LONG-TIME BEHAVIOR OF SOLUTIONS TO A QUASILINEAR PARABOLIC EQUATION

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Abstract: *In this paper, we study the first initial boundary value problem for a class of quasilinear degenerate parabolic equations involving weighted p-Laplacian operators. The existence and uniqueness of a weak solution with respect to initial values is ensured by an application of the Faedo - Galerkin approximation and compact method. Moreover, the long-time behavior of solutions to that problem is considered via the concept of global attractors in various bi-spaces.*

Keywords: *Prabolic equation, bi-spaces.*

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded open set with a sufficiently smooth boundary $\partial\Omega$. We are concerned with the following initial boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - div\left(a(x)|\nabla u|^{p-2}\nabla u\right) + f(u) = g(x), \ x \in \Omega, \ t > 0, \\ u(x,t) = 0, \ x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), \ x \in \Omega, \end{cases} \tag{1.1}$$

where the functions a, f, g satisfy

(H1) Let Σ be a closed subset of Ω such that $|\Sigma| = 0$. The function $a : \Omega \to \mathbb{R}$ satisfies the following conditions

i) $a(x) \in L^{\infty}(\Omega)$, ii) $a(x) = 0$ for $x \in \Sigma$,

iii) $a(x) > 0$ for $x \in \bar{\Omega} \setminus \Sigma$, iv) $\int_{\Omega} \frac{1}{[a(x)]^{\frac{n}{\alpha}}} dx < +\infty$ for some $\alpha \in \left(0, n(p-1)\right]$.

(H2) $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying

$$c_1 |u|^q - c_0 \leq f(u)u \leq c_2 |u|^q + c_0, \ q \geq 2, \tag{1.2}$$

$$f'(u) \geq -c_3, \ for\ all\ u \in \mathbb{R}, \tag{1.3}$$

where c_0, c_1, c_2, c_3 are positive constants.

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(H3) $g \in L^s(\Omega)$, where the positive number s is such that

$$\frac{s}{s-1} \in \left[1, p_\alpha^*\right), \; p_\alpha^* := \frac{pn}{n-p+\alpha}, \; and \; 2 \le p < n+\alpha. \tag{1.4}$$

Recently, motivated by [7], where a semilinear degenerate elliptic problem was studied, the diffusion coefficient $a(x)$ is allowed to have at most a finite number of zeroes. Then, the attention is paid to a semilinear degenerate parabolic problem in [15] where its degeneracy is considered in the sense that the measurable, nonnegative diffusion coefficient $a(x)$ is allowed to be possibly vanished on a nonempty closed subset Σ with zero measure. For the physical motivation, it might be related to media model which possibly are somewhere *"perfect" insulators*or*"perfect" conductors*. So this allows the coefficient a vanish somewhere or to be unbounded (see [12]), or it might be related to the upper box-counting dimension of the set K when $a := dist(x; K)$ with K a subset of Ω (see [1]).

This paper is motivated by [4, 5, 13, 14, 21, 22] when we study the asymptotic behavior of the weak solutions of the problem by analysing the existence and structure of its global attractors. During the last decade, many mathematicians have been studing problems associated with the p-Laplace operator which appears in a variety of physical fields (see [2, 3, 4, 5, 6, 9, 10, 11, 13, 14, 21, 18, 20]). Recently, Yang, Sun and Zhong [2] proved the existence of an $(L^2(\Omega), W_0^{1,p}(\Omega) \cap L^q(\Omega))$- global attractor by using a new a *priori* estimate method to testify the asymptotic compactness.

The problem (1.1) contains some important classes of parabolic equations, such as the semilinear heat equation (when $a(x) = constant > 0, p = 2$), semilinear degenerate parabolic equations (when $p = 2$) which was investigated in [15], the p-Laplacian equation (when $a(x) = 1$, $p \ge 2$ see [21]), etc.

The paper is organized as follows. In Section 2, we prove the existence and uniqueness of a global weak solution to problem (1.1. In Section 3, we study the existence of global attractors in various bi-spaces for the semigroup.

The problem is distinguished in two cases such as subcritical if $\alpha \in \left(0, p + \frac{n(p-2)}{2}\right)$ and supcriticalif $\alpha \in \left[p + \frac{n(p-2)}{2}, n(p-1)\right]$.

