SOME PROPERTIES OF SOLUTIONS TO 2D G-NAVIER-STOKES EQUATIONS

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ABSTRACT:

We consider the initial boundary value problem for 2D g-Navier-Stokes equations in bounded domains with homogeneous Dirichlet boundary conditions. We prove some important properties of solutions to the problem including the backward uniqueness property, the squeezing property.

Keywords: g -Navier-Stokes; strong solutions; backward uniqueness property and squeezing property.

MỘT SỐ TÍNH CHẤT CỦA NGHIỆM ĐỐI VỚI PHƯƠNG TRÌNH *g*-NAVIER-STOKES HAI CHIỀU

TÓM TẮT:

Chúng ta xét bài toán giá trị biên ban đầu cho phương trình *g*-Navier-Stokes 2 chiều trong miền giới hạn với điều kiện biên Dirichlet thuần nhất. Chúng tôi chứng minh một số tính chất quan trọng của nghiệm bao gồm tính chất duy nhất lùi, tính chất ép.

Từ khóa: g -Navier-Stokes; nghiệm mạnh; tính chất duy nhất lùi; tính chất ép.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary Γ. We consider the following two-dimensional (2D) non-autonomous g -Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(x,t) \text{ in } (0,T)\times\Omega, \\ \nabla\cdot(gu) = 0 \text{ in } (0,T)\times\Omega, \\ u = 0 \text{ on } (0,T)\times\Gamma, \\ u(x,0) = u_0(x),\ x\in\Omega, \end{cases} \tag{1.1}$$

where $u = u(x,t) = (u_1,u_2)$ is the unknown velocity vector, $p = p(x,t)$ is the unknown pressure, $\nu > 0$ is the kinematic viscosity coefficient, u_0 is the initial velocity.

The g-Navier-Stokes equations is a variation of the standard Navier-Stokes equations. More precisely, when $g \equiv \text{const}$ we get the usual Navier-Stokes equations.

The 2D g-Navier-Stokes equations arise in a natural way when we study the standard 3D problem in thin domains. We refer the reader to [8] for a derivation of the 2D g-Navier-Stokes equations from the 3D Navier-Stokes equations and a relationship between them. As mentioned in [7], good properties of the 2D g-Navier-Stokes equations can lead to an initiate of the study of the Navier-Stokes equations on the thin three dimensional domain $\Omega_g = \Omega \times (0, g)$.

In the last few years, the existence of both weak and strong solutions to 2D g-Navier-Stokes equations has been studied in $[2,3]$. The existence of time-periodic solutions to g-Navier-Stokes equations was studied recently in [4]. Moreover, the long-time behavior of solutions in terms of existence of global/uniform/pullback attractors has been studied extensively in both autonomous and non-autonomous cases, see e.g. $[1,5,6,7,8]$ and references therein. However, to the best of our knowledge, little seems to be known about other properties of solutions to the 2D g-Navier-Stokes equations. This is a motivation of the present paper.

The aim of this paper is to study some important properties of solutions to g-Navier-Stokes equations such as the backward uniqueness property, the squeezing property. To do this, we assume that the function g satisfies the following hypothesis:

(G) $g \in W^{1,\infty}(\Omega)$ such that

$$0 < m_0 \le g(x) \le M_0 \text{ for all } x = (x_1, x_2) \in \Omega, \text{ and } |\nabla g|_\infty < m_0 \lambda_1^{1/2},$$

where $\lambda_1 > 0$ is the first eigenvalue of the g-Stokes operator in Ω (i.e., the operator A defined in Section 2).

The paper is organized as follows. In Section 2, for convenience of the reader, we recall some auxiliary results on function spaces and inequalities for the nonlinear terms related to the g-Navier-Stokes equations. Section 3 proves a backward unique-ness result. In Section 4, we prove the squeezing property for the solutions on the global attractor.

