EXPONENTIAL STABILITY FOR SINGULAR SYSTEMS WITH INTERVAL TIME-VARYING DELAYS

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Received: 12 February 2019 / Accepted: 11 June 2019 / Published: June 2019

Abstract: *This paper deals with the problem of exponential stability for singular systems with interval time-varying delays. By constructing a set of improved Lyapunov-Krasovskii functionals combined with Newton-Leibniz formula, a new delay-dependent condition is established in terms of linear matrix inequalities (LMIs) which guarantees that the system is regular, impulse-free and exponentially stability.*

Keywords: *Singular system; exponential stability; interval time-varying delay; linear matrix inequality.*

1. Introduction

Singular systems have many applications in reality such as electrical circuit network, power systems, aerospace engineering, network control, economic systems [6,19]. Therefore, these systems have been extensively studied over the past few decades and have yielded significant results. Using Lyapunov's approach to reach out to scientists who have established stable standards for singular systems with time-delay and singular systems with time-varying in the form of linear matrix inequalities [21,18]. However, there are few results concerned with the problem of exponential stability of singular systems with interval time-varying delays and most of the delay-dependent results in the literature tackled only the case of constant or slowly time-varying delays.

In this paper, a class of singular systems with interval time-varying delays is considered. New delay-range-dependent exponential stability condition is established in terms of LMIs ensuring the regularity, impulse free and exponential stability of the system. Employing the idea of perturbation approach, we decompose the system into slow and fast subsystems. Then, the exponential decay of slow variables is proved by constructing an improved LKF. Using this, we prove the fast variables fall within exponential decay with the same decay rate by some new estima- tions specifically developed in this paper. The main contribution of this paper is that we derive a new delay-range-dependent criterion for the exponential stability of singular systems with interval time-varying discrete delays.

Notations: The following notations will be used throughout this paper. $\mathbb{R}^+$ denotes the set of all nonnegative real numbers; $\mathbb{R}^n$ denotes the $n-$dimensional Euclidean space with

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the norm $\|.\|$ and scalar product $\langle x, y\rangle = x^{\mathrm{T}} y$ of two vectors x, y; $\lambda_{\max}(A)(\lambda_{\min}(A)$, resp.$)$ denotes the maximal (the minimal, resp.) number of the real part of eigenvalues of A; A^{T} denotes the transpose of the matrix A and I denotes the identity matrix; $Q \geq 0$ ($Q > 0$, resp.) means that Q is semi-positive definite (positive definite, resp.) i.e. $x^T Qx \geq 0$ for all $x \in \mathbb{R}^n$ (resp. $x^T Qx > 0$ for all $x \neq 0$); $A \geq B$ means $A - B \geq 0$; $C^1([-\tau, 0], \mathbb{R}^n)$ denotes the set of $\mathbb{R}^n$ - valued continuous functions on $[-\tau, 0]$ with the norm $\|\varphi\|_\tau = \sup_{-\tau \leq t \leq 0} \|\varphi(t)\|$.

2. Preliminaries

Consider the following singular system with time varying delays

$$\begin{cases} E\dot{x}(t) = & Ax(t) + Dx(t - h(t)), \quad t \geq 0, \\ x(t) = & \varphi(t), \quad t \in [-h_2, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state; $E, A, D \in \mathbb{R}^{n \times n}$ are real known system matrices with appropriate dimensions; matrix E may be singular with $\operatorname{rank}(E) = r \leq n$. The time varying delays $h(t)$ is continuous functions satisfying $0 \leq h_1 \leq h(t) \leq h_2$ and $\dot{h}(t) \leq \mu < 1$, where h_1 and h_2 are lower and upper bounds of the time varying delays $h(t).\phi(t) \in C([-h_2, 0], \mathbb{R}^n)$ is the compatible initial function specifying the intial state the system.

The following definitions for singular time delay system are adopted (e.g. see [18]).

Definition 1. [4], [21]

i) The pair (E, A) is said to be regular if the characteristic polynomial det (sE − A) is not identically.

ii) The pair (E, A) is said to be impulse free if deg(det(sE − A)) = rank(E).

Definition 2. [21]

i) System (1) is said to be regular and impulse-free if the pair (E, A) regular and impulse.

ii) $\alpha-$ exponentially stable for $\alpha > 0$ if there exists $N > 0$ such that, for any compatible initial conditions $\phi(t)$ the solution $x(t, \phi)$ satisfies

$$\|x(t, \varphi)\| \leq N \|\varphi\|_{h_2} e^{-\alpha t}, \quad \forall t \geq 0.$$

iii)Exponentially admissible if it is regular, impulse-free and $\alpha -$exponentially stable.

We introduce the following technical well-known propositions, which will be used in the proof of our results.