This leads to the lack of a suitable compact embedding of $\mathcal{D}_0^{1,p}(\Omega,a)$ into $L^2(\Omega)$ in the supcritical case. Moreover, the solutions are at most in $\mathcal{D}_0^{1,p}(\Omega,a) \cap L^q(\Omega)$, so there is no compact embedding results for these cases which we need to prove the asymptotical compact for the semigroup. Therefore, the proof requires more involved techniques which makes it slightly complicated. In order to overcome these difficulties, we exploit the aproach in [5, 21, 22] which has been used recently for some kind of partial differential equations.

Notation: We use C to denote various constants whose values may change with each appearance. We write $\Omega(u \geq M) := \{x \in \Omega : u(x) \geq M\}$ and $\Omega(u \leq M) := \{x \in \Omega : u(x) \leq M\}$. By $\langle .,. \rangle$, we represent the both duality product and inner product. p', q' are conjugate of $p, q,$ respectively.

2. Existence and uniqueness of weak solutions

First of all, let us introduce the energy space $\mathcal{D}_0^{1,p}(\Omega, a)$ defined as the closure of $C_0^\infty(\Omega)$ in the norm $\|u\|_{\mathcal{D}_0^{1,p}(\Omega,a)} := \left(\int_\Omega a(x)|\nabla u|^p \, dx\right)^{1/p}$. Let $\mathcal{D}^{-1,p'}(\Omega, a)$ be the dual space of $\mathcal{D}_0^{1,p}(\Omega, a)$. We denote

$$\Omega_T := \Omega \times (0,T), \; V := L^p\left(0,T; \mathcal{D}_0^{1,p}(\Omega,a)\right) \cap L^q(\Omega_T), \; V^* := L^{p'}\left(0,T; \mathcal{D}^{-1,p'}(\Omega,a)\right) + L^{q'}(\Omega_T).$$

Definition 2.1. A function u is called a weak solution of (1.1) on $(0,T)$ if and only if

$$u \in V, \; \frac{\partial u}{\partial t} \in V^*, \; u|_{t=0} = u_0, \; a.e. \text{ in } \Omega,$$

and $\int_{\Omega_T} \left(\frac{d}{dt} u(t)\eta + a(x)|\nabla u|^{p-2} \nabla u \cdot \nabla \eta + f(u)\eta - g\eta \right) dxdt = 0,$ for all test functions $\eta \in V$ and $u_0 \in L^2(\Omega)$.

It is known that (see [4]) that if $u \in V$ and $\frac{\partial u}{\partial t} \in V^*$, then $u \in C\left([0,T]; L^2(\Omega)\right)$. This makes the initial condition in problem (1.1) meaningful. The following lemma is infered from Holder's inequality.

Lemma 2.1. *Under the assumption **(H1)**, the following embeddings holds*

(i) $\mathcal{D}_0^{1,p}(\Omega, a) \subset W_0^{1,\beta}(\Omega)$ *continuously if* $1 \leq \beta \leq \frac{pn}{n+\alpha}$,

(ii) $\mathcal{D}_0^{1,p}(\Omega, a) \subset\subset L^r(\Omega)$ *compactly if* $1 \leq r < p_\alpha^*$.

By Young's inequality, embedding $\mathcal{D}_0^{1,p}(\Omega,a) \subset\subset L^{\frac{s}{s-1}}(\Omega)$ and Lemma 2.1, we have

Lemma 2.2. *Let* $\Omega' \subset \Omega$ *and* $u \in \mathcal{D}_0^{1,p}(\Omega', a)$, *we have*

$$\left|\int_{\Omega'} gu dx\right| \leq \varepsilon \|u\|^p_{\mathcal{D}_0^{1,p}(\Omega',a)} + C(\varepsilon)\|g\|^{p'}_{L^s(\Omega')}, \; \forall \varepsilon > 0.$$

Lemma 2.3. [9] *Let* $1 < p < +\infty$. *There exist positive constants* c_p, C_p *such that for every* $\xi, \eta \in \mathbb{R}^n$

$$c_p N_p(\xi,\eta) \le (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta).(\xi-\eta) \le C_p N_p(\xi,\eta),$$

where $N_p(\xi,\eta) = \{|\xi| + |\eta|\}^{p-2}|\xi-\eta|^2$, *a dot denotes the Euclidean product in* $\mathbb{R}^n$.