2. PRELIMINARIES

Let $\mathbb{L}^2(\Omega, g) = (L^2(\Omega, g))^2$ and $\mathbb{H}_0^1(\Omega, g) = (H_0^1(\Omega, g))^2$ be endowed, respectively, with the inner products $(u, \mathrm{v})_g = \int_\Omega u.vg dx, u, v \in \mathbb{L}^2(\Omega, g)$, and

$$((u, \mathrm{v}))_g = \int_\Omega \sum_{j=1}^{2} \nabla u_j . \nabla v_j g dx, u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{H}_0^1(\Omega, g),$$

and norms $|u|^2 = (u,u)_g$, $\|u\|^2 = ((u,u))_g$. Thanks to assumption $(\mathbf{G})$, the norms $|.|$ and $\|.\|$ are equivalent to the usual ones in $(L^2(\Omega))^2$ and in $(H_0^1(\Omega))^2$.

Let

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^2 : \nabla \cdot (gu) = 0\}.$$

Denote by H_g the closure of $\mathcal{V}$ in $L^2(\Omega, g)$, and by V_g the closure of $\mathcal{V}$ in $H_0^1(\Omega, g)$ It follows that $V_g \subset H_g \equiv H_g' \subset V_g'$, where the injections are dense and continuous. We will use $\|.\|_*$ for the norm in V_g', and $\langle .,. \rangle$ for duality pairing between V_g and V_g'

We now define the trilinear form b by

$$b(u,v,w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j g dx,$$

whenever the integrals make sense. It is easy to check that if $u, v, w \in V_g$, then

$$b(u,v,w) = -b(u,w,v).$$

Hence

$$b(u,v,v) = 0, \forall u, v \in V_g.$$

Set

$$A : V_g \to V_g' \text{ by } \langle Au, v \rangle = ((u,v))_g,$$
$$B : V_g \times V_g \to V_g'$$

by $\langle B(u,v), w \rangle = b(u,v,w)$ and put $Bu = B(u,u)$.

Denote $D(A) = \{u \in V_g : Au \in H_g\}$, then $D(A) = H^2(\Omega, g) \cap V_g$ and

$$Au = -P_g \Delta u, \forall u \in D(A),$$

where P_g is the ortho-projector from $L^2(\Omega, g)$ onto H_g.

Using the Hölder inequality and the Ladyzhenskaya inequality (when $n = 2$)

$$|u|_{L^4} \le c|u|^{1/2} |\nabla u|^{1/2}, \ \forall u \in H_0^1(\Omega),$$

and the interpolation inequalities, as in [9] one can prove the following result.

Lemma 2.1. *If $n = 2$, then*

$$|b(u,v,w)| \le \begin{cases} c_1 |u|^{1/2} \|u\|^{1/2} \|v\| |w|^{1/2} \|w\|^{1/2}, & \forall u, v, w \in V_g, \\ c_2 |u|^{1/2} \|u\|^{1/2} \|v\|^{1/2} |Av|^{1/2} |w|, & \forall u \in V_g, v \in D(A), w \in H_g, \\ c_3 |u|^{1/2} |Au|^{1/2} \|v\| |w|, & \forall u \in D(A), v \in V_g, w \in H_g, \\ c_4 |u| \|v\| |w|^{1/2} |Aw|^{1/2}, & \forall u \in H_g, v \in V_g, w \in D(A), \end{cases} \tag{2.1}$$

and

$$|B(u,v)|+|B(v,u)|\le c_5\|u\|\|v\|^{1-\theta}|Av|^{\theta},\quad \forall u\in V_g; v\in D(A). \qquad (2.2)$$

where $\theta\in(0,1)$; $c_i, i=1,\dots,5,$ *are appropriate constants*.