Proposition 1. [16] *(Matrix Cauchy inequality) For any $M, N \in \mathbb{R}^{n \times n}$, $M = M^T > 0$ and $x, y \in \mathbb{R}^n$ then $2x^{\mathrm{T}} Ny \leq x^{\mathrm{T}} Mx + y^{\mathrm{T}} N^{\mathrm{T}} M^{-1} Ny$.*

Proposition 2. [10] *For any symmetric positive definite matrix* M , *scalar* $\nu > 0$ *and vector function* $\omega : [0,\nu] \to \mathbb{R}^n$ *such that the integrals concerned are well defined, then*

$$\left(\int_0^{\nu} \omega(s)ds\right)^{\mathrm{T}} M \left(\int_0^{\nu} \omega(s)ds\right) \le \nu \int_0^{\nu} \omega^{\mathrm{T}}(s) M \omega(s) ds.$$

Proposition 3. [16] *(Schur complement Lemma) For given matrices* X, Y, Z *with appropriate dimensions satisfying* $X = X^{\mathrm{T}}, Y = Y^{\mathrm{T}} > 0$. *Then* $\begin{bmatrix} X & Z^{\mathrm{T}} \\ Z & -Y \end{bmatrix} < 0$

if and only if $X + Z^{\mathrm{T}} Y^{-1} Z < 0$.

Lemma 1. *Let* $\tau > 0, \delta > 0, \gamma \in (0;1)$ *be given and* $\rho(t)$ *be a continuous function satisfying* $0 \le \rho(t) \le \gamma \bar{\rho}_\tau(t) + \delta, \forall t \ge 0$ *where* $\bar{\rho}_\tau(t) = \sup_{-\tau \le s \le 0} \rho(t+s)$. *Then*

$$\rho(t) \le \gamma \bar{\rho}_\tau(0) + \frac{\delta}{1-\gamma}, \quad \forall t \ge 0.$$

Proof. Note that $\rho(0) \le \gamma \bar{\rho}_\tau(0) + \delta < \gamma \bar{\rho}_\tau(0) + \dfrac{\delta}{1-\gamma} =: \eta$. We will prove $\rho(t) < \eta$ for all $t \ge 0$. Contrarily, assume that there exist $t_* > 0$ satisfying $\rho(t_*) = \eta, \rho(t) < \eta, \forall t \in [0, t_*)$, then $\rho(t_*) \le \eta$, where $\rho(t) = \sup_{0 \le s \le t} \rho(s)$. From the fact that $\gamma\eta + \delta < \eta$, we have $\rho(t_*) \le \gamma \max\{\bar{\rho}_\tau(0), \rho(t_*)\} + \delta \le \gamma \max\{\bar{\rho}_\tau(0), \eta\} + \delta < \eta$ which yields a contradiction. This shows that $\rho(t) \le \eta$ for all $t \le 0$.

By applying Lemma 1 for funtion $\rho(t) = e^{\lambda t} f(t)$ we obtain the following lemma.

Lemma 2. *Suppose that positive numbers* $\tau, \lambda, \delta_1, \delta_2, \delta_1 e^{\lambda\tau} < 1,$ *and continuous functions* $f(t)$ *satisfy* $0 \le f(t) \le \delta_1 \bar{f}_\tau(t) + \delta_2 e^{-\lambda t}, \quad \forall t \ge 0,$

where $\bar{f}_\tau(t) = \sup_{-\tau \le s \le 0} f(t+s)$. *Then* $f(t) \le \left[\delta_1 e^{\lambda\tau} \bar{f}_\tau(0) + \dfrac{\delta_2}{1-\delta_1 e^{\lambda\tau}}\right] e^{-\lambda t}, \quad \forall t \ge 0.$

3. A main result

For given $\alpha > 0$. We denote:

$$\Pi_{11} = A^{\mathrm{T}} P^{\mathrm{T}} + PA + Q + Q_1 + Q_2 + 2\alpha PE + X_1 E + E^{\mathrm{T}} X_1^{\mathrm{T}}, \Pi_{12} = PD - X_1 E + E^{\mathrm{T}} X_2^{\mathrm{T}} + Y_1 E - Z_1 E,$$

$$\Pi_{18} = A^{\mathrm{T}}[h_2 W_1 + (h_2 - h_1) W_2], \Pi_{22} = -(1-\mu) e^{-2\alpha h_2} Q + Y_2 E + E^{\mathrm{T}} Y_2^{\mathrm{T}} - X_2 E - E^{\mathrm{T}} X_2^{\mathrm{T}} - Z_2 E - E^{\mathrm{T}} Z_2^{\mathrm{T}},$$

$$\Pi_{28} = D^{\mathrm{T}}[h_2 W_1 + (h_2 - h_1) W_2], \Pi_{33} = -e^{-2\alpha h_1} Q_1, \Pi_{44} = -e^{-2\alpha h_2} Q_2, \Pi_{66} = -\gamma_{22}(W_1 + W_2)$$

$$\Pi_{88} = -[h_2 W_1 + (h_2 - h_1) W_2], \gamma_{21} = \frac{e^{2\alpha h_2} - 1}{2\alpha}, \gamma_{22} = \frac{e^{2\alpha h_2} - e^{2\alpha h_1}}{2\alpha}.$$