Putting $L_{p,a}u := -div\left(a(x)|\nabla u|^{p-2}\nabla u\right)$.

As a consequence of Lemma 2.3 and using similar arguments as in [16, Chapter 2], we get

Lemma 2.4. *The operator* $L_{p,a}$ *maps* $\mathcal{D}_0^{1,p}(\Omega,a)$ *into its dual* $\mathcal{D}^{-1,p'}(\Omega,a)$. *Moreover,*

(i) $L_{p,a}$ *is hemicontinuous, i.e., for all* $u,v,w \in \mathcal{D}_0^{1,p}(\Omega,a)$, *the mapping* $\lambda \mapsto \langle L_{p,a}(u+\lambda v), w\rangle$ *is continuous from* $\mathbb{R} \to \mathbb{R}$.

(ii) $L_{p,a}$ *is strongly monotone when* $p \ge 2$, *i.e.,*

$$\langle L_{p,a}u - L_{p,a}v, u-v\rangle \ge \delta \|u-v\|^p_{\mathcal{D}_0^{1,p}(\Omega,a)}, \text{ for all } u,v \in \mathcal{D}_0^{1,p}(\Omega,a).$$

Lemma 2.5. *Let* $\{u_n\}$ *be a bounded sequence in* $L^p\left(0,T;\mathcal{D}_0^{1,p}(\Omega,a)\right)$.

Then $\{u_n\}$ *converges almost everywhere in* Ω_T *up to a subsequence.*

Proof. First, we prove that, for a.e. $t \in [0,T]$, there exists $C(t) > 0$ such that $\|u_n(t)\|_{\mathcal{D}_0^{1,p}(\Omega,a)} \le C(t)$.

Indeed, if there exists a set $\Lambda \subset [0,T]$ with positive measure such that $\|u_n(t)\|_{\mathcal{D}_0^{1,p}(\Omega,a)} \to \infty$ as $n \to \infty$, for all $t \in \Lambda$,

then $\int_0^T \|u_n(t)\|_{\mathcal{D}_0^{1,p}(\Omega,a)}\, dt \ge \int_\Lambda \|u_n(t)\|_{\mathcal{D}_0^{1,p}(\Omega,a)}\, dt \to \infty$.

This implies a contradiction. On the other hand, it follows from Lemma 2.1. that $\{u_n(t)\}$ is precompact in $L^r(\Omega)$ for some $r \in [1, p_\alpha^*)$. Therefore, $u_n(t) \to u(t)$ a.e. in Ω.

Theorem 2.1. *Given* $u_0 \in L^2(\Omega)$. *We assume that (*H1), (H2)*, and* (H3) *hold. Then the problem (1.1) has a unique weak solution on the interval* $(0,T)$.

Moreover, the mapping $u_0 \mapsto u(t)$ *is continuous.*

Proof. **i) Existence.** Let $\{e_j\}_1^\infty$ be a basis of $\mathcal{D}_0^{1,p}(\Omega,a)\cap L^q(\Omega)$. We find the approximating solution $u_n(t)$ in the form $u_n(t)=\sum_{j=1}^{n} u_{nk}(t)e_k$.

We get u_n from solving the following problem

$$\int_\Omega \frac{du_n}{dt}e_k dx+\int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla e_k dx+\int_\Omega f(u_n)e_k dx=\int_\Omega g(x)e_k dx, \tag{2.2}$$

$$\int_\Omega u_n(0)e_k dx=\int_\Omega u_0 e_k dx. \tag{2.3}$$

Since $f\in C^1(\mathbb{R})$ and using the Peano theorem, this problem possesses a local solution $u_{\{nk\}}(t)$. By multiplying by $u_{nk}(t)$ in (2.2) and summing k =1 to n, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)}+\int_\Omega a(x)|\nabla u_n|^p dx+\int_\Omega f(u_n)u_n dx=\int_\Omega g u_n dx. \tag{2.4}$$

We now establish some a *priori* estimates for u_n. Using (1.2) and Lemma 2.2 yield

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)}+\|u_n\|_{\mathcal{D}_0^{1,p}(\Omega,a)}+c_1\|u_n\|^q_{L^q(\Omega)}\le c_0|\Omega|+\varepsilon\|u_n\|_{\mathcal{D}_0^{1,p}(\Omega,a)}+C(\varepsilon)\|g\|^{p'}_{L^s(\Omega)}, \tag{2.5}$$

for all $\varepsilon>0$.

