A family of modified Newton iteration method for solving nonlinear algebraic equations

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***Abstract*— In this study, a modified Newton iteration version for solving nonlinear algebraic equations is formulated using a correction function derived from convergence order condition of iteration. If the second order of convergence is selected, we get a family of the modified Newton iteration method. Several forms of the correction function are proposed in checking the effectiveness and accuracy of the present iteration method. For illustration, approximate solutions of four examples of nonlinear algebraic equations are obtained and then compared with those obtained from the classical Newton iteration method.**

***Index Terms*—nonlinear algebraic equation, modified Newton iteration, correction function.**

1 INTRODUCTION

Finding solutions of nonlinear algebraic equation is one of the most important tasks in computations and analysis of applied mathematical and engineering problems [1,2]. The iteration algorithm for nonlinear algebraic systems can be classified into two main groups: bracketing techniques and fixed point methods. The bracketing techniques can be addressed as the well-known bisection [3,4], Regula Falsi method [5], Cox method [6]. The group of fixed point methods includes a long list of research contributions, among them are Halley method [7], Jaratt method [8], King's method [9].

The Newton method is a well-known technique for solving non-linear equations. It can be considered as an improved version of the classical fixed point method with iteration function containing the information of derivative at each iteration step. The Newton method has a fast convergence rate of iteration process when a starting point is on the neighborhood of the exact solution of equation under consideration. The development contributions of Newton method are archived based on the improvement of convergence order, accuracy and computational time [10-14]. In a work by Frontini and Sormani [10,11], a modification of the Newton’s method which produces iterative methods with order of convergence three has been proposed to find multiple roots of a nonlinear algebraic equations. In [12], a research on the fourth-order convergence of Newton method was carried out by Chun and Ham. In their approach, per iteration requires two evaluations of the function and one of its first-derivative. For the order of convergence five, analyses of convergence and numerical tests were presented in [13], and based on these analyses, a class of new multi-step iterations was developed. The higher-order convergence analysis problem of the Newton method is an interesting topic for future researches in order to obtain solutions of nonlinear algebraic systems with effectiveness and high precision.

The objective of the present paper is to generalize the classical Newton formula by introducing a new correction function $h(t)$ that plays as a correction coefficient for the ratio of $f(x)$ to $f'(x)$ at per iteration step. The form of $h(t)$ depends on the convergence condition of iteration method. In our study, the second-order convergence condition is used to obtain a family of modified Newton iteration method.

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2 FORMULATION OF MODIFIED NEWTON ITERATION METHOD

In this section we are concerned with solving the algebraic equation of the form

$$f(x)=0 \tag{1}$$

in which the function $f(x)$ is continuous on the interval $B\equiv(a,b)\subset\mathbb{R}$, and has non-zero continuous derivative, i.e. $f'(x)\neq 0$ for $x\in(a,b)$. Assume that Eq. (1) has a single solution α in (a,b). To find the solution α, one can use the following classical Newton iteration formula

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \tag{2}$$

Let $e_n=x_n-\alpha$ be a difference value between the exact solution α and approximate solution value at n-th iteration step. It is well-known that the formula (2) has the second-order convergence with the solution error at (n+1)-th iteration step being e_{n+1},

$$e_{n+1}=c_2e_n^2+O(e_n^3) \tag{3}$$

where the notation $O(e_n^3)$ denotes the higher-order terms than e^2. The coefficient c_2 in Eq. (2) is defined as

$$c_2=\frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)} \tag{4}$$

with assuming that the second-order derivative of $f(x)$ at $x=\alpha$ exists.

We have the following theorem for iteration:

Theorem 1. Given a differential function $f(x)$ defined on an interval $B\equiv(a,b)\subset\mathbb{R}$ with single solution α belonging to B, i.e. $f(\alpha)=0$. If $h(t)$ is an arbitrary continuous differential function of argument t with $h(0)=1$ and $|h'(0)|<+\infty$, and x_0 is a starting point close to α, the iteration determined by

$$x_{n+1}=x_n-h(u_n)\frac{f(x_n)}{f'(x_n)} \tag{5}$$

has the second-order convergence with solution error e_{n+1} at (n+1)-th iteration step

$$e_{n+1}=\left(c_2h(0)-h'(0)\right)e_n^2+O(e_n^3) \tag{6}$$

where the coefficient c_2 determined by (4), and

$$u_n=\frac{f(x_n)}{f'(x_n)} \tag{7}$$

Proof. Expanding the Taylor series of $f(x_n)=f(\alpha+e_n)$ about the solution point α and noting $f(\alpha)=0$, we obtain

