), i=1,2.$

The paper is organized as follows. In Section 1, we introduce the notation along with some definitions and give some assumptions on the forcing term g as well as the memory kernel $\kappa(\cdot)$ (or $\mu(\cdot)$). Section 2 is devoted to showing the finiteness of the fractal dimension of the global attractor by using the fractal dimension theorem given by A.O. Celebi et al [4]. The last section is the conclusion.

3. Finite Dimensionality of the Global Attractor

First, we recall the definition of the weak solution of problem (1) as well as the theorem about the existence of a global attractor of the problem. By an application of a Galerkin scheme, we can get the existence and uniqueness of local solutions to the problem (4) as follows.

Theorem 3.1. Assume that hypotheses *(H1)-(H2)* hold. Then for any $z_0 = (u_0, \eta_0) \in \mathcal{H}_1$, the problem (4) has a unique solution $z = (u, \eta^t)$ on the interval [0;T] satisfying $z \in C([0,T]; \mathcal{H}_1)$ for all $T > 0$. Furthermore, if $z_0 \in \mathcal{H}_2$, then the problem (4) has a unique solution $z = (u, \eta^t)$, it satisfies that $z \in C([0,T]; \mathcal{H}_1)$ for all $T > 0$.

From [8], we have the existence of global attractors $\mathcal{A}$.

Theorem 3.2. [8] Assume that conditions *(H1)-(H2)* hold. The semigroup $\{S(t)\}_{t\geq 0}$ generated by (4) has a global attractor $\mathcal{A}$ in $\mathcal{H}_2$.

We first give a lemma for describing the finite dimensionality of a set.

Lemma 3.3. [4, Theorem 3, p. 441]

Let B be a bounded set in Hilbert space X. The map $V : B \to X$ satisfies that $B \subset V(B)$, and

$$\| V(v) - V(\bar{v}) \|_X \leq l \| v - \bar{v} \|_X, \quad v, \bar{v} \in B,$$

$$\| Q_N V(v) - Q_N V(\bar{v}) \|_X \leq \delta \| v - \bar{v} \|_X \quad (\delta < 1),$$

where Q_N is the orthogonal projection of X onto the subspace $X_N^{\perp}$, and X_N is spanned by the first N basis elements of X, that is, the first N eigenfunctions of the problem

$$-\Delta\phi(x) = \lambda\phi(x), \quad \phi(x + Le_i) = \phi(x), i = 1,2,3...n, \quad \int_\Omega \phi(x)dx = 0. \tag{5}$$

Then the fractal dimension of B satisfies where k is the Gaussian constant.

Theorem 3.4. *The fractal dimension of the global attractor $\mathcal{A}$ for the semigroup $\{S(t)\}_{t\geq 0}$ generated by (4) is finite.*

Proof. To prove that for some $t_1 > 0$, the operator $S(t_1) =: V$ satisfies the conditions of Lemma 2.3. Assume that

$$(u(x,0), \eta^0(x,s)) = (u_0(x), \eta_0^u(x,s)), \quad (v(x,0), \eta^{0,v}(x,s)) = (v_0(x), \eta_0^v(x,s)) \in \mathcal{A}.$$

Theorem 3.2 implied that $(u(x,t), \eta^{t,u}(x,s)), (v(x,t), \eta^{t,v}(x,s)) \in \mathcal{A}$, where

$\eta^{t,u}(x,s)=\int_0^s u(x,t-\tau)d\tau,\quad \eta^{t,v}(x,s)=\int_0^s v(x,t-\tau)d\tau.$ Putting $w=u-v$, then $\eta^{t,w}(x,s)=\int_0^s w(x,t-\tau)d\tau,$ and $(w(x,t),\eta^{t,w}(x,s))$ satisfies the equation

$$w_t-\Delta w_t-\int_0^\infty \mu(s)\Delta\eta^{t,w}(s)ds+k_0\left(|u|^{p-1}u-|v|^{p-1}v\right)=0. \tag{6}$$

Multipying (6) by $-\Delta w$ in $\dot{L}^2_{\text{per}}(\Omega)$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\| w\|_1^2+\| w\|_2^2+\|\eta^{t,w}\|_{2,\mu}^2\right)-\int_0^\infty \mu'(s)\|\eta^{t,w}\|_2^2\,ds\\&+k_0\left(|u|^{p-1}u-|v|^{p-1}v,-\Delta w\right)=0.\end{aligned} \tag{7}$$