It follows that

$$\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)}+(2-2\varepsilon)\|u_n\|_{\mathcal{D}_0^{1,p}(\Omega,a)}+2c_1\|u_n\|^q_{L^q(\Omega)}\le 2c_0|\Omega|+2C(\varepsilon)\|g\|^{p'}_{L^s(\Omega)}. \tag{2.6}$$

After an integration in t, this leads to

$$\|u_n(t)\|^2_{L^2(\Omega)}+(2-2\varepsilon)\int_0^t\|u_n\|_{\mathcal{D}_0^{1,p}(\Omega,a)}dt+2c_1\|u_n\|^q_{L^q(\Omega_t)}\le\|u_n(0)\|^2_{L^2(\Omega)}+2tc_0|\Omega|+2tC(\varepsilon)\|g\|^{p'}_{L^s(\Omega)}. \tag{2.7}$$

We deduce from the last inequality that $\{u_n\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$, $\{u_n\}$ is bounded in $L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,a))$, $\{u_n\}$ is bounded in $L^q(\Omega_T)$.

We see that the local solution u_n can be extended to the interval $[0,T]$.

On the other hand, $L_{p,a}u_n$ defines an element of $\mathcal{D}^{-1,p'}(\Omega,a)$, determined by duality $\langle L_{p,a}u_n,w\rangle=\int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla w dx$, for all $w\in\mathcal{D}_0^{1,p}(\Omega,a)$.

Taking (H1) into account and the boundedness of u_n in $L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,a))$.

We deduce that $L_{p,a}u_n$ is bounded in $L^{p'}(0,T;\mathcal{D}^{-1,p'}(\Omega,a))$ since

$$\left|\int_0^T \langle L_{p,a}u_n, v\rangle\, dt\right| = \left|\int_0^T \int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla v dxdt\right| \le \int_0^T\int_\Omega (a(x)^{\frac{p-1}{p}}|\nabla u_n|^{p-1})(a(x)^{\frac{1}{p}}|\nabla v|)dxdt$$
$$\le \|u_n\|^{p/p'}_{L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,a))}\|v\|_{L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,a))},$$

for any $v \in L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,a))$. In addition, from (1.2), we deduce

$$|f(u)| \le C(|u|^{q-1}+1). \tag{2.8}$$

Combining with the boundedness of $\{u_n\}$ in $L^q(\Omega_T)$ it implies that $\{f(u_n)\}$ is bounded in $L^{q'}(\Omega_T)$. We can rewrite the first equation of (1.1) in V^* as

$$u'_n = g(x) - L_{p,a}u_n - f(u_n). \tag{2.9}$$

Therefore, $\{u'_n\}$ is bounded in V^*. Due to Alaoglu weak-star compactness theorem (see [18])

$$u'_n \rightharpoonup u' \text{ in } V^*,$$
$$L_{p,a}u_n \rightharpoonup \xi_1 \text{ in } L^{p'}(0,T;\mathcal{D}^{-1,p'}(\Omega,a)), \tag{2.10}$$
$$f(u_n) \rightharpoonup \xi_2 \text{ in } L^{q'}(\Omega_T), \tag{2.11}$$

for all $T > 0$.

Thanks to Lemma 2.5 and Lemma 1.3, p.12 in [16] and due to $f(u_n)$ is continuous and bounded.

We have
$$f(u_n) \rightharpoonup \xi_2 = f(u) \text{ in } L^{q'}(\Omega_T). \tag{2.12}$$

We now show that $\xi_1 = L_{p,a}u$. It follows from Lemma 2.4 that

$X_n := \int_0^T \langle L_{p,a}u_n - L_{p,a}v, u_n - v\rangle\, dt \ge 0,$ for every $v \in L^p(0,T;\mathcal{D}_0^{1,p}(\Omega,a))$.