Theorem 1. *Given* $\alpha > 0$. *System (1) is* $\alpha-$*exponentially admissible if there exist symmetric positive definite matrices* Q, Q_i, W_i $i = 1,2,$ *and matrices* $P, X_i, Y_i, Z_i,$ $i = 1,2,$ *satisfying the following LMIs:*

$$PE = E^{\mathrm{T}} P^{\mathrm{T}} \geq 0, \tag{2}$$

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & Z_1E & -Y_1E & \gamma_{21}X_1 & \gamma_{22}Y_1 & \gamma_{22}Z_1 & \Pi_{18} \\ * & \Pi_{22} & Z_2E & -Y_2E & \gamma_{21}X_2 & \gamma_{22}Y_2 & \gamma_{22}Z_2 & \Phi_{28} \\ * & * & \Pi_{33} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Pi_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma_{21}W_1 & 0 & 0 & 0 \\ * & * & * & * & * & \Pi_{66} & 0 & 0 \\ * & * & * & * & * & * & -\gamma_{22}W_2 & 0 \\ * & * & * & * & * & * & * & \Pi_{88} \end{bmatrix} < 0, \tag{3}$$

$$(1-\mu)Q - Q_2 < 0. \tag{4}$$

Proof. Step 1: We prove the regularity and impulse-free of system (1).

Since $\mathrm{rank}(E) = r \leq n$, these exists two nonsingular matrices M, N such that $\bar{E} = MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. We denote

$$\bar{A} = MAN = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \bar{P} = N^{\mathrm{T}}PM^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \bar{Q} = N^{\mathrm{T}}QN = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_{22} \end{bmatrix},$$

$$\bar{Q}_i = N^{\mathrm{T}}Q_iN = \begin{bmatrix} Q_{i11} & Q_{i12} \\ Q_{i12}^{\mathrm{T}} & Q_{i22} \end{bmatrix}, \bar{X}_i = N^{\mathrm{T}}X_iM^{-1} = \begin{bmatrix} X_{i11} & X_{i12} \\ X_{i21} & X_{i22} \end{bmatrix}, \bar{Y}_i = N^{\mathrm{T}}Y_iM^{-1} = \begin{bmatrix} Y_{i11} & Y_{i12} \\ Y_{i21} & Y_{i22} \end{bmatrix},$$

$$\bar{Z}_i = N^{\mathrm{T}}Z_iM^{-1} = \begin{bmatrix} Z_{i11} & Z_{i12} \\ Z_{i21} & Z_{i22} \end{bmatrix}, \quad \bar{W}_i = M^{-\mathrm{T}}W_iM^{-1} = \begin{bmatrix} W_{i11} & W_{i12} \\ W_{i12}^{\mathrm{T}} & W_{i22} \end{bmatrix}, i = 1,2.$$

From (2) we have

$$\bar{P}\bar{E} = \bar{E}^{\mathrm{T}}\bar{P}^{\mathrm{T}} \geq 0. \tag{5}$$

Using the expression of $\bar{E}$ and $\bar{P}$, we obtain $P_{21} = 0, P_{11} \geq 0$. From (3) using the Proposition 3, we have

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & Z_1E & -Y_1E \\ * & \Pi_{22} & Z_2E & -Y_2E \\ * & * & \Pi_{33} & 0 \\ * & * & * & \Pi_{44} \end{bmatrix} < 0, \tag{6}$$

Pre-multiplying $\begin{bmatrix} I & I & I & I \end{bmatrix}$ and post-multiplying $\begin{bmatrix} I & I & I & I \end{bmatrix}^{\mathrm{T}}$ on both sides of (6), we obtain

$$(A+D)^{\mathrm{T}} P^{\mathrm{T}} + P(A+D) + 2\alpha PE + (1-e^{-2\alpha h_1})Q_1 + (1-e^{-2\alpha h_2})[(1-\mu)Q + Q_2] < 0 \tag{7}$$

and hence $(A+D)^{\mathrm{T}} P^{\mathrm{T}} + P(A+D) < 0$. By Lemma 3, which proves that P is nonsingular, and hence $\bar{P}$ is nonsingular, then $P_{11} > 0$. On the other hand, from $\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ * & \Pi_{22} \end{bmatrix} < 0$ pre-multiplying $\mathrm{diag}\{N^{\mathrm{T}}, N^{\mathrm{T}}\}$ and post-multiplying $\mathrm{diag}\{N, N\}$ we obtain

$$\begin{bmatrix} \bar{\Pi}11 & \bar{\Pi}12 \\ * & \bar{\Pi}22 \end{bmatrix} < 0 \tag{8}$$

where

$\bar{\Pi}_{11} = \bar{A}^{\mathrm{T}} \bar{P}^{\mathrm{T}} + \bar{P}\bar{A} + \bar{Q} + \bar{Q}_1 + \bar{Q}_2 + 2\alpha \bar{P}\bar{E} + \bar{X}_1\bar{E} + \bar{E}^{\mathrm{T}}\bar{X}_1^{\mathrm{T}}, \bar{\Pi}_{12} = \bar{P}\bar{D} - \bar{X}_1\bar{E} + \bar{E}^{\mathrm{T}}\bar{X}_2^{\mathrm{T}} + \bar{Y}_1\bar{E} - \bar{Z}_1\bar{E},$