$$f(x_n)=f'(\alpha)\left(e_n+c_2e_n^2\right)+O(e_n^3) \tag{8}$$

From Eq. (8), the derivative $f'(x_n)$ can be derived as follows

$$f'(x_n)=f'(\alpha)(1+2c_2e_n)+O(e_n^2) \tag{9}$$

Using Eqs. (8) and (9), the ratio u_n of $f(x_n)$ to $f'(x_n)$ can be estimated as follows

$$u_n=\frac{f(x_n)}{f'(x_n)}=\frac{f'(\alpha)\left(e_n+c_2e_n^2\right)+O(e_n^3)}{f'(\alpha)(1+2c_2e_n)+O(e_n^2)} \tag{10}$$
$$\approx e_n-c_2e_n^2+O(e_n^3)$$

where the expression of u_n is retained at the second-order of the error e_n.

The Taylor expansion of $h(u_n)$ in the neighborhood of zero point gives

$$h(u_n)=h(0)+h'(0)u_n+\frac{1}{2}h''(0)u_n^2+O(u_n^3) \tag{11}$$

Substituting Eq. (10) into Eq. (11) for u_n, and the result into Eq. (5), we get

$$x_{n+1}=x_n-h(0)e_n+\left(c_2h(0)-h'(0)\right)e_n^2+O(e_n^3) \tag{12}$$

Eq. (12) can be rewritten in the form of solution error

$$e_{n+1}=\left(1-h(0)\right)e_n+\left(c_2h(0)-h'(0)\right)e_n^2+O(e_n^3) \tag{13}$$

The expression (13) shows that the second-order condition of iteration (5) is satisfied if the correction function $h(t)$ is selected so that three following conditions must be fulfilled:

i. $h(t)$ is continuous differential function on some open interval $I\subset\mathbb{R}$.

ii. $h(0)=1$

iii. $|h'(0)|<+\infty$, i.e. the value of derivative $h'(0)$ must be finite.

From the second condition ii., Eq. (13) is reduced to a simpler form

$$e_{n+1}=\left(c_2h(0)-h'(0)\right)e_n^2+O(e_n^3) \tag{14}$$

The proof is complete.

3 THE CHOICE OF CORRECTION FUNCTIONS

The addition of the correction function $h(t)$ gives a generalized form of the classical Newton iteration method. The Newton method is recovered if the function $h(t)$ is taken to be unity, i.e. $h(t)=1$. The importance of the function $h(t)$ is that it decides the magnitude of coefficient $c_2h(0)-h'(0)$ of solution error in the expression (14). In the case that value of e_n is very small, and can neglect the higher-order than 3 of e_n, the error e_{n+1} at (n+1)-th iteration step can be estimated as a quadratic function of e_n:

$$e_{n+1} \approx \hat{e}_{n+1} = \left(c_2 - h'(0)\right)e_n^2 \tag{15}$$

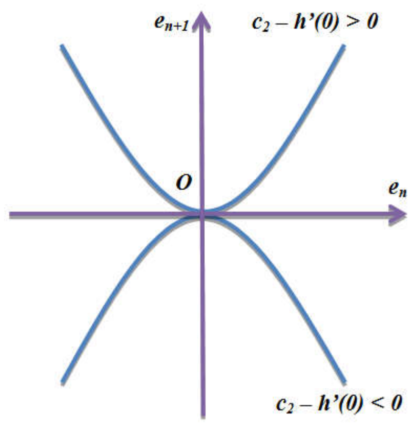

Figure 1. The function e_{n+1} as a quadratic function of e_n when neglecting the higher-order terms than 3.

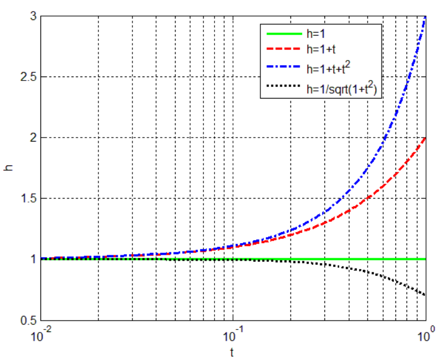

Figure 2. Graphs of four chosen correction functions

The expression (15) shows that the sign of the estimated error value $\hat{e}_{n+1}$ depends on the sign of the coefficient $c_2-h'(0)$. If $c_2 \geq h'(0)$, the estimated error $\hat{e}_{n+1}$ increases in e_n^2. Fig. 1 illustrates the behavior of the function $\hat{e}_{n+1}$ when e_n is varying for two cases: $c_2-h'(0)>0$ and $c_2-h'(0)<0$. In numerical computation practice, if the initial value of solution is selected close to the desired solution, after several numbers of iterations, the value of $\hat{e}_{n+1}$ becomes very small. If $c_2-h'(0)=0$, the estimated error $\hat{e}_{n+1}$ will vanish, therefore the solution error e_{n+1} is now a function of at least order 3 of the previous step solution error e_n. However the choice of $h(t)$ in this case is very difficult because in almost cases of algebraic equations, the desired solution α is not known exactly.

