FINITE-DIMENSIONAL ASYMPTOTIC BEHAVIOR OF A PREDATOR-PREY MODEL WITH DIFFUSION

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ABSTRACT. The predator-prey model is an essential tool in mathematical ecology and specifically for our understanding of interacting populations in the natural environment. We consider a predator-prey model with variable carrying capacity and cross-diffusion. First, we recall some recent results on the existence and regularity of inertial manifolds for nonautonomous evolution equations involving sectorial operators. We then apply the inertial manifold theory to study the finite-dimensional asymptotic behavior of the above-mentioned predator-prey model.

Keywords. admissible space, asymptotic behavior, inertial manifold, predator-prey model, spectral gap condition

DÁNG ĐIỆU TIỆM CẬN HỮU HẠN CHIỀU CỦA MỘT MÔ HÌNH THÚ-MỒI VỚI KHUẾCH TÁN

TÓM TẮT: Mô hình thú-mồi là một công cụ thiết yếu trong sinh thái toán học và đặc biệt quan trọng cho sự hiểu biết của chúng ta về các quần thể tương tác trong môi trường tự nhiên. Chúng tôi xét một mô hình thú-mồi với sức chứa môi trường sống biến thiên và khuếch tán chéo. Trước hết, chúng tôi nhắc lại một số kết quả gần đây về sự tồn tại và tính chính quy của đa tạp quán tính đối với phương trình tiến hóa không ôtônôm chứa toán tử quạt. Sau đó, chúng tôi áp dụng lý thuyết đa tạp quán tính vào nghiên cứu dáng điệu tiệm cận hữu hạn chiều của mô hình thú-mồi nói trên.

Từ khóa. không gian chấp nhận được, dáng điệu tiệm cận, đa tạp quán tính, mô hình thú-mồi, điều kiện kẽ hở phổ

1. INTRODUCTION

Prey-predator dynamic is an essential tool in mathematical ecology, specifically for our understanding of interacting populations in the natural environment. This relationship will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance.

The investigation of the asymptotic behavior of solutions to infinite-dimensional dynamical systems generated by evolutionary partial differential equations in large time is one of the central problems of dynamical systems. An important tool for such investigation is the concept of *inertial manifolds* introduced in 1985 by C. Foias, G.R. Sell & R. Temam [3] (see also [4]) when they studied the asymptotic behavior of solutions to Navier-Stokes equations. An inertial manifold for an evolution equation is a (Lipschitz) finite-dimensional manifold which is positively invariant and exponentially attracts all other solutions of the equation. This fact permits to invoke the reduction principle to study the asymptotic behavior of the solutions to evolution equations in infinite-dimensional spaces by comparing with that of the induced equations in spaces of finite-dimension. Inertial manifolds for evolution equations have been systematically studied in many works, see [5, 8, 10, 11, 12, 13] and the literature cited therein.

We consider the following predator-prey model (the detailed treatments of this equation will be performed in Section 3)

$$\begin{cases} \dfrac{\partial u}{\partial t} - D_1\Delta u = ru\left(1-\dfrac{u}{K(t)}\right) - quv, & t>s, \quad 0<x<\pi \\ \dfrac{\partial v}{\partial t} - D_2\Delta v = -dv + cuv, & t>s, \quad 0<x<\pi \end{cases} \tag{1.1}$$

With the change of variables $u = U + \bar{u}$, $v = V + \bar{v}$, where $(\bar{u}, \bar{v})$ is a stationary solution of the system, we can write the above system in the form

$$\dot{x}(t) + Ax(t) = f(t, x(t)),$$

where

$$x := \begin{bmatrix} U \\ V \end{bmatrix}, A := -\begin{bmatrix} D_1\Delta - r & 0 \\ 0 & D_2\Delta - d \end{bmatrix} - \begin{bmatrix} 2r - \bar{v}q & -\bar{u}q \\ c\bar{v} & c\bar{u} \end{bmatrix},$$

$$f(t,x) := \begin{bmatrix} -\dfrac{2r\bar{u}U + rU^2}{K(t)} - qUV \\ cUV \end{bmatrix}.$$

We will prove that, the linear operator $-A$ is a *sectorial operator* having a sufficiently large gap between two spectral parts.