$\bar{\Pi}_{22} = -(1-\mu)e^{-2\alpha h_2}\bar{Q} + \bar{Y}_2\bar{E} + \bar{E}^{\mathrm{T}}\bar{Y}_2^{\mathrm{T}} - \bar{X}_2\bar{E} - \bar{E}^{\mathrm{T}}\bar{X}_2^{\mathrm{T}} - \bar{Z}_2\bar{E} - \bar{E}^{\mathrm{T}}\bar{Z}_2^{\mathrm{T}}$

Applying Lemma 4, from (4) we obtain

$$\begin{bmatrix} P_{22}A_{22} + A_{22}^{\mathrm{T}}P_{22}^{\mathrm{T}} + Q_{22} + Q_{122} + Q_{222} & P_{22}D_{22} \\ * & -(1-\mu)e^{-2\alpha h_2}Q_{22} \end{bmatrix} < 0, \tag{9}$$

Which gives $P_{22}A_{22} + A_{22}^{\mathrm{T}}P_{22}^{\mathrm{T}} < 0,$ and hence P_{22} and A_{22} are nonsingular matrices. Implies system (1) is regular and impulse free.

Next, we can choose two nonsingular matrices M, N such that $\bar{E} = MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ and $\bar{A} = MAN = \begin{bmatrix} A_{11} & 0 \\ 0 & I_{n-r} \end{bmatrix}$.

Step 2: Decompose the system and exponential estimate for slow variables. Under variable transformation

$$y(t) = N^{-1}x(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \tag{10}$$

where $y_1(t) \in \mathbb{R}^r, y_2(t) \in \mathbb{R}^{n-r}$. The corresponding transformed system (2.3) is given by

$$\begin{cases} \bar{E}\dot{y}(t) & = \bar{A}y(t) + \bar{D}y(t-h(t)), \quad t \geq 0, \\ y(t) & = N^{-1}\varphi(t) := \psi(t), \quad t \in [-h_2, 0], \end{cases} \tag{11}$$

In other word, under the transformation $y(t) = N^{-1}x(t)$, system (1) is decomposed into the following slow and fast subsystems

$$\dot{y}_1(t) = A_{11}y_1(t) + D_{11}y_1(t-h(t)) + D_{12}y_2(t-h(t)), \tag{12}$$

$$0 = y_2(t) + D_{21}y_1(t - h(t)) + D_{22}y_2(t - h(t)) \tag{13}$$

System (12) and (13) are referred to as slow and fast subsystems and $y_1(t) \in \mathbb{R}^r, y_2 \in \mathbb{R}^{n-r}$ are called slow and fast variables, respectivily. We now prove the exponential stability of slow subsystem (8). For this, we construct the following LKF

$$V(y_t) = V_1 + V_2 + V_3 + V_4 + V_5 + V_6 \tag{14}$$

where

$$V_1 = y^{\mathrm{T}}(t)\bar{P}\bar{E}y(t), \qquad V_4 = \int_{t-h(t)}^{t} e^{2\alpha(s-t)} y^{\mathrm{T}}(s)\bar{Q}y(s)ds,$$

$$V_2 = \int_{t-h_1}^{t} e^{2\alpha(s-t)} y^{\mathrm{T}}(s)\bar{Q}_1 y(s)ds, \qquad V_5 = \int_{-h_2}^{0}\int_{t+s}^{t} e^{2\alpha(u-t)} \dot{y}^{\mathrm{T}}(u)\bar{E}^{\mathrm{T}}\bar{W}_1\bar{E}\dot{y}(u)duds,$$

$$V_3 = \int_{t-h_2}^{t} e^{2\alpha(s-t)} y^{\mathrm{T}}(s)\bar{Q}_2 y(s)ds, \qquad V_6 = \int_{-h_2}^{-h_1}\int_{t+s}^{t} e^{2\alpha(u-t)} \dot{y}^{\mathrm{T}}(u)\bar{E}^{\mathrm{T}}\bar{W}_2\bar{E}\dot{y}(u)duds.$$

It is easy to see that

$$\lambda_1 \left\|y_1(t)\right\|^2 \le V(t, y_t) \le \lambda_2 \left\|y_t\right\|^2, \tag{15}$$

where y_t denotes the segment $\{y(t+s) : s \in [-h_2; 0]\}, \lambda_1 = \lambda_{\min}(P_{11})$ and

$$\lambda_2 = \lambda_{\max}(P_{11}) + h_1\lambda_{\max}(\bar{Q}_1) + h_2\lambda_{\max}(\bar{Q}_2) + h_2\lambda_{\max}(\bar{Q}) + \left[h_2^2\lambda_{\max}(\bar{W}_1) + (h_2^2 - h_1^2)\lambda_{\max}(\bar{W}_2)\right].$$

