

FRÉCHET SUBDIFFERENTIALS OF THE MINIMUM TIME FUNCTION AT POINTS IN THE TARGET

Nguyen Van Luong, Nguyen Thi Thu

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Abstract: *We derive formulas for computing the Fréchet subdifferential and Fréchet singular differential at points in the target of the minimum time function associated with a system governed by differential inclusions.*

Keywords: *Minimum time function, Fréchet subdifferentials, differential inclusions.*

1. Introduction and Preliminaries

We consider the following differential inclusion

$$\begin{cases} x'(t) \in F(x(t)), \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \quad \text{a.e. } t \in [0, \tau] \quad (1.1)$$

for some $\tau > 0$, where $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a multifunction satisfying the following standard conditions:

(F1) $F(x)$ is nonempty, convex, and compact for each $x \in \mathbb{R}^n$.

(F2) F is locally Lipschitz, i.e. for each compact set $K \subset \mathbb{R}^n$, there exists a constant $L > 0$ such that

$$F(x) \subset F(y) + L \|y - x\| \overline{B}, \text{ for all } x, y \in K.$$

(F3) there exists $\gamma > 0$ such that $\max\{\|v\|: v \in F(x)\} \leq \gamma(1 + \|x\|)$, for all $x \in \mathbb{R}^n$.

A solution of (1.1) is an absolutely continuous function $x(\cdot)$ defined on $[0, \tau]$ satisfying (1.1) with the initial value $x(0) = x_0$. We also say that $x(\cdot)$ is a trajectory of F starting at x_0 . Under the above assumptions on F , if $x(\cdot)$ is a trajectory of F defined on $[0, \tau]$ then there exists a constant $M > 0$ such that $\|x(t) - x_0\| \leq Mt$ for all $t \in [0, \tau]$. Without loss of generality, we fix the constant M for all trajectories and for all $\tau > 0$ considered in this paper. We next recall a result regarding C^1 trajectories of F .

Theorem 1.1. (See, e.g., [21] and pages 115-117 in [1]). *Assume (F1)-(F3). Let $E \subset \mathbb{R}^N$ be compact. Then there exists $\tau > 0$ such that associated to every $x \in E$ and*

Nguyen Van Luong, Nguyen Thi Thu
 Faculty of Natural Sciences, Hong Duc University
 Email: nguyenvanluong@hdu.edu.vn (✉)

$v \in F(x)$ is a trajectory $x(\cdot)$ defined on $[0, \tau]$ with $x'(0) = v$. Moreover, we have $\|x'(t) - v\| \leq LMt$ for all $t \in [0, \tau]$.

We assume that a closed subset K of \mathbb{R}^n is given which is called the target. The minimum time function $T: \mathbb{R}^n \rightarrow [0, +\infty]$ associated with the differential inclusion (1.1) to the target K is defined as follows. If $x \notin K$ then

$$T(x) := \inf \{t > 0 : \exists x(\cdot) \text{ satisfying (1.1) with } x(0) = x \text{ and } x(t) \in K\} \quad (1.2)$$

If there is no trajectory of F starting at x can reach K , then. If $x \in K$ then we set $T(x) = 0$. It is well-known that, under assumptions (F1) - (F3), the infimum in (1.2) is attained and the minimum time function T is lower semicontinuous [21].

The minimum time function is an important optimal value function of optimal control theory. This function has been widely studied since the beginning stages of optimal control theory [2, 3, 4, 5, 6, 7, 8, 9, 12, 16, 21]. In general, the minimum time function is not differentiable. Generalized differentiations of the minimum time function and their applications have been investigated by several researchers [11, 13, 14, 15, 18] and references therein). The aim of this paper is to present formulas for computing the Fréchet subdifferential and Fréchet singular subdifferential of the minimum time function at points in the target. Our results generalize the corresponding results in [17, 19].

The rest of this section is devoted to some basic concepts of nonsmooth analysis. Standard references are in [10, 20]. We denote by $\|\cdot\|$ the Euclidean norm in \mathbb{R}^n , by $\langle \cdot, \cdot \rangle$ the inner product. We also denote by $B(x, r)$ the open ball of radius $r > 0$ centered at x and $IB = B(0, 1)$.

Let $S \subset \mathbb{R}^n$ be a closed set and let $x \in S$. The Fréchet normal cone to S at x , written $\hat{N}_S(x)$, is the set

$$\hat{N}_S(x) := \left\{ \zeta \in \mathbb{R}^n : \limsup_{y \in S, y \rightarrow x} \frac{\langle \zeta, y - x \rangle}{\|y - x\|} \leq 0 \right\}.$$

Elements in $\hat{N}_S(x)$ are called Fréchet normals to S at x . In other words, $\zeta \in \hat{N}_S(x)$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\langle \zeta, y - x \rangle \leq \varepsilon \|y - x\|$, $\forall y \in B(x, \delta)$.