Since the global attractor $\mathcal{A}$ is bounded in $\dot{H}^2_{\text{per}}(\Omega)$, that is,

$$\max_{x\in\Omega}|u|,\quad \max_{x\in\Omega}|v|,\quad \| u\|_2^2=\| u\|^2_{\dot{H}^2_{\text{per}}(\Omega)},\quad \| v\|_2^2=\| v\|^2_{\dot{H}^2_{\text{per}}(\Omega)}\le M_0, \tag{8}$$

where M_0 is a constant.

Thus,

$$\begin{aligned}k_0\left(|u|^{p-1}u-|v|^{p-1}v,-\Delta w\right)&\le k_0(p+1)\int_\Omega\left(|u|^{p-1}+|v|^{p-1}\right)|w||\Delta w|\,dx\\&\le k_0(p+1)M_1\int_\Omega |w||\Delta w|\,dx\\&\le\frac{k_0(p+1)M_1}{2}(\| w\|^2+\| w\|_2^2)\le M_2(\| w\|_1^2+\| w\|_2^2).\end{aligned} \tag{9}$$

From (7) and (9), we get

$$\begin{aligned}\frac{d}{dt}\left(\| w\|_1^2+\| w\|_2^2+\|\eta^{t,w}\|_{2,\mu}^2\right)&\le 2M_2\left(\| w\|_1^2+\| w\|_2^2\right)\\&\le 2M_2\left(\| w\|_1^2+\| w\|_2^2+\|\eta^{t,w}\|_{2,\mu}^2\right).\end{aligned} \tag{10}$$

then Gronwall inequality implies that

$$\| w\|_1^2+\| w\|_2^2+\|\eta^{t,w}\|_{2,\mu}^2\le\left(\| w_0\|_1^2+\| w_0\|_2^2+\|\eta_0^{t,w}\|_{2,\mu}^2\right)e^{2M_2t}. \tag{11}$$

For some $t_1>0$ we define $l=:e^{2M_2t_1}$, the constant that we need in Lemma 3.3.

Next, taking the inner product of (6) in $\dot{L}^2_{\text{per}}(\Omega)$ with $-\Delta Q_N w(x,t)$, we commute the operator $-\Delta$ with the projection Q_N and get

$$\frac{1}{2}\frac{d}{dt}\left(\| \nabla Q_N w\|^2+\| \Delta Q_N w\|^2+\int_\Omega \mu(s)\| \Delta Q_N \eta^{t,w}(s)\|^2\, ds\right)$$
$$-\int_0^\infty \mu'(s)\| \Delta Q_N \eta^{t,w}(s)\|^2\, ds+k_0\left(|u|^{p-1}u-|v|^{p-1}v,-\Delta Q_N w\right)=0. \tag{12}$$

As in (8) and (9), we have

$$k_0\left(|u|^{p-1}u-|v|^{p-1}v,-\Delta Q_N w\right)\le k_0(p+1)M_1\| \nabla w\|\| \nabla Q_N w\|. \tag{13}$$

By the Poincaré inequality, we have

$\| \nabla Q_N \phi\|\le \lambda_{N+1}^{-\frac{1}{2}}\| \Delta Q_N \phi\|,\quad \phi\in X_N^\perp,$ where λ_N is the N-th eigenvalue of the periodic eigenvalue problem (5).

Thus,

$$\begin{aligned} M_2\| \nabla w\|\| \nabla Q_N w\| &\le M_2\| \nabla w\|\| \Delta Q_N w\|\lambda_{N+1}^{-\frac{1}{2}} \\ &\le M_2\lambda_{N+1}^{-\frac{1}{2}}(\| \nabla w\|^2+\| \Delta Q_N w\|^2). \end{aligned} \tag{14}$$

Putting (13) and (14) into (12) we have

$$\frac{d}{dt}E_Q(t)-2\int_0^\infty \mu'(s)\| Q_N\eta^{t,w}(s)\|_2^2\, ds\le 2M_2\lambda_{N+1}^{-\frac{1}{2}}(\| w\|_1^2+\| Q_N w\|_2^2),$$

where $E_Q(t)=\| Q_N w\|_1^2+\| Q_N w\|_2^2+\| Q_N \eta^{t,w}\|_{2,\mu}^2$.