Taking derivative of V_1 in t along the trajectory of the system, we have

$$\dot{V}_1 = 2y^{\mathrm{T}}(t)\bar{P}\bar{E}\dot{y}(t) = 2y^{\mathrm{T}}(t)\bar{P} \times \left[\bar{A}y(t) + \bar{D}y(t - h(t))\right] \tag{16}$$

$$= y^{\mathrm{T}}(t)\left[\bar{P}\bar{A} + \bar{A}^{\mathrm{T}}\bar{P}^{\mathrm{T}} + 2\alpha\bar{P}\bar{E}\right]y(t) + 2y^{\mathrm{T}}(t)\bar{P}\bar{D}y(t - h(t)) - 2\alpha V_1.$$

The time-derivative of $V_k, k = 2, 3, \ldots, 8,$ are computed and estimated as follows

$$\dot{V}_2 = y^{\mathrm{T}}(t)\bar{Q}_1 y(t) - e^{-2\alpha h_1} y^{\mathrm{T}}(t - h_1)\bar{Q}_1 y(t - h_1) - 2\alpha V_2; \tag{17}$$

$$\dot{V}_3 = y^{\mathrm{T}}(t)\bar{Q}_2 y(t) - e^{-2\alpha h_2} y^{\mathrm{T}}(t - h_2)\bar{Q}_2 y(t - h_2) - 2\alpha V_3;$$

$$\dot{V}_4 = y^{\mathrm{T}}(t)\bar{Q}y(t) - (1 - \dot{h}(t))e^{-2\alpha h(t)} y^{\mathrm{T}}(t - h(t))\bar{Q}y(t - h(t)) - 2\alpha V_4$$

$$\le y^{\mathrm{T}}(t)\bar{Q}y(t) - (1 - \mu)e^{-2\alpha h_2} y^{\mathrm{T}}(t - h(t))\bar{Q}y(t - h(t)) - 2\alpha V_4;$$

$$\dot{V}_5 = h_2\dot{y}^{\mathrm{T}}(t)\bar{E}^{\mathrm{T}}\bar{W}_1\bar{E}\dot{y}(t) - \int_{t-h_2}^{t} e^{2\alpha(s-t)}\dot{y}^{\mathrm{T}}(s)\bar{E}^{\mathrm{T}}\bar{W}_1\bar{E}\dot{y}(s)ds - 2\alpha V_7, \tag{18}$$

$$\dot{V}_6 = (h_2 - h_1)\dot{y}^{\mathrm{T}}(t)\bar{E}^{\mathrm{T}}\bar{W}_2\bar{E}\dot{y}(t) - \int_{t-h_2}^{t-h_1} e^{2\alpha(s-t)}\dot{y}^{\mathrm{T}}(s)\bar{E}^{\mathrm{T}}\bar{W}_2\bar{E}\dot{y}(s)ds - 2\alpha V_8.$$

Using the following identities

$$2\left[y^{\mathrm{T}}(t)\bar{X}1+y^{\mathrm{T}}(t-h(t))\bar{X}2\right]\left[\bar{E}y(t)-\bar{E}y(t-h(t))-\int_{t-h(t)}^{t}\bar{E}\dot{y}(s)ds\right]=0, \tag{19}$$

$$2\left[y^{\mathrm{T}}(t)\bar{Y}1+y^{\mathrm{T}}(t-h(t))\bar{Y}2\right]\left[\bar{E}y(t-h(t))-\bar{E}y(t-h_2)-\int_{t-h_2}^{t-h(t)}\bar{E}\dot{y}(s)ds\right]=0,$$

$$2\left[y^{\mathrm{T}}(t)\bar{Z}1+y^{\mathrm{T}}(t-h(t))\bar{Z}2\right]\left[\bar{E}y(t-h_1)-\bar{E}y(t-h(t))-\int_{t-h(t)}^{t-h}\bar{E}\dot{y}(s)ds\right]=0.$$

From (16) to (19), we have

$$\begin{aligned}\dot{V}(t,y_t)+2\alpha V(t,y_t)&\le \eta^{T}(t)\bar{\Pi}\eta(t)+\eta^{\mathrm{T}}(t)\Omega^{\mathrm{T}}\bar{U}\Omega\eta(t)+\frac{e^{2\alpha h_2}-e^{2\alpha h_1}}{2\alpha}\eta^{\mathrm{T}}(t)\bar{Y}(\bar{W}_1+\bar{W}_2)^{-1}\bar{Y}^{\mathrm{T}}\eta(t)\\&+\frac{e^{2\alpha h_2}-e^{2\alpha h_1}}{2\alpha}\eta^{\mathrm{T}}(t)\bar{Z}\bar{W}_2^{-1}\bar{Z}^{\mathrm{T}}\eta(t)+\frac{e^{2\alpha h_2}-1}{2\alpha}\eta^{\mathrm{T}}(t)\bar{X}\bar{W}_1^{-1}\bar{X}^{T}\eta(t)\\&\le \eta^{\mathrm{T}}(t)[\bar{\Pi}+\gamma_{21}\bar{X}\bar{W}_1^{-1}\bar{X}^{\mathrm{T}}+\gamma_{22}\bar{Z}\bar{W}_2^{-1}\bar{Z}^{\mathrm{T}}+\gamma_{22}\bar{Y}(\bar{W}_1+\bar{W}_2)^{-1}\bar{Y}^{\mathrm{T}}+\Omega^{\mathrm{T}}\bar{U}\Omega]\eta(t),\end{aligned} \tag{20}$$