Recently, the existence of inertial manifolds for mild solutions to evolution equations involving a sectorial operator was proved by the work [10]. The purpose of this paper is to apply results [10, 11] to describe the finite-dimensional asymptotic behavior of a predator-prey model with cross-diffusion.

2. INERTIAL MANIFOLDS FOR EVOLUTION EQUATIONS REVISITED

2.1 Preliminaries

We start by the definition of sectorial operators.

Definition 2.1. A closed, linear and densely defined operator $S : X \supset D(S) \to X$ in Banach space X is called a *sectorial operator* (of (σ,ω)-type) if there exist real numbers $\omega \in \mathbb{R}, \sigma \in \left(0, \dfrac{\pi}{2}\right)$ and $M \geq 1$ such that

$$\sum\nolimits_{\sigma}(\omega) := \left\{ z \in \mathbb{C} : |\arg(z-\omega)| < \sigma + \frac{\pi}{2}, z \neq \omega \right\} \subset \rho(S);$$

$$\|R(\lambda, S)\| \leq \frac{M}{|\lambda - \omega|} \quad \text{for all } \lambda \in \sum\nolimits_{\sigma}(\omega). \tag{2.1}$$

To prove the existence of an inertial manifold, we suppose the following assumption.

Assumption A. *Let A be a closed linear operator on a Banach space X such that −A is a sectorial operator of* (σ,ω)*-type with* $0<\sigma<\frac{\pi}{2}$ *and* $\omega<0$. *We suppose that the spectrum* $\sigma(-A)$ *of* $-A$ *can be decomposed as follows:*

$$\sigma(-A)=\sigma_u(-A)\cup\sigma_c(-A)\subset C_-$$

with $\omega_u<\omega_c<\omega<0$ *where*

$$\omega_u:=\sup\{\operatorname{Re}\lambda:\lambda\in\sigma_u(-A)\},\omega_c:=\inf\{\operatorname{Re}\lambda:\lambda\in\sigma_c(-A)\}, \quad (2.2)$$

and $\sigma_c(-A)$ *is compact.*

ASSUMPTION A allows us to choose real numbers κ and μ such that

$$\omega_u<\kappa<\mu<\omega_c<0 \qquad (2.3)$$

We recall the Riesz projection (or spectral projection) *P* corresponding to $\sigma_c(-A)$, defined by $P=\frac{1}{2\pi i}\int_{l^+}R(\lambda,-A)d\lambda$, where l^+ is a closed regular curve contained in $\rho(-A)$, surrounding $\sigma_c(-A)$ and positively oriented. Denoting by $(e^{-tA})_{t\geq 0}$ the analytic semigroup generated by $-A$. We now recall some properties, called *dichotomy estimates*, of the analytic semigroup $(e^{-tA})_{t\geq 0}$.

Proposition 2.2 (see Nguyen – Bui [9]). *Let P be the Riesz projection and choose* $\kappa<\mu<0$ *being the real numbers as in* (2.3). *For* $\theta>0$*, the following dichotomy estimates hold true:*

$$\| e^{-tA}P \|\leq M_1e^{-\mu|t|} \qquad \text{for all } t\in R, \qquad (2.4)$$

$$\left\| A^{\theta}e^{-tA}P\right\|\leq M_2e^{-\mu|t|} \qquad \text{for all } t\in R, \qquad (2.5)$$

$$\left\| e^{-tA}(I-P)\right\|\leq Me^{\kappa t} \qquad \text{for all } t\geq 0, \qquad (2.6)$$

$$\left\| A^{\theta}e^{-tA}(I-P)\right\|\leq Nt^{-\theta}e^{\kappa t} \qquad \text{for all } t>0. \qquad (2.7)$$

We end the

Proposition 2.3. *Let* O *be a sectorial operator on a Banach space* X *with a discrete spectrum described as*

$$0>\lambda_1\geq\lambda_2\geq..., \quad \text{each with finite multiplicity and } \lim_{k\to\infty}\lambda_k=-\infty \qquad (2.8)$$

and let H be bounded linear operator on X such that norm $\|H\|$ *is small enough. Then, the operator O+H is a sectorial operator satisfying the* ASSUMPTION A.