Next, we define the function $\Phi_Q(t)=E_Q(t)+\nu a\Lambda_Q$,

where $\Lambda_Q(t)$ is defined in Lemma 3.2 in [8] (with $(Q_N w, Q_N\eta^{t,w}(S))$ in place of $(u,\eta^t(s))$) and we can see that $\frac{1}{2}E_Q(t)\le \Phi_Q(t)\le 2E_Q(t)$. $\Phi_3(t)-\Delta Q_N w_t$,

(15)

then using similar arguments as in (2.7) in [8], one can deduce that

$$a^2(\| Q_N w_t\|_1^2+\| Q_N w_t\|_2^2)\le a^2\left(4\| Q_N\eta^{t,w}\|_{2,\mu}^2+2k_0^2\| Q_N w\|_2^2\right). \tag{16}$$

Summation of (15) and (16), we obtain

$$\frac{d}{dt}\Phi_Q(t)+\left(2\delta-\frac{\nu a\delta\mu(0)}{2}-\nu\right)\| Q_N\eta^{t,w}\|_{2,\mu}^2+\left(\frac{\nu a}{2}-2M_2\lambda_{N+1}^{-\frac{1}{2}}\right)\| Q_N w\|_2^2$$
$$+\left(a^2-\frac{\nu a^2}{4}\right)\left(\| Q_N w_t\|_1^2+\| Q_N w_t\|_2^2\right)\le 2M_2\lambda_{N+1}^{-\frac{1}{2}}\| w\|_1^2.$$

Choose $\nu, a > 0$ small enough and N large enough that $\frac{\nu a}{2} - 2M_2\lambda_{N+1}^{-\frac{1}{2}} > 0$ and using (11), then there exists a constant $\gamma_4 > 0$ such that $\frac{d}{dt}\Phi_Q(t) + \gamma_4\Phi_Q(t) \le 2M_2\lambda_{N+1}^{-\frac{1}{2}}\left(\| w_0 \|_1^2 + \| w_0 \|_2^2 + \| \eta_0^{t,w} \|_{2,\mu}^2\right)e^{2M_2t}$, and the Gronwall inequality implies that

$$\begin{aligned}\Phi_3(t) &\le \Phi_3(0)e^{-\gamma_4 t} + \frac{2M_2\lambda_{N+1}^{-\frac{1}{2}}}{2M_2 + \gamma_4}e^{2M_2t} \\ &\le \left(\| w_0 \|_1^2 + \| w_0 \|_2^2 + \| \eta_0^{t,w} \|_{2,\mu}^2\right)\left(\frac{2M_2\lambda_{N+1}^{-\frac{1}{2}}}{2M_2 + \gamma_4}e^{2M_2t} - e^{-\gamma_4 t}\right)\end{aligned}$$

so we have $\| Q_N w_t \|_2^2 \le \Phi_3(t) \le \| w_0 \|_2^2 \left(\frac{2M_2\lambda_{N+1}^{-\frac{1}{2}}}{2M_2 + \gamma_4}e^{2M_2t} - e^{-\gamma_4 t}\right)$.

Choose N and $t_1 > 0$ such that $\frac{2M_2\lambda_{N+1}^{-\frac{1}{2}}}{2M_2 + \gamma_4}e^{2M_2t} - e^{-\gamma_4 t} \le \delta < 1.$

Therefore, for $t = t_1$ we have $S(t_1)$ satisfies the conditions of Lemma 3.3, then the fractal dimension of the global attractor $\mathcal{A}$ satisfies that $d_F(\mathcal{A}) \le N \ln\left(\frac{8k^2l^2}{1-\delta^2}\right) / \ln\left(\frac{2}{1-\delta^2}\right)$.

The proof is complete.

4. Conclusions

In this paper, we study the long-time behavior of the solution of nonclassical diffusion equation lacking instantaneous damping with hereditary memory. Using the methods proposed by Dafermos (1980) and the ideas of V. Pata et al (2020), we improved the evaluation techniques, and proved the existence of a fractal dimensional finite attractor. The result of this paper is a development, complementing the results in [8].

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