where

$$\eta^{\mathrm{T}}(t)=\left[y^{\mathrm{T}}(t)\quad y^{\mathrm{T}}(t-h(t))\quad y^{\mathrm{T}}(t-h_1)\quad y^{\mathrm{T}}(t-h_2)\right],\bar{X}=\left[\bar{X}_1^{\mathrm{T}}\quad \bar{X}_2^{\mathrm{T}}\quad 0\quad 0\right]^{\mathrm{T}},\quad \bar{Y}=\left[\bar{Y}_1^{\mathrm{T}}\quad \bar{Y}_2^{\mathrm{T}}\quad 0\quad 0\right]^{\mathrm{T}},$$

$$\bar{Z}=\left[\bar{Z}_1^{\mathrm{T}}\quad \bar{Z}_2^{\mathrm{T}}\quad 0\quad 0\right]^{\mathrm{T}},\quad \bar{U}=h_2\bar{W}1+(h_2-h_1)\bar{W}2,\Omega=\left[\bar{A}\quad \bar{D}\quad 0\quad 0\right],$$

and $\bar{\Pi}=\begin{bmatrix}\bar{\Pi}_{11} & \bar{\Pi}_{12} & \bar{Z}_1\bar{E} & -\bar{Y}_1\bar{E}\\ * & \bar{\Pi}_{22} & \bar{Z}_2\bar{E} & -\bar{Y}_2\bar{E}\\ * & * & -e^{2\alpha h_1}\bar{Q}_1 & 0\\ * & * & * & -e^{2\alpha h_2}\bar{Q}_2\end{bmatrix}.$

On the other hand, by pre-multiplying $\mathrm{diag}\{N^{\mathrm{T}},N^{\mathrm{T}},N^{\mathrm{T}},N^{\mathrm{T}},I,I,I,I\}$ and post-multiplying $\mathrm{diag}\{N,N,N,N,I,I,I,I\}$ on both sides of (3) , we obtain

$$\begin{bmatrix}\bar{\Pi}_{11} & \bar{\Pi}_{12} & \bar{Z}_1\bar{E} & -\bar{Y}_1\bar{E} & \gamma_{21}N^{\mathrm{T}}X_1 & \gamma_{22}N^{\mathrm{T}}Y_1 & \gamma_{22}N^{\mathrm{T}}Z_1 & N^{\mathrm{T}}\Pi_{18}\\ * & \bar{\Pi}_{22} & \bar{Z}_2\bar{E} & -\bar{Y}_2\bar{E} & \gamma_{21}N^{\mathrm{T}}X_2 & \gamma_{22}N^{\mathrm{T}}Y_2 & \gamma_{22}N^{\mathrm{T}}Z_2 & N^{\mathrm{T}}\Pi_{28}\\ * & * & -e^{2\alpha h_1}\bar{Q}_1 & 0 & 0 & 0 & 0 & 0\\ * & * & * & -e^{2\alpha h_2}\bar{Q}_2 & 0 & 0 & 0 & 0\\ * & * & * & * & -\gamma_{21}W_1 & 0 & 0 & 0\\ * & * & * & * & * & \Pi_{66} & 0 & 0\\ * & * & * & * & * & * & -\gamma_{22}W_2 & 0\\ * & * & * & * & * & * & * & \Pi_{88}\end{bmatrix}<0. \tag{21}$$

By using Proposition 3 for (21), it can be shown that

$$\bar{\Phi}+\gamma_{21}\bar{X}\bar{W}_1^{-1}\bar{X}^{\mathrm{T}}+\gamma_{22}\bar{Z}\bar{W}_2^{-1}\bar{Z}^{\mathrm{T}}+\gamma_{22}\bar{Y}(\bar{W}_1+\bar{W}_2)^{-1}\bar{Y}^{\mathrm{T}}+\Omega^{\mathrm{T}}\bar{U}\Omega<0. \tag{22}$$

From (20) and (22), we have

$$\dot{V}(t,y_t)+2\alpha V(t,y_t)\le 0, t\ge 0$$

and hence

$$V(y_t)\le V(\psi)e^{-2\alpha t}\le \lambda_2\|\psi\|_{h_2}^2 e^{-2\alpha t}, \ge 0.$$

Taking (15) into account, we obtain

$$\|y_1(t)\|\le\sqrt{\frac{\lambda_2}{\lambda_1}}\|\psi\|_{h_2} e^{-\alpha t}=:\upsilon_1\|\psi\|_{h_2} e^{-\alpha t},\quad t\ge 0. \tag{23}$$

This proves that the slow variable, that is, the first $r-$dimensional component $y_1(t)$ of the state vector $y(t)$ is $\alpha-$ exponentially stable.