For every $u,v\in D(A)$, then

$$|B(u,v)|\le c_6\begin{cases}|Au|\|v\|,\\ \|u\||Av|.\end{cases} \qquad (2.3)$$

Lemma 2.2. [3] Let $u\in L^2(0,T;V_g)$, *then the function Cu defined by*

$$((Cu(t),v)_g=\left(\left(\frac{\nabla g}{g}.\nabla\right)u,v\right)_g=b\left(\frac{\nabla g}{g},u,v\right),\forall v\in V_g,$$

belongs to $L^2(0,T;H_g)$, *and therefore also belongs to* $L^2(0,T;V_g')$. *Moreover,*

$$|Cu(t)|\le\frac{|\nabla g|_\infty}{m_0}.\|u(t)\|,\ \text{for a.e. } t\in(0,T),$$

and

$$\|Cu(t)\|_*\le\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}.\|u(t)\|,\ \text{for a.e. } t\in(0,T).$$

Since

$$-\frac{1}{g}(\nabla.g\nabla)u=-\Delta u-(\frac{\nabla g}{g}.\nabla)u,$$

we have

$$(-\Delta u,v)_g=((u,v))_g+((\frac{\nabla g}{g}.\nabla)u,v)_g=(Au,v)_g+((\frac{\nabla g}{g}.\nabla)u,v)_g,\forall u,v\in V_g.$$

We recall the definition of strong solutions to problem (1.1)

Definition 2.1. *A function* u *is called a strong solution to problem* (1.1) *on the interval* $(0,T)$ *if*

$$\begin{cases}u\in C([0,T];V_g)\cap L^2(0,T;D(A)), du/dt\in L^2(0,T;H_g),\\ \dfrac{d}{dt}u(t)+\nu Au(t)+\nu Cu(t)+B(u(t),u(t))=f(t)\text{ in } H_g,\ \text{for a.e. } t\in(0,T),\\ u(0)=u_0.\end{cases}$$

Theorem 2.1. [2] *For any* T > 0, $u_0\in V_g$, *and* $f\in L^2(0,T;H_g)$ *given, problem* (1.1) *has a unique strong solution* u *on* $(0,T)$. *Moreover, the strong solutions depend continuously on the initial data in* V_g.

We recall here some *a priori* estimates of strong solutions frequently used later.

3. BACKWARD UNIQUENESS PROPERTY

Let u, v solve respectively the g-Navier-Stokes equations

$$\begin{cases} \dfrac{du}{dt} + \nu Au + \nu Cu + B(u,u) = f, \\ u = u_0. \end{cases} \qquad (3.1)$$

$$\begin{cases} \dfrac{dv}{dt} + \nu Av + \nu Cv + B(v,v) = f, \\ v = v_0. \end{cases} \qquad (3.2)$$

Two solutions u, v are called a backward uniqueness property if $u(t_1) = v(t_1)$ then $u(t) = v(t)$ for all time $t < t_1$.

Lema 3.1. [9] *If a function* $w \in L^\infty(0,T;V) \cap L^2(0,T;D(A))$ *satisfy*

$$\frac{dw}{dt} + Aw = h(t, w(t)), t \in (0,T),$$

where h *is function from* $(0,T) \times V$ *into* H *such that* $|h(t,w(t))| \le k(t) \| w(t) \|$, *for* $a.e, t \in (0,T), k \in L^2(0,T),$ *and* $w(T) = 0$ *then* $w(t) = 0, 0 < t < T.$

Theorem 3.1. *Under the assumptions of Theorem* (2.1) *, then the strong solutions of* g *-Navier-Stokes have a backward uniqueness property*

Proof. Denote $w = u - v$, we have

$$\begin{aligned} \frac{dw}{dt} + \nu Aw &= -B(u,u) + B(v,v) - \nu Cw \\ &= -B(u,w) - B(w,v) - \nu Cw. \end{aligned}$$

Using (2.2) and Lemma 2.2, we obtain

$$\begin{aligned} |h(t,w(t))| &:= |-B(u,w) - B(w,v) - \nu Cw| \\ &\le \left(c_5 \|u\|^{1-\theta} |Au|^\theta + c_5 \|v\|^{1-\theta} |Av|^\theta + \nu \frac{|\nabla g|_\infty}{m_0} \right) \|w\|. \end{aligned}$$

Applying Lemma 3.1 with

$$k(t) = c_5 \|u\|^{1-\theta} |Au|^\theta + c_5 \|v\|^{1-\theta} |Av|^\theta + \nu \frac{|\nabla g|_\infty}{m_0},$$

we have the proof.