Moreover, we have

$$\int_0^T \langle L_{p,a}u_n, u_n\rangle\, dt = \int_0^T\int_\Omega a(x)|\nabla u_n|^p\, dxdt = \int_0^T\int_\Omega (gu_n - f(u_n)u_n - u_n' u_n)dxdt \tag{2.13}$$
$$= \int_0^T\int_\Omega (gu_n - f(u_n)u_n)dxdt + \frac{1}{2}\|u_n(0)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_n(T)\|^2_{L^2(\Omega)}.$$

Therefore,

$$X_n = \int_0^T\int_\Omega (gu_n - f(u_n)u_n)dxdt + \frac{1}{2}\|u_n(0)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_n(T)\|^2_{L^2(\Omega)} = -\int_0^T \langle L_{p,a}u_n, v\rangle dt - \int_0^T \langle L_{p,a}v, u_n - v\rangle dt.$$

Since $u_n(0) \to u_0$ in $L^2(\Omega)$, and by the lower semi-continuity $\|\cdot\|_{L^2(\Omega)}$ we get

$$\|u(T)\|_{L^2(\Omega)} \le \liminf_{n\to+\infty}\|u_n(T)\|_{L^2(\Omega)}.$$

Meanwhile, by the Lebesgue dominated theorem, we have

$$\int_0^T \int_\Omega (gu - f(u)u)dxdt = \lim_{n\to+\infty} \int_0^T \int_\Omega (gu_n - f(u_n)u_n)dxdt. \tag{2.14}$$

Putting this together with (2.13) and (2.14), we obtain

$$\lim_{n\to+\infty} \sup X_n \le \int_0^T \int_\Omega (gu - f(u)u)dxdt + \frac{1}{2}\left(\|u(0)\|^2_{L^2(\Omega)} - \|u(T)\|^2_{L^2(\Omega)} \right) - \int_0^T \langle \xi_1, v\rangle dt - \int_0^T \langle L_{p,a}v, u - v\rangle dt. \tag{2.15}$$

In view of (2.9), we deduce

$$\int_0^T \int_\Omega (gu - f(u)u)dxdt + \frac{1}{2}\left(\|u(0)\|^2_{L^2(\Omega)} - \|u(T)\|^2_{L^2(\Omega)} \right) = \int_0^T \langle \xi_1, u\rangle dt.$$

Taking (2.15) into account, we get the estimate

$$\int_0^T \langle \xi_1 - L_{p,a}v, u - v\rangle\, dt \ge 0, \forall v \in L^p(0,T; \mathcal{D}_0^{1,p}(\Omega, a)).$$

We choose $v = u - \delta w, \delta > 0$, so

$$\int_0^T \langle \xi_1 - L_{p,a}(u - \delta w), w\rangle dt \ge 0, \quad \forall w \in L^p(0,T; \mathcal{D}_0^{1,p}(\Omega, a)).$$

Letting $\delta \to 0$, we get $\int_0^T \langle \xi_1 - L_{p,a}u, w\rangle dt = 0, \forall w \in L^p(0,T; \mathcal{D}_0^{1,p}(\Omega, a)),$

Thus $\xi_1 = L_{p,a}u.$ We now prove $u(0) = u_0$.

By taking the test functions $\varphi \in C^1([0,T]; \mathcal{D}_0^{1,p}(\Omega, a) \cap L^q(\Omega))$ such that $\varphi(T) = 0$,

we have $-\int_0^T \langle u_n, \varphi'\rangle dt + \int_0^T \langle L_{p,a}u_n, \varphi\rangle dt + \int_0^T \langle f(u_n) - g, \varphi\rangle dt = \langle u_n(0), \varphi(0)\rangle.$

Let $n \to \infty$, we obtain

$$-\int_0^T \langle u, \varphi'\rangle dt + \int_0^T \langle L_{p,a}u, \varphi\rangle dt + \int_0^T \langle f(u) - g, \varphi\rangle dt = \langle u_0, \varphi(0)\rangle, \tag{2.16}$$

since $u_n(0) \to u_0$. On the other hand, from the " limiting equation", we have

$$-\int_0^T \langle u, \varphi'\rangle dt + \int_0^T \langle L_{p,a}u, \varphi\rangle dt + \int_0^T \langle f(u) - g, \varphi\rangle dt = \langle u(0), \varphi(0)\rangle. \tag{2.17}$$

Comparing (2.16) with (2.17), we get $u(0) = u_0$.

ii) Uniqueness and continuous dependence on the initial data. Let us denote by u and v two weak solutions of (1.1) with initial data $u_0, v_0 \in L^2(\Omega)$, respectively. Then