Besides the assumptions on the linear operator *A*, we need the φ-Lipschitz property of the nonlinear term f in the following definition.

Definition 2.4. Let E be an admissible Banach function space on $\mathbb{R}$ and let $\varphi \in E$ be positive function. Put $X_\theta := D(A^\theta)$ for $\theta \in [0,1)$. Then, a function $f : \mathbb{R} \times X_\theta \to X$ is said to be φ-Lipschitz if f satisfies

$$\|f(t,x)\| \le \varphi(t)\left(1+\|A^\theta x\|\right), \qquad \text{for } a.e. t \in \mathbb{R} \text{ and for all } x \in X_\theta, \qquad (2.9)$$

$$\|f(t,x_1) - f(t,x_2)\| \le \varphi(t)\|A^\theta(x_1 - x_2)\| \quad \text{for } a.e. t \in \mathbb{R} \text{ and for all } x_1, x_2 \in X_\theta. \qquad (2.10)$$

Moreover, we need the following assumption on the function φ.

Assumption B. *Let E be an admissible Banach function space on $\mathbb{R}$ and let $\varphi \in E$ be positive function. We consider positive function $\varphi \in E$ such that*

$$R(\varphi,\theta) := \sup_{t\in\mathbb{R}} \left(\int_{t-1}^{t} \frac{\varphi(\tau)^{\frac{1+\theta}{2\theta}}}{(t-\tau)^{\frac{1+\theta}{2}}} d\tau \right)^{\frac{2\theta}{1+\theta}} < \infty \text{ ,where } 0 < \theta < 1. \quad (2.11)$$

We assume further that the nonlinear mapping $u \mapsto f(t,u)$ is of class C^1.

2.2 Inertial Manifolds for Evolution Equations

Consider the evolution equations

$$\begin{cases} \dot{x}(t) + Ax(t) = f(t,x(t)), & t > s, \\ x(s) = x_s, \end{cases} \qquad (2.12)$$

where $-A$ is a sectorial operator on the Banach space X with a gap in its spectrum, and $f : \mathbb{R} \times X_\theta \to X$ is a nonlinear operator for $X_\theta := D(A^\theta)$ being the domain of the fractional power A^θ for $0 \le \theta < 1$.

To prove the existence of inertial manifolds for parabolic evolution equations, instead of (2.12)we consider the integral equation

$$u(t) = e^{-(t-s)A}u(s) + \int_s^t e^{-(t-\xi)A} f(\xi,u(\xi))d\xi \text{ for } a.e. t \ge s. \qquad (2.13)$$

By a *solution* of equation (2.13) we mean a strongly measurable function $u(t)$ defined on an interval J with the values in X_θ that satisfies (2.13) for $t,\ s \in J$. We note that the solution u to equation (2.13) is called a *mild solution* of equation (2.12).

We suppose that A satisfies ASSUMPTION A and consider the Riesz projection P. We recall here that an *inertial manifold* for equation (2.13) is a collection of Lipschitz manifold. $M = (M_t)_{t\in\mathbb{R}}$ in X (each M_t is the graph of a Lipschitz mapping $\Phi_t : PX \to QX_\theta$) which is positively invariant, and which has the asymptotic completeness property, *i.e.*, for any solution $u(\cdot)$ of (2.13) and any fixed $s \in \mathbb{R}$, there is a positive constant H such that

$$\operatorname{dist}_{X_\theta}\left(u(t), M_t\right) \le He^{-\gamma(t-s)} \quad \text{for } t \ge s, \qquad (2.14)$$

where $\gamma > 0$, and $\operatorname{dist}_{X_\theta}$ denotes the Hausdorff semi-distance generated by the norm in X_θ.