Let $f: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be an extended real-value function. The effective domain of f is the set $dom(f) := \{x \in \mathbb{R}^n : f(x) < +\infty\}$ and the epigraph of f is the set

$$epi(f) := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : x \in dom(f), \alpha \geq f(x)\}.$$

Let $x \in dom(f)$. The Fréchet subdifferential of f at x is the set

$$\hat{\partial} f(x) := \left\{ \zeta \in \mathbb{R}^n : \liminf_{y \rightarrow x} \frac{f(y) - f(x) - \langle \zeta, y - x \rangle}{\|y - x\|} \geq 0 \right\}$$

In other words, $\zeta \in \hat{\partial} f(x)$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\langle \zeta, y - x \rangle \leq f(y) - f(x) + \varepsilon \|y - x\|$, $\forall y \in B(x, \delta)$.

The Fréchet subdifferential of f at x can also be defined as follows:

$$\hat{\partial} f(x) = \left\{ \zeta \in \mathbb{R}^n : (\zeta, -1) \in \hat{N}_{\text{epi}(f)}(x, f(x)) \right\}$$

Elements in $\hat{\partial} f(x)$ are called *Fréchet subgradients* of f at x

The *Fréchet singular subdifferential* of f at x is the set

$$\hat{\partial}^\infty f(x) := \left\{ \zeta \in \mathbb{R}^n : (\zeta, 0) \in \hat{N}_{\text{epi}(f)}(x, f(x)) \right\}$$

Elements in $\hat{\partial}^\infty f(x)$ are called *Fréchet singular subgradients* of f at x . In other words, $\zeta \in \hat{\partial}^\infty f(x)$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle \zeta, y - x \rangle \leq \varepsilon (\|y - x\| + |\beta - f(x)|), \quad \forall y \in B(x, \delta), (y, \beta) \in \text{epi}(f)$$

2. Fréchet Subdifferentials of the minimum time function

This section presents the main results of this paper. In the next theorem, we provide a formula for computing the Fréchet subdifferential of the minimum time function at point in the target. This result generalizes [17, Theorem 4.1]. Note that the result in [17, Theorem 4.1] is for linear control system. Here, we prove the result for a more general setting.

Theorem 2.1. *Assume that the multifunction F satisfies (F1)-(F3). Let $x_0 \in K$. Then*

$$\hat{\partial} T(x_0) = \hat{N}_K(x_0) \cap \left\{ \zeta \in \mathbb{R}^n : h(x_0, \zeta) \geq -1 \right\} \tag{1.3}$$

Proof. Assume that $\zeta \in \hat{\partial} T(x_0)$. Then, for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$T(y) - \langle \zeta, y - x_0 \rangle \geq -\varepsilon \|y - x_0\|, \quad \forall y \in B(x_0, \eta). \tag{1.4}$$

Since $T(x) = 0$ whenever $x \in K$, it follows from (2.2) that

$$\langle \zeta, y - x_0 \rangle \leq \varepsilon \|y - x_0\|, \quad \forall y \in K \cap B(x_0, \eta)$$

Hence, $\zeta \in \hat{N}_K(x_0)$.

Since $F(x_0)$ is compact, there exists $w \in F(x_0)$ such that

$$\langle \zeta, w \rangle = \min_{v \in F(x_0)} \langle \zeta, v \rangle = h(x_0, \zeta)$$

By theorem 1.1, there exists a C^1 trajectory $y(\cdot)$ of $-F$ on $[0, T]$ for some $T > 0$ satisfying $y(0) = x_0$, $\dot{y}(0) = -w$ and $\|y(t) - x_0\| \leq Mt$ for all $t \in [0, T]$.

There are two cases needed to consider.

Case 1. There exists $\delta > 0$ such that $y(t) \in K \cap B(x_0, \eta)$ for all $t \in [0, \delta]$. Plugging $y := y(t)$ with $t \in [0, \delta]$ into (1.4), we have

$$\langle \zeta, y(t) - x_0 \rangle \leq \varepsilon \|y(t) - x_0\| \leq M\varepsilon t.$$

Equivalently,

$$\left\langle \zeta, \frac{y(t) - x_0}{t} \right\rangle \leq M\varepsilon$$

Letting $t \rightarrow 0+$, we get $\langle \zeta, y'(0) \rangle \leq M\varepsilon$, i.e., $\langle \zeta, -w \rangle \leq M\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $h(\zeta, x_0) = \langle \zeta, w \rangle \geq 0$.