We here consider a special case of choosing the correction function $h(t)$: $h(0)=1$ and $h'(0)=0$. For this case, the estimated error $\hat{e}_{n+1}$ is

$$\hat{e}_{n+1} = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}e_n^2 \tag{16}$$

It is seen that the estimated error $\hat{e}_{n+1}$ in Eq. (16) does not depend on the behavior of the function $h(t)$ for $t \neq 0$ provided that the conditions $h(0)=1$ and $h'(0)=0$ are satisfied. Two examples of $h(t)$ in this case are

(f1): $h(t)=\dfrac{1}{\sqrt{1+t^2}}$

(f2): $h(t)=1+t^2$

Noting that the choice $h(t)\equiv 1$ in the classical form of Newton method is such a condition situation. In several studies, the function $h(t)$ can be chosen as some constants, for example, $h(t)\equiv 2/3$ in [15].

Another special case of $h(t)$ is presented here that satisfies conditions $h(0)=1$ and $h'(0)=1$. In this case, an example of $h(t)$ is taken to be:

(f3): $h(t)=1+t$

This case shows that the estimated error $\hat{e}_{n+1}$ only depends on the nature of the function $f(x)$, i.e. depends on the quantity $c_2=\dfrac{1}{2}\dfrac{f''(\alpha)}{f'(\alpha)}$. If this quantity is large, the estimated $\hat{e}_{n+1}$ will be large, too. The graphs of the function h(t)=1 (for classical Newton method) and three functions (f1), (f2) and (f3) are plotted in Fig. 2. If the iterations are convergent and magnitude of derivative of $f(x)$ at each iteration step is finite, it can be examined that the ratio $f(x_n)/f'(x_n)$ is quite small. This leads to the fact that the argument t of the correction functions is small [here, the argument t represents for $f(x_n)/f'(x_n)$]. In Fig. 2, t is taken in the interval [0,1]. Several examples for illustrating the effectiveness of the modified Newton iteration method using above correction functions will be presented in next section.

4 EXAMPLES

4.1 Example 1

Consider the following polynomial equation

$$x^3+4x^2-10=0 \tag{17}$$

We here use the classical Newton iteration formula and modified Newton formulae with three forms of the correction function $h(t)$: $h(t)=1+t$, $h(t)=1+t^2$, $h(t)=1/\sqrt{1+t^2}$. The obtained results for Eq. (17) with different values of the starting point x_0 of iteration are given in Tab. 1. The obtained approximate solution is 1.365230013 with tolerance $\varepsilon=10^{-9}$ for all of iterations. The stopping criteria of iterations are $|x_{n+1}-x_n|<\varepsilon$ and $|f(x_{n+1})|<\varepsilon$. For the same tolerance ε, the effectiveness of iterations is demonstrated by the number of iteration steps to obtain the desired solution of the equation (17).

TABLE 1. Approximate solution values and corresponding number of iteration steps at several values of starting point x_0 (No.: number of iteration steps, NaN: divergence).

x_0	Newton	No.	$h=1+t$	No.	$h=1+t+t^2$	No.	$h=1/\sqrt{1+t^2}$	No.
0.5	1.365230013	6	NaN	16	1.365230013	7	1.365230013	4
1.0	1.365230013	4	1.365230013	5	1.365230013	4	1.365230013	4
1.5	1.365230013	4	1.365230013	4	1.365230013	4	1.365230013	4
2.0	1.365230013	5	1.365230013	5	1.365230013	5	1.365230013	5
2.5	1.365230013	5	1.365230013	5	NaN	9	1.365230013	6
3.0	1.365230013	6	1.365230013	6	NaN	7	1.365230013	6
3.5	1.365230013	6	NaN	13	NaN	13	1.365230013	7
4.0	1.365230013	6	NaN	17	NaN	22	1.365230013	8