4. SQUEEZING PROPERTY

We write P for the orthogonal projection onto the finite-dimensional subspace, and Q for the projection onto its orthogonal complement. Then by $[12]$, we can define a continuous semigroup $S(t)$ of g -Navier-Stokes equations and it has global attractor in V_g.

Definition 4.1. *Write* $S=S(1)$. *Then the squeezing property holds if, for each* $0<\delta<1$, *there exists a finite rank orthogonal projection* $P(\delta)$, *with orthogonal complement* $Q(\delta)$, *such that for every* $u, v \in \mathcal{A}$ *either*

$$|Q(Su-Sv)| \leq |P(Su-Sv)| \tag{4.1}$$

or, if not, then

$$|Su-Sv| < \delta |u-v| \tag{4.2}$$

Theorem 4.1. *If* $f \in H_g$ *then the squeezing property holds for the 2D* $g-$*Navier-Stokes equations.*

Proof. The equation for the difference $w(t)=u(t)-v(t)$ is

$$\frac{dw}{dt}+\nu Aw+\nu Cw+B(u,w)+B(w,v)=0, \tag{4.3}$$

and we will write

$$p=P_n w, \quad q=Q_n w, \quad w=p+q.$$

First, we take the inner product of (4.3) with p, using

$$b(u,w,p)=b(u,p+q,p)=b(u,q,p),$$

we have

$$\frac{1}{2}\frac{d|p|^2}{dt}+\nu\|p\|^2+\nu(Cw,p)_g=-b(u,q,p)-b(w,v,p).$$

Using the bounds on b in Lemma 2.1 and the existence of an absorbing set in H_g, V_g and $D(A)$, we can obtain

$$\frac{1}{2}\frac{d|p|^2}{dt}+\nu\|p\|^2+\nu(Cw,p)_g \geq -C(|u|^{1/2}|Au|^{1/2}|q\|\|p\|-|w\|\|p\|\|v|^{1/2}|Av|^{1/2})$$

$$\geq C(|q|\lambda^{1/2}|p|+|w|\lambda^{1/2}|p|),$$

where $\lambda=\lambda_n$. Using Lemma 2.2, we obtain

$$\frac{1}{2}\frac{d|p|^2}{dt}+\nu\|p\|^2 \geq -C\left[|q|\lambda^{1/2}|p|+\left(1+\nu\frac{|\nabla g|_\infty}{m_0}|w|\lambda^{1/2}|p|\right)\right]$$

$$\geq -C\lambda^{1/2}|p|(|q|+|p|),$$

so that

$$|p|\frac{d|p|}{dt}\geq -|p|(\nu\lambda|p|+C\lambda^{1/2}(|q|+|p|))$$
$$\geq -\lambda^{1/2}|p|(\nu\lambda^{1/2}|p|+C|q|+C|p|). \qquad (4.4)$$

Now take the inner product of (4.3) with q, we have

$$\frac{1}{2}\frac{d|q|^2}{dt}+\nu\|q\|^2+\nu(Cw,q)_g=-b(u,p,q)-b(w,v,q)$$
$$\leq C|p|\|q\|+C|w|\|q\|.$$

Using Lemma 2.2, we obatain

$$\frac{1}{2}\frac{d|q|^2}{dt}+\nu\|q\|^2 \leq C|p|\|q\|+\left(C+\nu\frac{|\nabla g|_\infty}{m_0}\right)|w|\|q\|\leq C\|q\|(|p|+|q|),$$

so that

$$|q|\frac{d|q|}{dt} \leq \|q\|(-\nu\lambda^{1/2}|q|+C|p|+C|q|).$$

Provided that the expressionin the parentheses is negative,

$$(\nu\lambda^{1/2}-C)|q|>C|p|,$$

then we have

$$|q|\frac{d|q|}{dt} \leq \lambda^{1/2}|q|(-\nu\lambda^{1/2}|q|+C|p|+C|q|). \qquad (4.5)$$