Step 3: The exponential decay of fast variables. In this step, we will prove the fast variables are fallen into exponential decay with the same decay rate α. Let us denote $p(t)=A_{h21}y_1(t-h(t))$ by pre-multiplying the second equation of (8) with $2y_2^{\mathrm{T}}(t)P_{22}$, we obtain

$$0=2y_2^{\mathrm{T}}(t)P_{22}y_2(t)+2y_2^{\mathrm{T}}(t)P_{22}D_{22}y_2(t-h(t))+2y_2^{\mathrm{T}}(t)P_{22}p(t). \tag{24}$$

Consider the following function

$$J(t):=(1-\mu)y_2^{\mathrm{T}}(t)Q_{22}y_2(t)-(1-\mu)e^{-2\alpha h_2}y_2^{\mathrm{T}}(t-h(t))Q_{22}y_2(t-h(t)) \tag{25}$$

Then, from (24) we have,

$$\begin{aligned}J(t)&=y_2^{\mathrm{T}}(t)[P_{22}+P_{22}^{\mathrm{T}}+(1-\mu)Q_{22}]y_2(t)+2y_2^{\mathrm{T}}(t)P_{22}p(t)\\&\quad-(1-\mu)e^{-2\alpha h_2}y_2^{\mathrm{T}}(t-h(t))Q_{22}y_2(t-h(t))+2y_2^{\mathrm{T}}(t)P_{22}D_{22}y_2(t-h(t)).\end{aligned}$$

Applying Proposition 1 to this yields that,

$$\begin{aligned}J(t)&\le y_2^{\mathrm{T}}(t)[P_{22}+P_{22}^{\mathrm{T}}+(1-\mu)Q_{22}]y_2(t)+y_2^{\mathrm{T}}(t)Q_{122}y_2(t)+p^{\mathrm{T}}(t)P_{22}^{\mathrm{T}}Q_{122}^{-1}P_{22}p(t)\\&\quad+2y_2^{\mathrm{T}}(t)P_{22}D_{22}y_2(t-h(t))-(1-\mu)e^{-2\alpha h_2}y_2^{\mathrm{T}}(t-h(t))Q_{22}y_2(t-h(t))\\&\le\begin{bmatrix}y_2(t)\\y_2(t-h(t))\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}J_{11}&P_{22}D_{22}\\&-(1-\mu)e^{-2\alpha h_2}Q_{22}\end{bmatrix}\begin{bmatrix}y_2(t)\\y_2(t-h(t))\end{bmatrix}+p^{\mathrm{T}}(t)P_{22}^{\mathrm{T}}Q_{122}^{-1}P_{22}p(t)\end{aligned} \tag{26}$$

where $J_{11}=P_{22}+P_{22}^{\mathrm{T}}+(1-\mu)Q_{22}+Q_{122}$. On the other hand, from (4) it follows

$$\begin{bmatrix}J_{11}+Q_{222}&P_{22}D_{22}\\ *&-(1-\mu)e^{-2\alpha h_2}Q_{22}\end{bmatrix}<0.$$

Hence, from (26) we have

$$J(t)\le -y_2^{\mathrm{T}}(t)Q_{222}y_2(t)+p^{\mathrm{T}}(t)P_{22}^{\mathrm{T}}Q_{122}^{-1}P_{22}p(t) \tag{27}$$

It follows from (25), (27) and applying propositon 2, we have

$$y_2^{\mathrm{T}}(t)[(1-\mu)Q_{22}+Q_{222}]y_2(t)\le(1-\mu)y_2^{\mathrm{T}}(t-h(t))e^{-2\alpha h_2}Q_{22}y_2(t-h(t))+p^{\mathrm{T}}(t)P_{22}^{\mathrm{T}}Q_{122}^{-1}P_{22}p(t) \tag{28}$$

By pre - and post - multiplying with N^{T}, N, it follows from (4) $(1-\mu)Q_{22} < Q_{222}$. Therefore

$$2(1-\mu)y_2^{\mathrm{T}}(t)Q_{22}y_2(t) \le (1-\mu)e^{-2\alpha h_2} \sup_{-h_2\le s\le 0} y^{\mathrm{T}}(t+s)Q_{22}y(t+s) + p^{\mathrm{T}}(t)P_{22}^{\mathrm{T}}Q_{122}^{-1}P_{22}p(t) \tag{29}$$

Observe that, for all $t \ge 0$, if $t-h(t) \ge 0$ then

$$\|y_1(t-h(t))\|^2 \le \frac{\lambda_2}{\lambda_1}\|\psi\|_{h_2}^2 e^{-2\alpha(t-h(t))} \le \frac{\lambda_2}{\lambda_1}\|\psi\|_{h_2}^2 e^{2\alpha h_2}e^{-2\alpha t}.$$