Assume that equation (2.13) has an inertial manifold. We now rewrite the solution in the form $u(t) = p(t) + q(t)$, where $p(t) \in Pu(t)$, and $q(t) \in Qu(t)$, where $Q := I - P$. Then evolution equation (2.12) can be rewritten as a system of differential equations

$$\begin{cases} p^{\cdot}(t) + Ap(t) = Pf(t, p(t) + q(t)), \\ q^{\cdot}(t) + Aq(t) = Qf(t, p(t) + q(t)), \\ p\big|_{t=s} = p_s \equiv Pu_s, \quad q\big|_{t=s} = q_s \equiv Qu_s. \end{cases} \qquad (2.15)$$

Thanks to the positively invariant property of inertial manifolds, the condition $(p_s, q_s) \in \mathrm{M}_S$ implies that $(p(t), q(t)) \in M_t$, for $t > s$. Therefore, the solution lying in M_t can be found in two stages: at first we solve the problem

$$\dot{p}(t) + Ap(t) = Pf(t, p(t) + \Phi_t(p(t)), \qquad p\big|_{t=s} = p_s, \qquad (2.16)$$

and then we take $u(t) = p(t) + \Phi_t(p(t))$. Thus, the qualitative behavior of solutions lying in inertial manifolds is completely determined by the properties of ordinary differential equation (2.16) in the finite-dimensional space PX. The ordinary differential equation (2.16) is said to be *inertial form* of evolution equation (2.12).

We summarize results Nguyen – Bui [10, Theorem 3.5] and Nguyen – Bui – Do [11, Theorem 2.9] on the existence and regularity of inertial manifolds as follows. This result is a key mathematical tool to study the asymptotic behavior of predator-prey models, which is the main goal of this paper:

Theorem 2.5. *Let the linear operator A satisfy* ASSUMPTION A *and* φ *belongs to some admissible space E. Let f be* φ *-Lipschitz satisfy* ASSUMPTON B. *If*

$$k < 1, \quad \frac{MkM_2^2N_2}{(1-k)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty + k < 1. \qquad (2.17)$$

where

$$k := \begin{cases} \dfrac{NN_1 + M_1N_2}{1-e^{-\alpha}}\|\Lambda_1\varphi\|_\infty + NR(\varphi,\theta)\left[\dfrac{1-\theta}{\alpha(1+\theta)}\left(1-e^{-\alpha\frac{1+\theta}{1-\theta}}\right)\right]^{\frac{1-\theta}{1+\theta}} & \text{if } 0 < \theta < 1, \\ \dfrac{MN_1 + M_1N_2}{1-e^{-\alpha}}\|\Lambda_1\varphi\|_\infty & \text{if } \theta = 0. \end{cases} \qquad (2.18)$$

and $\alpha := \frac{\mu-\kappa}{2}$, *then equation (2.13) has a C^1-smooth interial manifold.*

Proof. See Nguyen – Bui [10, Theorem 3.5] and Nguyen – Bui – Do [11, Theorem 2.9].

Remark 2.1.

1. For $0<\theta<1$, the condition (2.17) is fulfilled if (1) the difference $\mu-\kappa$ is sufficiently large; (2) the norm $\|\Lambda_1\varphi\|_\infty = \sup_{t\in\mathbb{R}} \int_{t-1}^{t} \varphi(\tau)d\tau$ is sufficiently small. On the other hand, if $\theta=0$, then for the fulfillment of the condition (2.17) we need only the fact that the norm $\|\Lambda_1\varphi\|_\infty$ is sufficiently small.