We now choose n large enough that

$$\nu\lambda^{1/2}-C>2C. \qquad (4.6)$$

Now, either (4.1) holds, and so there is nothing to prove, or it does not, in which case

$$|Qw(1)|>|Pw(1)|. \qquad (4.7)$$

In this case, using (4.6), we have

$$(\nu\lambda^{1/2}-C)|Qw(t)|>2C|Pw(t)|, \qquad (4.8)$$

holds for $t=1$. Since $w(t)$ is continuous into H_g, then (4.8) holds in a neighbourhood of $t=1$. We consider two possibilities: If (4.8) holds for all $t\in[\frac{1}{2},1]$, then we have, by (4.6),

$$(\nu\lambda^{1/2}-C)|q|-C|p|>\frac{1}{2}(\nu\lambda^{1/2}-C)|q|>C|q|,$$

for $t \in [\frac{1}{2},1]$, and so (4.5) becomes

$$\frac{d}{dt}|q| \le -\lambda C|q|,$$

which gives

$$|q(1)| \le e^{-\frac{1}{2}\lambda C}\left|q\left(\frac{1}{2}\right)\right|.$$

Since (4.7) holds, this implies that

$$|w(1)| \le 2e^{-\frac{1}{2}\lambda C}\left|q\left(\frac{1}{2}\right)\right| \le 2e^{-\frac{1}{2}\lambda C}\left|w\left(\frac{1}{2}\right)\right|,$$

and using the Lipschitz property of strong solutions, $\left|w\left(\frac{1}{2}\right)\right| \le L\left(\frac{1}{2}\right)|w(0)|$, we have

$$|w(1)| \le 2L\left(\frac{1}{2}\right)e^{-\frac{1}{2}\lambda C}|w(0)|.$$

This gives (4.2), provided that $\lambda = \lambda_n$ is chosen large enough. If (4.8) does not hold on all of $t \in [\frac{1}{2},1]$, then it holds on $t \in [t_0,1]$, with

$$(\nu\lambda^{1/2} - C)|Qw(t_0)| = 2C|Pw(t_0)|. \tag{4.9}$$

In this case we take

$$\Phi(t) = \Phi\big(p(t),q(t)\big) = \big(|p|+|q|\big)\exp\left(\frac{\nu\lambda^{1/2}|q|}{C(|p|+|q|)}\right).$$

From (4.4), (4.5) holds, we have

$$\frac{d\Phi}{dt} \le 0; \quad \forall\, t \in [t_0,1].$$

Thus $\Phi(1) \le \Phi(t_0)$. However, at $t=1$ we have (4.8), so that

$$\Phi(1) \ge |q(1)|\, e^{\nu\lambda^{1/2}/C},$$

and at $t=t_0$ the equality (4.9) hold, which gives

$$2C\big(|p(t_0)|+|q(t_0)|\big) = \big(\nu\lambda^{1/2}+C\big)|q(t_0)|,$$

and so

$$\Phi(t_0) = \frac{\nu\lambda^{1/2} + C}{2C} | q(t_0) | e^{2\nu\lambda^{1/2}/(\nu\lambda^{1/2}+C)}.$$

It follows that

$$| q(1) | \le e^{-\nu\lambda^{1/2}/C} \frac{\nu\lambda^{1/2} + C}{2C} e^2 | q(t_0) |,$$

and using once more the Lipschitz property of strong solutions, we obtain

$$| q(1) | \le e^{-\nu\lambda^{1/2}/C} \frac{\nu\lambda^{1/2} + C}{2C} e^2 L(1) | w(0) | .$$

Since $| p(1) | < | q(1) |,$ it certainly follows that

$$| w(1) | \le e^{-\nu\lambda^{1/2}/C} \frac{\nu\lambda^{1/2} + C}{C} e^2 L(1) | w(0) | .$$

This gives (4.2), provided that $\lambda = \lambda_n$ is chosen large enough, and the theorem is proved.

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