It is seen from Tab. 1 that, as the starting point x_0 is increasing from 0.5 to 4.0, the maximum iteration step number of the classical Newton method is 6 whereas that of modified Newton method depends on the choice of the correction function $h(t)$. If the function $h(t)=1+t$ is selected, the number 17 of iteration steps is not enough to reach the desired solution when the starting point is taken far from 1.365230013 (approximate solution point). In the narrow range of starting point from 1.0 to 3.0, the solution 1.365230013 still can be attained with several iteration steps similar to the classical Newton method. For the case $h(t)=1+t^2$, the domain of starting points for iteration should be chosen [1.0, 2.0] that even though is narrower than the case $h(t)=1+t$. For the chosen function $h(t)=1/\sqrt{1+t^2}$, the obtained results of iteration step number are nearly the same as the classical Newton method. Fig. 3 is the basin of attraction in 1D for Eq. (17) for different values of starting point x_0 in two cases: the classical Newton iteration formula and modified Newton formula with $h(t)=1/\sqrt{1+t^2}$. If x_0 is far from 1.365230013, the number of iteration steps will increase.

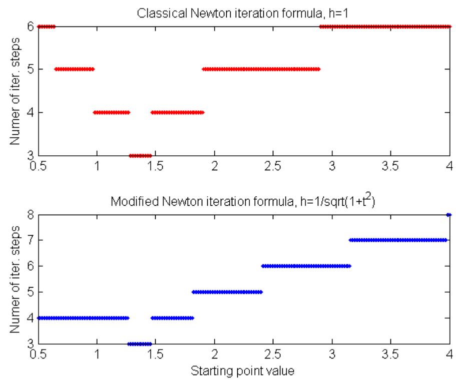

Figure 3. Basin of attraction in 1D illustration for Example 1 in a range of starting points

4.2 Example 2

The second example is to solve the following equation

$$x^2-e^x-3x+2=0 \tag{18}$$

Two correction functions are selected, $h(t)=1+t$ and $h(t)=1/\sqrt{1+t^2}$. The numerical results for this example are presented in Tab. 2. The basins of attraction for Example 2 in two cases of $h(t)$ are plotted in Fig. 4 in the domain [-4, 4] of starting point. Tab. 2 reveals that the choice of $h(t)=1/\sqrt{1+t^2}$ is better than that of $h(t)=1+t$ because the number of iteration steps of the modified method is nearly equal to that of the classical Newton method whereas the choice $h(t)=1+t$ yields several positions of starting point which lead to the divergence, for examples, $x_0=-2$, $x_0=-1.5$, $x_0=-1$.

TABLE 2. Approximate solution values and corresponding number of iteration steps of Example 2

x_0	Newton	No.	$h=1+t$	No.	$h=1/\sqrt{1+t^2}$	No.
-2.0	0.2575302854	5	NaN	13	0.2575302854	6
-1.5	0.2575302854	4	NaN	14	0.2575302854	6
-1.0	0.2575302854	4	NaN	17	0.2575302854	5
-0.5	0.2575302854	4	0.2575302854	7	0.2575302854	4
0	0.2575302854	3	0.2575302854	5	0.2575302854	3
0.5	0.2575302854	3	0.2575302854	5	0.2575302854	3
1.0	0.2575302854	3	0.2575302854	7	0.2575302854	3
1.5	0.2575302854	4	0.2575302854	9	0.2575302854	5
2.0	0.2575302854	4	0.2575302854	8	0.2575302854	5
2.5	0.2575302854	5	0.2575302854	5	0.2575302854	5
3.0	0.2575302854	5	0.2575302854	7	0.2575302854	7

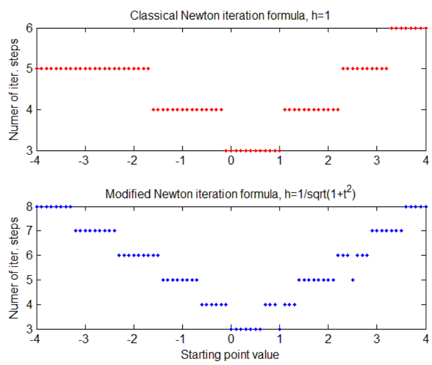

Figure 4. Basin of attraction in 1D illustration for Example 2 in a range of starting points