2. If the nonlinear term f is locally φ-Lipschitz, i.e., f is φ-Lipschitz in some ball B_ρ (in X_θ) centered at 0 with radius ρ for some constant $\rho>0$, we can use the cut-off procedures as follows. Let $\vartheta(.)$ be an infinitely differentiable function on $[0,\infty)$ such that $\vartheta(s)=1$ *for* $0\le s\le 1, \vartheta(s)=0$ for $s\ge 2$, $0\le\vartheta(s)\le 1$ and $|\vartheta'(s)|\le 2$ for $s\in[0,\infty)$. We define the cut-off mapping

$$f_\rho(t,u) := \vartheta\left(\frac{\|A^\theta u\|}{\rho}\right) f(t,u) \qquad \textit{for all } u\in D(A^\theta). \qquad (2.19)$$

We have (see [10, Lemma 3.7]), *if* $f(t,u)$ *is locally* φ*-Lipschitz in a ball* B_ρ*, then*

$f_\rho(t,u)$ is $\left(\frac{2\rho^2+5\rho+2}{\rho}\varphi\right)$ *– Lipschitz.*

We then consider the following abstract Cauchy problem

$$\begin{cases} \dfrac{dx(t)}{dt} + Ax(t) = f_\rho(t,x(t)), & t>s, \\ x(s) = x_s, & s\in\mathbb{R}, \end{cases} \qquad (2.20)$$

where $f_\rho(t,.)$ defined as in (2.19). Since $f_\rho(t,.)$ is $\tilde{\varphi}$– Lipschitz for $\tilde{\varphi} := \frac{(2\rho^2+5\rho+2)}{\rho}\varphi$,

we can apply Theorem 2.5 to obtain that, if conditions in Remark 2.1–(1) hold for $\tilde{\varphi}$ then there exists an inertial manifold of for mild solutions to equation (2.19).

3. FINITE-DIMENSIONAL ASYMPTOTIC BEHAVIOR OF A PREDATOR-PREY MODEL WITH CROSS-DIFFUSION

In this section, we will study the predator-prey population model with cross-diffusion which is described by the following partial differential equations of parabolic type (see, e.g., J.D. Murray [6, 7])

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - D_1 \Delta u = ru\left(1 - \dfrac{u}{K(t)}\right) - quv, & t > s,\ 0 < x < \pi, \\
\dfrac{\partial v}{\partial t} - D_2 \Delta v = -dv + cuv & t > s,\ 0 < x < \pi, \\
\dfrac{\partial u(t,0)}{\partial t} = \dfrac{\partial u(t,\pi)}{\partial t} = 0, & t > s, \\
u(s,x) = u_s(x) & 0 < x < \pi, \\
v(s,x) = v_s(x) & 0 < x < \pi,
\end{cases}
\tag{3.1}
$$

where $u = u(t,x)$ and $v = v(t,x)$ are the prey and predator populations (depending on time- variable t and space-variable x), respectively; the positive constants D_1 and D_2 are diffusion coefficients; the positive-valued functions r is the birth rate of the prey, q is the death rate of prey by predator, and c is the growth rate of the predator in presence of the prey; whereas d isthe death rate of the predator in absence of the prey, lastly the positive function $K(t)$ represents the carrying capacity of the environment. The terminology "diffusion" in this context represents the occurrence of displacements of the predator (to catch the prey) and of the prey (to run away from the predator). Hereafter, the notion Δ denotes for Laplace operator with relevant Neumann boundary conditions.

Let $\left(\bar{u}, \bar{v}\right)$ be a stationary solution of the system, e.g., $\bar{u} = 0$ and $\bar{v}$ is a solution to the eigenvalue problem relative to Laplace-Neumann operator $\Delta v = \dfrac{d}{D_2} v$. We can change to new variables by putting $u = U + \bar{u},\ v = V + \bar{v}$. We arrive at

$$
\frac{\partial}{\partial t}\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} D_1\Delta & 0 \\ 0 & D_2\Delta \end{bmatrix}\begin{bmatrix} U + \bar{u} \\ V + \bar{v} \end{bmatrix} + \begin{bmatrix} r(U+\bar{u}) - \dfrac{r(U+\bar{u})^2}{K(t)} - q(V+\bar{v})(U+\bar{u}) \\ c(U+\bar{u})(V+\bar{v}) - d(V+\bar{v}) \end{bmatrix}
$$