Case 2. There exists $\delta > 0$ such that $y(t) \notin K$ for all $t \in [0, \delta]$. Fix $t \in [0, \delta]$ such that $y(s) \in B(x_0, \eta)$ for all $s \in [0, t]$. Set $x(s) = y(t-s), s \in [0, t]$. Then $x(\cdot)$ is a trajectory of F with $x(t) = x_0$. By the principle of optimality, for all $s \in [0, t]$,

$$T(y(s)) \leq T(y(0)) + s = T(x_0) + s = s.$$

In (1.4), we take $y := y(s)$ with $s \in [0, t]$,

$$T(y(s)) - \langle \zeta, y(s) - x_0 \rangle \geq -\varepsilon \|y(s) - x_0\| \geq -M\varepsilon s.$$

Equivalently, $s - \langle \zeta, y(s) - x_0 \rangle \geq -M\varepsilon s$ which implies that

$$\left\langle \zeta, -\frac{y(s) - x_0}{s} \right\rangle \geq -1 - M\varepsilon.$$

Let $s \rightarrow 0+$, we get $\langle \zeta, -y'(0) \rangle \geq -1 - M\varepsilon$.

Since $\varepsilon > 0$ is arbitrary, $h(x_0, \zeta) = \langle \zeta, w \rangle \geq -1$.

Assume now that $\zeta \in \hat{N}_K(x_0)$ with $h(x_0, \zeta) \geq -1$. We will show that $\zeta \in \hat{\partial}T(x_0)$, i.e., for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$T(y) - \langle \zeta, y - x_0 \rangle \geq -\varepsilon \|y - x_0\|, \quad \forall y \in B(x_0, \eta). \quad (1.5)$$

We may assume that $\zeta \neq 0$. Set $c = M\|\zeta\| + 1$. For $\varepsilon > 0$, let $\varepsilon_0 \in (0, \varepsilon/c)$. Since $\zeta \in \hat{N}_K(x_0)$, there exists $\eta_0 > 0$ such that

$$\langle \zeta, y - x_0 \rangle \leq \varepsilon_0 \|y - x_0\|, \quad \forall y \in K \cap B(x_0, \eta_0).$$

Since $\varepsilon_0 < \varepsilon$ and $T(x) = 0$ for all $x \in K$, (1.3) holds for any $y \in K \cap B(x_0, \eta_0)$.

$$\text{Set now } \eta := \min \left\{ \frac{\eta_0}{c}, \frac{\varepsilon - c\varepsilon_0}{cL\|\zeta\|^2}, 1 \right\}.$$

It is enough to show that (1.5) holds for all $y \in B(x_0, \eta) \setminus K$. Assume to the contrary that there exists $y_0 \notin K$ such that $\|y_0 - x_0\| < \eta$ and

$$T(y_0) < \langle \zeta, y_0 - x_0 \rangle - \varepsilon \|y_0 - x_0\|. \quad (1.6)$$

It follows from (1.6) and the Cauchy-Schwarz inequality that

$$T(y_0) \leq \|\zeta\| \cdot \|y_0 - x_0\| < \infty. \quad (1.7)$$

Set $T_1 := T(y_0)$ and let $x(\cdot)$ be an optimal trajectory for y_0 . Then $y_1 := x(T_1) \in K$.

We have, for all $t \in [0; T_1]$, that

$$\begin{aligned} \|x(t) - x_0\| &\leq \|x(t) - y_0\| + \|y_0 - x_0\| \\ &\leq Mt + \|y_0 - x_0\| \\ &\leq (M\|\zeta\| + 1)\|y_0 - x_0\| = c\|y_0 - x_0\|. \end{aligned} \quad (1.8)$$

In particular, $\|y_1 - x_0\| \leq c\|y_0 - x_0\| < \eta_0$, i.e., $y_1 \in K \cap B(x_0, \eta_0)$. Thus,

$$\langle \zeta, y_1 - x_0 \rangle \leq \varepsilon_0 \|y_1 - x_0\| \leq \varepsilon_0 c \|y_0 - x_0\|. \quad (1.9)$$

Let $y(\cdot)$ be a measurable function which is the projection of $x'(\cdot)$ on the set $F(x_0)$ restricted to $[0; T_1]$. Since F is locally Lipschitz,

$$\|y(t) - \dot{x}(t)\| \leq L\|x(0) - x(t)\| \leq LMt + L\|y_0 - x_0\| \quad \text{a.e. } t \in [0, T_1] \quad (1.10)$$