4.3 Example 3: Equation in complex domain

We consider the following simple equation in complex domain

$$z^3-1=0 \tag{19}$$

It is seen that Eq. (19) has three solutions $z_1=1$, $z_2=\left(-1+i\sqrt{3}\right)/2$ and $z_3=\left(-1-i\sqrt{3}\right)/2$. In the complex plane, three solutions are three vertices of an equilateral triangle. The iteration formulae can provide insight of the nature of iteration processes for approximate solutions of nonlinear equations. Using the Newton formula, we have the following iteration series for Eq. (19)

$$z_{n+1}=z_n-\frac{z_n^3-1}{3z_n^2}=\frac{2z_n^3+1}{3z_n^2} \tag{20}$$

Similarly, the following modified Newton iteration formula is formulated

$$z_{n+1}=z_n-\frac{1}{\sqrt{1+\left(\frac{z_n^3-1}{3z_n^2}\right)^2}}\frac{z_n^3-1}{3z_n^2} \tag{21}$$

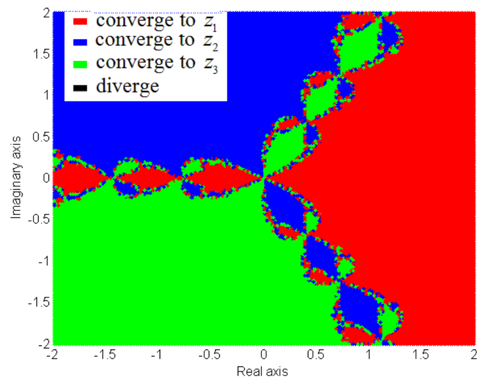

Figure 5. Basin of attraction of classical Newton iteration formula for Example 3

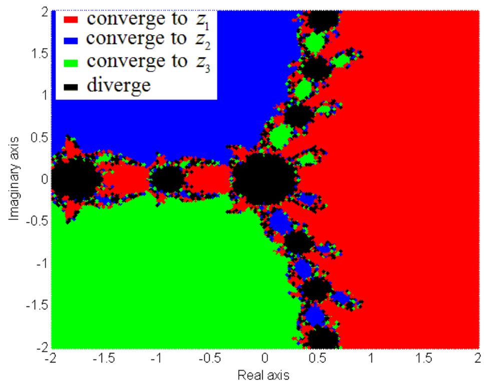

Figure 6. Basin of attraction of modified Newton iteration formula for Example 3 with $h(t)=1/\sqrt{1+t^2}$

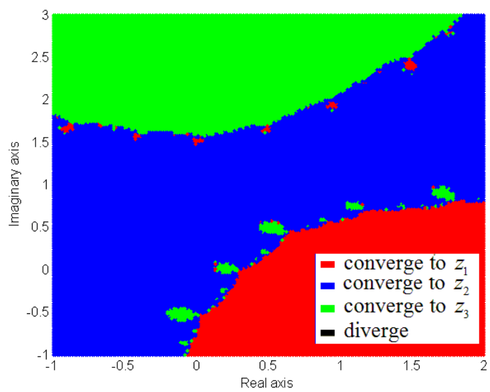

Figure 7. Basin of attraction of classical Newton iteration formula for Example 4.

The selection of a starting point for iteration is important because it affects to the convergence and approximate solution values of the iteration process. In Fig. 5, if the starting point is dropped on the red color region, the solution $z_1 = 1$ can be obtained from the iteration process. In the blue region, however, the iteration solution series tend to the second solution $z_2 = \left(-1 + i\sqrt{3}\right)/2$. The third solution $z_3 = \left(-1 - i\sqrt{3}\right)/2$ can be obtained if the starting point is taken in the green region. It is observed that in the 2D domain [-2, 2]x[-2, 2] with 200x200 starting points, there exist a number of points at which the iteration process is divergent. In Fig. 5, divergence points belong to the black region.

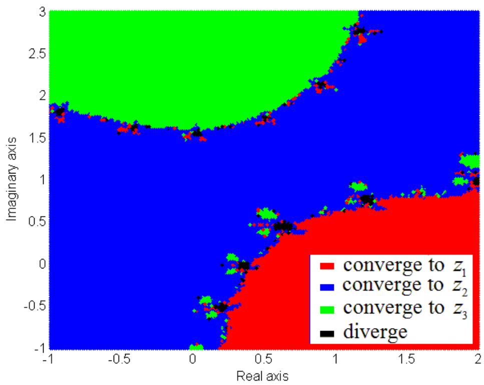

Figure 8. Basin of attraction of modified Newton iteration formula for Example 4 with $h(t) = 1/\sqrt{1+t^2}$

Fig. 6 exhibits the difference between the convergent domain of the modified iteration method and that of the classical Newton method. The distribution of convergent points of Fig. 6 is quite different from that of Fig. 5. The black color region becomes larger, i.e. the number of divergent points increases if using the modified version of Newton method. For a set of points lying on the neighborhood of desired solutions, the estimated errors of the classical Newton method and modified version with $h = 1/\sqrt{1+t^2}$ are nearly the same and this can be seen from Eq. (14) because of $h'(0) = 0$.