$$
= \left(\begin{bmatrix} D_1\Delta - r & 0 \\ 0 & D_2\Delta - d \end{bmatrix} + \begin{bmatrix} 2r - \bar{v}q & -\bar{u}q \\ c\bar{v} & c\bar{u} \end{bmatrix} \right) \begin{bmatrix} U \\ V \end{bmatrix} + \begin{bmatrix} -\dfrac{2r\bar{u}U + rU^2}{K(t)} - qUV \\ cUV \end{bmatrix}.
$$

The above system can be rewritten in an operator form as $\dfrac{dx(t)}{dt} + Ax(t) = f(t, x(t))$, where

$$
x := \begin{bmatrix} U \\ V \end{bmatrix}, \quad A := -\begin{bmatrix} D_1\Delta - r & 0 \\ 0 & D_2\Delta - r \end{bmatrix} - \begin{bmatrix} 2r - \bar{v}q & -\bar{u}q \\ c\bar{v} & c\bar{u} \end{bmatrix},
$$

$$
f(t,x) := \begin{bmatrix} -\dfrac{2r\bar{u}U + rU^2}{K(t)} - qUV \\ cUV \end{bmatrix}.
$$

In this problem we can choose the power $\theta = 0$. We define operators L_i as

$$L_j(\phi) := D_i\phi'' - r \text{ for all } \phi \in D(L_i) := \left\{\phi \in C^2[0,\pi] : \phi'(0) = \phi'(\pi) = 0\right\}.$$

It can be seen that (see K.J. Engel & R. Nagel [2, 4.8 Example]) L_i, for $i = 1,2$, is sectorial operator and L_i generates an analytic semigroup $\left(e^{tL_i}\right)_{t\geq 0}$. Moreover, the spectrum of L_i is consisted of eingenvalues given by

$$\sigma\left(L_1\right) = \left\{-r, -D_1 1^2 - r, -D_1 2^2 - r, ..., -D_1 n^2 - r, ...\right\},$$
$$\sigma\left(L_2\right) = \left\{-d, -D_2 1^2 - d, -D_2 2^2 - d, ..., -D_2 n^2 - d, ...\right\}.$$

Note that $\sigma(L_i) \subset (-\infty, 0)$. Therefore, the linear operator

$$L := \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \text{ on the domain } D(L) := D(L_1) \times D(L_2) \subset \mathrm{X} := C[0,\pi] \times C[0,\pi]$$

is the generator of an analytic semigroup $\left(\begin{vmatrix} e^{tL_1} & 0 \\ 0 & e^{tL_2} \end{vmatrix}\right)_{t\geq 0}$. Furthermore, L has spectrum of the form

$$\sigma(L) = \left\{-r, -D_1 1^2 - r, -D_1 2^2 - r, ..., -D_1 n^2 - r, ... \right.$$
$$\left. -d, -D_2 1^2 - d, -D_2 2^2 - d, ..., -D_2 n^2 - d, ...\right\}. \qquad (3.2)$$

Next, we put

$$B := \begin{bmatrix} 2r - \bar{v}q & -\bar{u}q \\ c\bar{v} & c\bar{u} \end{bmatrix}.$$

Then, the operator B is a linear bounded operator on X. We consider $-A = L + B$ with $D\left(A\right) = D\left(L\right)$. Now, the predator-prey model (3.1) can be rewritten as the following abstract Cauchy problem

$$\begin{cases} \dfrac{dx(t)}{dt} + Ax(t) = f(t, x(t)), t > s, \\ x(s) = x_s, s \in \mathbb{R}. \end{cases} \qquad (3.3)$$

To investigate the existence of an admissibly inertial manifold for the predator-prey model with coss-diffusion, we need to show that the linear operator is a sectorial operator satisfies the ASSUMPTION A and the nonlinear term is φ-Lipschitz.