Using (1.7) - (1.10) and having in mind that $h(x_0, \zeta) \geq -1$, one has

$$\begin{aligned} T(y_0) - \langle \zeta, y_0 - x_0 \rangle &= T_1 - \langle \zeta, y_0 - y_1 \rangle - \langle \zeta, y_1 - x_0 \rangle \\ &= T_1 + \int_0^{T_1} \langle \zeta, x'(t) \rangle dt - \langle \zeta, y_1 - x_0 \rangle \\ &= T_1 + \int_0^{T_1} \langle \zeta, y(t) \rangle dt + \int_0^{T_1} \langle \zeta, x'(t) - y(t) \rangle dt - \langle \zeta, y_1 - x_0 \rangle \\ &\geq T_1 + \int_0^{T_1} h(x_0, \zeta) dt - \|\zeta\| \int_0^{T_1} \|x'(t) - y(t)\| dt - \varepsilon_0 c \|y_0 - x_0\| \\ &\geq -L\|\zeta\| (MT_1 + \|y_0 - x_0\|) T_1 - \varepsilon_0 c \|y_0 - x_0\| \\ &\geq -L\|\zeta\| (M\|\zeta\| + 1) \|y_0 - x_0\| T_1 - \varepsilon_0 c \|y_0 - x_0\| \\ &\geq -cL\|\zeta\|^2 \|y_0 - x_0\|^2 - \varepsilon_0 c \|y_0 - x_0\| \\ &\geq -\varepsilon_0 \|y_0 - x_0\|. \end{aligned}$$

which contradicts to (1.6). This ends the proof.

In the following theorem, we present a formula for computing the Fréchet singular differential of the minimum time function at point in the target. This result generalizes [19, Theorem 3.4]. Note that the result in [19, Theorem 3.4] is for the bilateral minimum time function - a special case of the general minimum time function considered in this paper.

Theorem 2.2. *Assume that the multifunction F satisfies (F1)-(F3). Let $x_0 \in K$.*

Then

$$\hat{\partial}^\infty T(x_0) = \hat{N}_K(x_0) \cap \{\zeta \in \mathbb{R}^n : h(x_0, \zeta) \geq 0\}. \quad (1.11)$$

Proof. Let $\zeta \in \hat{\partial}^\infty T(x_0)$. Then, for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\langle \zeta, y - x_0 \rangle \leq \varepsilon (\|y - x_0\| + \beta), \quad \forall y \in B(x_0, \eta), \beta \geq T(y). \quad (1.12)$$

Taking $y \in B(x_0, \eta) \cap K$ and $\beta = T(y) = 0$ in (1.12), we have

$$\langle \zeta, y - x_0 \rangle \leq \varepsilon \|y - x_0\|.$$

It follows that $\zeta \in \hat{N}_K(x_0)$. We are now going to show that $h(x_0, \zeta) \geq 0$. Let $w \in F(x_0)$ be such that $\langle \zeta, w \rangle = \min_{v \in F(x_0)} \langle \zeta, v \rangle = h(x_0, \zeta)$.

By Theorem 1.1, there exists $T > 0$ and a C^1 trajectory $y(\cdot)$ on $[0, T]$ of $-F$ satisfying $y(0) = x_0, y'(0) = -w$ and $\|y(t) - x_0\| \leq Mt$ for all $t \in [0, T]$.

There are two possible cases.

Case 1. There exists $\delta > 0$ such that $y(t) \in K \cap B(x_0, \eta)$ for all $t \in [0, \delta]$. Plugging $y := y(t)$ and $\beta := T(y) = 0$ with $t \in [0, \delta]$ into (1.12), we have

$$\langle \zeta, y(t) - x_0 \rangle \leq \varepsilon \|y(t) - x_0\| \leq M\varepsilon t,$$

equivalently,
$$\left\langle \zeta, \frac{y(t) - x_0}{t} \right\rangle \leq M\varepsilon.$$

Letting $t \rightarrow 0+$, we get $\langle \zeta, -w \rangle = \langle \zeta, y'(0) \rangle \leq M\varepsilon$ which implies that $h(\zeta, x_0) = \langle \zeta, w \rangle \geq 0$ as $\varepsilon > 0$ is arbitrary.

Case 2. There exists $\delta > 0$ such that $y(t) \notin K$ for all $t \in [0, \delta]$. Fix $t \in [0, \delta]$ such that $y(s) \in B(x_0, \eta)$ for all $s \in [0, t]$. Set $x(s) = y(t - s), s \in [0, t]$. Then $x(\cdot)$ is a trajectory of F with $x(t) = x_0$.