4.4 Example 4: Another complex equation

Let us solve the following complex equation:

$$z^3 - (1+3i)z^2 + (3i-2)z + 2 = 0 \qquad (22)$$

Eq. (22) has three solutions, $z_1 = 1$, $z_2 = i$, and $z_3 = 2i$ at different positions in the complex plane. The basins of attraction of the Newton and modified formulae for Eq. (22) are presented in Figs. 7 and 8. The distribution of starting points is not symmetric. The red, blue and green color regions show the convergence of both iteration methods for z_1, z_2, z_3, respectively. Also, the black region is the divergent one of iterations.

5 CONCLUSIONS

Solving nonlinear algebraic equations plays an important role in areas of applied mathematics because this is usually a final stage in dealing with a series of implementation processes to find solutions of problems of mathematics and engineering. The Newton iteration method is simple and can be easy to implement to a specified algebraic equation. The our present study gives a family of iteration methods in which the classical Newton formula is a special case. The following results can be drawn from the family of modified Newton iteration method:

- The order of convergence of modified iterations in the family with different forms of the correction function is still remained to be two as the classical Newton method, as shown in Theorem 1. According to the definition of convergence order of iteration methods and Theorem 1, we have $\lim_{n\to+\infty} \frac{|e_{n+1}|}{|e_n|^2} = |c_2 - h'(0)| \geq 0$ that has a finite value because $h'(0)$ is finite. This means that the convergence of modified Newton method is quadratic.
- The obtained results show that the choice of correction functions affects to the convergence of the modified iterations and the number of iteration steps can grow considerably if the starting point is far from the desired solution of the nonlinear equation. In general, the number of iteration steps of modified Newton method is larger than that of the classical Newton method. If an appropriate correction function is chosen, however, the difference between the iteration step numbers of modified and classical Newton methods may be not considerable.
- The basins of attraction in 1D and 2D demonstrate convergent regions of iterations in which a starting point can approach to exact solutions. Our study has proposed the use of several forms of the correction function. It is seen that the correction function $h = 1/\sqrt{1+t^2}$ can be a good choice for our iteration formulae because this function possesses a property that $h'(0) = 0$ leading to the estimated error of iteration solution

being the same as that of the classical Newton iteration formula. Consequently, we have the following iteration formula:

$$x_{n+1} = x_n - \frac{|f'(x_n)|}{\sqrt{[f(x_n)]^2 + [f'(x_n)]^2}} \frac{f(x_n)}{f'(x_n)} \quad (23)$$

- Two other proposed modified iteration versions of the classical Newton formula also can be used to find solution of algebraic equations:

$$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n) + f'(x_n)} \text{ for } h(t) = 1/(1+t) \quad (24)$$

$$x_{n+1} = x_n - \frac{f(x_n) f'(x_n)}{[f(x_n)]^2 + [f'(x_n)]^2} \quad (25)$$

for $h(t) = 1/(1+t^2)$

- More formulae for the modified Newton iteration method can be established based on the methodology of this study.

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Họ các phương pháp lặp Newton cải tiến giải phương trình đại số phi tuyến

Nghiêm Xuân Lực, Nguyễn Như Hiếu

***Tóm tắt* - Trong nghiên cứu này, một phiên bản cải tiến của phương pháp lặp Newton để gải phương trình đại số phi tuyến được trình bày, trong đó có sử dụng một hàm hiệu chỉnh. Hàm hiệu chỉnh này thu được từ điều kiện hội tụ của phép lặp. Theo đó, nếu bậc hội tụ của phép lặp là hai, ta có thể thu được họ các phép lặp Newton có chứa cả phép lặp Newton truyền thống. Các tác giả lựa chọn một vài dạng hàm hiệu chỉnh khác nhau để kiểm tra tính hiệu quả và độ chính xác của phép lặp đề nghị. Một số ví dụ minh họa cho ta nghiệm xấp xỉ của bài toán giải phương trình đại số phi tuyến là khá tin cậy và có độ chính xác cao.**

***Từ khóa* - phương trình đại số phi tuyến, phép lặp Newton cải tiến, hàm hiệu chỉnh.**