Firstly, thanks to the Proposition 2.3 we obtain that the linear operator $-A = L + B$ is a sectorial operator and satisfies the ASSUMPTION A (by the choice of $L = O$, $B = H$, and X $:= C[0,\pi] \times C[0,\pi]$).

Secondly, denote by B_ρ the ball centered at 0 with radius ρ in the space X . We next verify that the nonlinear term is a φ-Lipschitz function for some function φ. For $x(t) = \begin{bmatrix} U(t) \\ V(t) \end{bmatrix} \in \mathrm{X}$ $:= C[0,\pi] \times C[0,\pi]$, we have

$$\left\| f\left(t,x(t)\right)\right\| = \left\| \begin{bmatrix} -\dfrac{2r\bar{u}U(t)+r\left[U(t)\right]^2}{K(t)} - qUV \\ cU(t)V(t) \end{bmatrix} \right\|$$

$$\leq \left(\frac{\left|2r\bar{u}\right| + r\rho}{K(t)} + \left(q+c\right)\rho \right)\left(1 + \left\| \begin{bmatrix} U(t) \\ V(t) \end{bmatrix} \right\| \right). \qquad (3.4)$$

Taking $x_1(t) = \begin{bmatrix} U_1(t) \\ V_1(t) \end{bmatrix}, x_2(t) = \begin{bmatrix} U_2(t) \\ V_2(t) \end{bmatrix} \in X := C\left[0,\pi\right] \times C\left[0,\pi\right],$ we have

$$\left\| f\left(t,x_1\right) - f(t,x_2)\right\| = \left\| \begin{bmatrix} -\dfrac{2r\bar{u}\left[U_2(t)-U_1(t)\right] + r\left[U_2(t)^2 - U_1(t)^2\right]}{K(t)} - qU_1(t)V_1(t) - U_2(t)V_2(t) \\ cU_1(t)V_1(t) - U_2(t)V_2(t) \end{bmatrix} \right\|$$

$$= \left(\frac{\left|2r\bar{u}\right| + r\rho}{K(t)} + \left(q+c\right)\rho \right)\left(\left\| \begin{bmatrix} U_1(t) \\ V_1(t) \end{bmatrix} - \begin{bmatrix} U_2(t) \\ V_2(t) \end{bmatrix} \right\| \right). \qquad (3.5)$$

From (3.4) and (3.5) we estimate

$$\max\left\{ \frac{\left|2r\bar{u}\right| + r\rho}{K(t)} + \left(q+c\right)\rho, \frac{\left|2r\bar{u}\right| + 2r\rho}{K(t)} + \left(q+c\right)\rho \right\} = \frac{\left|2r\bar{u}\right| + 2r\rho}{K(t)} + \left(q+c\right)\rho.$$

Put

$$\varphi(t) = \frac{\left|2r\bar{u}\right| + 2r\rho}{K(t)} + \left(q+c\right)\rho \quad \text{for all } t \in \mathbb{R}, \qquad (3.6)$$

We infer that the function $f(t,x)$ is φ-Lipschitz where the function $\varphi(.)$ is defined by (3.6).

Apply Remark 2.1 – (2) we have the function

$$f_p(t,x) = \vartheta\left(\frac{\|x\|}{\rho}\right) f(t,x) \quad \text{for all } t \in \mathbb{R},$$

is φ-Lipschitz (note that, we chosen $R = \rho$), here

$$\varphi(t) := \frac{2\rho^2 + 5\rho + 2}{\rho}\varphi(t)$$

$$= \frac{2\rho^2 + 5\rho + 2}{\rho}\left(\frac{\left|2r\bar{u}\right| + 2r\rho}{K(t)} + \left(q+c\right)\rho \right)$$

$$= \frac{\left(2\rho + 5 + 2\rho^{-1}\right)\left(\left|2r\bar{u}\right| + 2r\rho\right)}{K(t)} + \left(2\rho^2 + 5\rho + 2\right)\left(q+c\right). \qquad (3.7)$$