By the principle of optimality, $T(y(s)) \leq T(y(0)) + s = T(x_0) + s = s$ for all $s \in [0, t]$

In (1.4), taking $y := y(s)$ and $\beta = s \geq T(y(s))$ with $s \in [0, t]$, we have

$$\langle \zeta, y(s) - x_0 \rangle \leq \varepsilon (\|y(t) - x_0\| + s) \leq \varepsilon(M + 1)s.$$

Equivalently, $\left\langle \zeta, \frac{y(s) - x_0}{s} \right\rangle \geq \varepsilon(M + 1)$.

Let $s \rightarrow 0+$, we get $\langle \zeta, y'(0) \rangle \leq \varepsilon(M + 1)$ which yields $h(x_0, \zeta) = \langle \zeta, w \rangle \geq 0$.

Now, let $\zeta \in \hat{N}_K(x_0)$ be such that $h(x_0, \zeta) \geq 0$. We prove that $\zeta \in \hat{\partial}^\infty T(x_0)$.

Assume that $\zeta \notin \hat{\partial}^\infty T(x_0)$. Then, there exists a constant $C > 0$ and sequences $\{y_n\} \subset \text{dom}(T)$, $\{\beta_n\} \subset [0, \infty)$ such that $y_n \rightarrow x_0$ and for all n that $y_n \neq x_0$, $\beta_n \geq T(y_n)$,

$$\langle \zeta, y_n - x_0 \rangle > C(\|y_n - x_0\| + \beta_n).$$

Hence, $\langle \zeta, y_n - x_0 \rangle > C(\|y_n - x_0\| + T(y_n))$, $\forall n$. (1.13)

Set $T_n := T(y_n)$ for each n . It follows from (1.13) that

$$T_n \leq \frac{1}{C} \|\zeta\| \|y_n - x_0\| < \infty.$$

Moreover, since $\zeta \in \hat{N}_K(x_0)$, by (1.13), we may assume that $y_n \notin K$, i.e., $T_n > 0$ for all n . For each n , let $x_n(\cdot)$ be an optimal trajectory for y_n . Set $z_n = x_n(T_n)$. Then $z_n \in K$.

We have, for all $t \in [0, T_n]$, that

$$\|x_n(t) - x_0\| \leq \|x_n(t) - y_n\| + \|y_n - x_0\| \leq MT_n + \|y_n - x_0\|.$$

In particular, $\|z_n - x_0\| \leq MT_n + \|y_n - x_0\| \leq \left(\frac{M}{C} \|\zeta\| + 1\right) \|y_n - x_0\|$,

which implies that $z_n \rightarrow x_0$ as $n \rightarrow \infty$. Since $\zeta \in \hat{N}_K(x_0)$, for any $\varepsilon > 0$ and for n sufficiently large, one has $\langle \zeta, z_n - x_0 \rangle \leq \varepsilon \|z_n - x_0\| \leq \varepsilon (MT_n + \|y_n - x_0\|)$.

For each n , let $a_n(\cdot)$ be the projection of $x'_n(\cdot)$ on $F(x_0)$ restricted on $[0, T_n]$. By Lipschitz continuity of F , one has

$$\|a_n(t) - x'_n(t)\| \leq L \|x_n(t) - x_0\| \leq LMT_n + L \|y_n - x_0\|.$$

For n sufficiently large,

$$\begin{aligned} C(\|y_n - x_0\| + T_n) &< \langle \zeta, y_n - x_0 \rangle \\ &= \langle \zeta, y_n - z_n \rangle + \langle \zeta, z_n - x_0 \rangle \\ &= -\int_0^{T_n} \langle \zeta, x'(t) \rangle dt + \langle \zeta, z_n - x_0 \rangle \\ &= \int_0^{T_n} \langle \zeta, a_n(t) - x'_n(t) \rangle dt - \int_0^{T_n} \langle \zeta, a_n(t) \rangle dt + \langle \zeta, z_n - x_0 \rangle \end{aligned}$$

$$\begin{aligned} &\leq \|\zeta\| \int_0^{T_n} \|a_n(t) - x'_n(t)\| dt + \langle \zeta, z_n - x_0 \rangle \\ &\leq L \|\zeta\| (MT_n + \|y_n - x_0\|) T_n + \varepsilon (MT_n + \|y_n - x_0\|) \\ &\leq C_0 (T_n + \varepsilon) (\|y_n - x_0\| + T_n), \end{aligned}$$

for some constant $C_0 > 0$. This yields $C < C_0(T_n + \varepsilon)$. Letting $n \rightarrow \infty$ and then letting $\varepsilon \rightarrow 0+$ in both sides of the latter inequality, we get $C \leq 0$. This contradiction ends the proof.

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