Otherwise, $\|y_1(t-h(t))\|^2 \le \|\psi\|_{h_2}^2 \le \|\psi\|_{h_2}^2 e^{-2\alpha(t-h(t))} \le \|\psi\|_{h_2}^2 e^{2\alpha h_2}e^{-2\alpha t}$ Therefore

$$\|y_1(t-h(t))\| \le \upsilon_1 e^{\alpha h_2}\|\psi\|_{h_2} e^{-\alpha t}, \quad t \ge 0.$$

From (29), we obtain

$$2(1-\mu)y_2^{\mathrm{T}}(t)Q_{22}y_2(t) \le (1-\mu)e^{-2\alpha h_2} \sup_{-h_2\le s\le 0} y^{\mathrm{T}}(t+s)Q_{22}y(t+s) + \frac{\lambda_{max}(P_{22}^{\mathrm{T}}P_{22})}{\lambda_{min}(Q_{122})}\left[\|D_{21}\|e^{\alpha h_2}e^{-\alpha t}\right]^2 \upsilon_1^2\|\psi\|_{h_2}^2$$

Let us denote $f(t) = y_2^{\mathrm{T}}(t)Q_{22}y_2(t), \quad t \ge -h_2,$ and $\bar{f}_{h_2}(t) = \sup_{-h_2\le s\le 0} f(t+s),$ then

$$f(t) \le \delta_1\bar{f}_{h_2}(t) + \delta_2 e^{-2\alpha t}, \quad t \ge 0 \tag{30}$$

where

$$\delta_1 = \frac{e^{-2\alpha h_2}}{2}, \delta_2 = \frac{\lambda_{max}\left(P_{22}^{\mathrm{T}}P_{22}\right)\|D_{21}\|^2 e^{2\alpha h_2}\upsilon_1^2\|\psi\|_{h_2}^2}{2\lambda_{min}\left(Q_{122}\right)(1-\mu).}.$$

Note that $\tau_{(\alpha,h)} = (1-\mu)$, hence

$$\delta_1 e^{2\alpha h_2} = \frac{e^{-2\alpha h_2}e^{2\alpha h_2}}{2} = \frac{1}{2} < 1.$$

By Lemma 2, it follows from (30) that

$$f(t) \le \left[\delta_1 e^{2\alpha h_2}\bar{f}_{h_2}(0) + \frac{\delta_2}{1-\delta_1 e^{2\alpha h_2}}\right]e^{-2\alpha t}, \quad t \ge 0.$$

Furthermore, it can be verified that

$$\bar{f}_{h_2}(0) \le \lambda_{max}(Q_{22})\|\psi\|_{h_2}^2, \quad \frac{\delta_2}{1-\delta_1 e^{2\alpha h_2}} = \frac{\lambda_{max}\left(P_{22}^{\mathrm{T}}P_{22}\right)\|D_{21}\|^2 e^{2\alpha h_2}\upsilon_1^2\|\psi\|_{h_2}^2}{\lambda_{min}\left(Q_{122}\right)(1-\mu)}.$$

And thus

$$\lambda_{min}\left(Q_{22}\right)\|y_2(t)\|^2 \le \left[\frac{\lambda_{max}(Q_{22})}{2} + \frac{\lambda_{max}\left(P_{22}^{\mathrm{T}}P_{22}\right)\left(D_{21}\right]^2 e^{2\alpha h_2}\upsilon_1^2}{\lambda_{min}\left(Q_{122}\right)(1-\mu)}\right]\|\psi\|_{h_2}^2 e^{-2\alpha t}, \quad t \ge 0.$$

Consequently

$$\|y_2(t)\| \le \upsilon_2\|\psi\|_{h_2} e^{-\alpha t}, \quad t \ge 0$$

where

$$\upsilon_2 = \left[\frac{\lambda_{max}(Q_{22})}{2} + \frac{\lambda_{max}(P_{22}^{T}P_{22})\|D_{21}\|^2 e^{2\alpha h_2}\upsilon_1^2}{\lambda_{min}(Q_{122})(1-\mu)} \right]^{\frac{1}{2}}.$$

This shows that the fast variable $y_2(t)$ is fallen into exponential decay with decay rate α. Note that $x(t) = Ny(t)$ we readily obtain

$$\|x(t,\varphi)\| \le \upsilon \|\varphi\|_{h_2} e^{\alpha t}, \quad t \ge 0$$

where $\upsilon = \|N\|\|N\|^{-1}\sqrt{\upsilon_1^2 + \upsilon_2^2}$. This completes the proof.

Remark: The result of this paper has resolved exponential stability for singular systems with interval time-varying delays while in the [21] tackled only the case of constant. Also, there were some results have given the exponential stability standards but it only applies to nonsingular systems (see. [5, 11, 20]). Therefore, the result in Theorem 1 is more advanced than the other paper' results.

4. Conslusion

In this paper, we have studied the exponential stability of singular systems with interval time-varying delays. On the basis of the new lemma and constructing a set of new Lyapunov-Krasovskii functionals, sufficient conditions for the exponential stability is established in terms of LMIs which guarantee that the system is regular, impulse-free and exponentially stability.

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