Obviously, in the real environment of nature, the carrying capacity of the environment K should depend on time, e.g., in the spring there should be more edibles for the preys than in the winter. Therefore, we consider the carrying capacity of the environment as a function of time. We refer the reader to M.K.A. Al-Moqbali *et al.* [1] for more detailed treatment on the population models with variable carrying capacity. In this paper we consider carrying capacity with logistic growth, that is carrying capacity satisfies

$$\frac{dK}{dt} = \beta K\left(1 - \frac{K}{K^*}\right), \tag{3.8}$$

where β and K^* are positive constants. To prove the existence of an inertial manifold, we assume that

$$K(t) = \frac{K^*}{1 + be^{-\beta|t|}} \qquad \text{for all } t \in \mathbb{R}.$$

Hence

$$\begin{aligned}\varphi(t) &= \frac{\left(2\rho + 5 + 2\rho^{-1}\right)\left(\left|2r\bar{u}\right| + 2\rho r\right)}{K^*}\left(1 + be^{-\beta|t|}\right) + \left(2\rho^2 + 5\rho + 2\right)\left(q + c\right) \\ &= \frac{\left(2\rho + 5 + 2\rho^{-1}\right)\left(\left|2r\bar{u}\right| + 2\rho r\right)}{K^*} e^{-\beta|t|} \\ &\quad + \frac{\left(2\rho + 5 + 2\rho^{-1}\right)\left(\left|2r\bar{u}\right| + 2\rho r\right)}{K^*} + \left(2\rho^2 + 5\rho + 2\right)\left(q + c\right)\end{aligned} \tag{3.9}$$

Put

$$h_1 := \frac{\left(2\rho + 5 + 2\rho^{-1}\right)\left(\left|2r\bar{u}\right| + 2\rho r\right)}{K^*}$$

$$h_2 := \frac{\left(2\rho + 5 + 2\rho^{-1}\right)\left(\left|2r\bar{u}\right| + 2\rho r\right)}{K^*} + \left(2\rho^2 + 5\rho + 2\right)\left(q + c\right)$$

Then (3.9) becomes

$$\varphi(t) = h_1 e^{-\beta|t|} + h_2 \qquad \text{for all } t \in \mathbb{R} \tag{3.10}$$

Now we can consider the following cut-off evolution equation in which the nonlinear term $f_p(t,x)$ is φ-Lipschitz for φ as in (3.10),

$$\begin{cases} \dot{x}(t) + \mathrm{A}x(t) = f_p\left(t, x(t)\right), & t > s, \\ x(s) = x_s, & s \in \mathbb{R}. \end{cases} \tag{3.11}$$

We have that $\varphi \in M$ since

$$\left\|\Lambda_1 \varphi\right\|_\infty = \sup_{t\in\mathbb{R}} \int_{t-1}^{t} \left|\varphi(\tau)\right| d\tau \le \frac{h_1}{\beta} + h_2. \tag{3.12}$$

Applying Theorem 2.5 we conclude that if gap $\mu - \kappa$ is sufficiently large and the norm $\left\|\Lambda_1\varphi\right\|_\infty$ is sufficiently small, then there exists a C^1-smooth inertial manifold for the problem (3.11). Finally, for the predator-prey model under consideration, this inertial manifold is finite dimension.

4 CONCLUSIONS

This paper has applied the theory of inertial manifolds to study the finite-dimensional asymptotic behavior of a predator-prey model with cross-diffusion. The paper lists recently published results on the existence and regularity of inertial manifolds for nonautonomous evolution equations. Then, using appropriate linear operators and function spaces, the predator-prey model is rewritten as an evolution equation in a Banach space. The existence of the inertial manifold and the inertial form allows to conclude about the finite-dimensional asymptotic behavior of the ecological model under consideration. Future work will study finite-dimensional asymptotic behavior of a predator-prey model with finite delay.

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