$w := u\text{-}v$ satisfies $\begin{cases} w_t + L_{p,a}\, u - L_{p,a}\, v + f(u) - f(v) = 0, \\ w|_{\partial\Omega} = 0, \\ w(0) = u_0 - v_0. \end{cases}$

Hence $\frac{1}{2}\frac{d}{dt}\left(\|w\|^2_{L^2(\Omega)} \right) + \langle L_{p,a}u - L_{p,a}v, u - v\rangle + \int_\Omega \big(f(u) - f(v)\big)(u - v)\, dx = 0.$ By using (1.3) and Lemma 2.4 $\frac{d}{dt}\left(\|w\|^2_{L^2(\Omega)} \right) \le 2c_3 \left(\|w\|^2_{L^2(\Omega)} \right)$.

An application of the Gronwall inequality leads to $\|w\|^2_{L^2(\Omega)} \le \|w(0)\|^2_{L^2(\Omega)} e^{2c_3 t}$. This implies the uniqueness (if $u_0 = v_0$) and the continuous dependence of the solution.

3. Global attractors

Theorem 2.1 allows us to construct a continuous (nonlinear) semigroup $S(t): L^2(\Omega) \to L^2(\Omega)$ associated to problem (1.1) as follows $S(t)u_0 := u(t),$ where $u(t)$ is the unique weak solution of (1.1) with the initial data u_0.

The exponent p_α^* plays a crucial role in the classical Sobolev embedding, i.e., $p_\alpha^* > 2$ when $\alpha \in (0, p + \frac{n(p-2)}{2})$, $p_\alpha^* \in [\frac{n}{n-1}, 2]$ *when* $\alpha \in [p + \frac{n(p-2)}{2}, n(p-1)]$. The main objective of this section is to show the existence of global attractors of the semigroup $S(t)$ generated by problem (1.1) in various bi-spaces. We will combine the so-called uniformly compact method and the method introduced in [5], [21], [22] to solve this problem. The following Proposition is the existence of bounded absorbing set.

Proposition 3.1. *The semigroup* $\{S(t)\}_{t\ge 0}$ *has an* $(L^2(\Omega), \mathcal{D}_0^{1,p}(\Omega,a) \cap L^q(\Omega))$*-bounded absorbing set* B_0*, i.e., there is a positive constant* ρ*, such that for any bounded subset* B *in* $L^2(\Omega)$*, there is a positive constant* T *which depends only on* L^2*-norm of* B *such that* $\int_\Omega a(x)|\nabla u|^p dx + \int_\Omega |u|^q dx \le \rho$*, for all* $t \ge T$ *and where* u *is the unique weak solution of (1.1) with the initial datum* u_0.

Proof. Multiplying the first equation in (1.1) by u and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,a)} + \int_\Omega f(u)u dx = \int_\Omega g u dx. \tag{3.1}$$

Combining with (1.2) and using Lemma 2.2 yields

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + (2-2\varepsilon)\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,a)} + 2c_1\|u\|^q_{L^q(\Omega)} \le 2c_0|\Omega| + 2C(\varepsilon)\|g\|^{p'}_{L^s(\Omega)},$$

for ε small enough. Due to $q \ge 2$, we deduce from the last inequality that

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + C\|u\|^2_{L^2(\Omega)} \le C(\|g\|_{L^s(\Omega)}, c_0, |\Omega|). \tag{3.2}$$

Applying the Gronwall inequality, we obtain

$$\|u(t)\|^2_{L^2(\Omega)} \le \|u(0)\|^2_{L^2(\Omega)} e^{-Ct} + C(\|g\|_{L^s(\Omega)}, c_0, |\Omega|)(1 - e^{-Ct}). \tag{3.3}$$

We see that, from (3.3), $\{S(t)\}_{t\geq 0}$ has an $(L^2(\Omega), L^2(\Omega))$-bounded absorbing set, i.e., for any bounded set B in $L^2(\Omega)$ there exists $T_1 = T_1(B)$ such that $\| S(t)u_0 \|^2_{L^2(\Omega)} \leq \rho_0$, (3.4)

for all $t \geq T_1$, $u_0 \in B$, where the constant ρ_0 is independent of u_0. Going back to (3.1) and integrating over $[t,t+1]$ with $t \geq T_1$, we derive

$$\int_t^{t+1} \| u \|^p_{\mathcal{D}_0^{1,p}(\Omega,a)} + \int_\Omega f(u)udx - \int_\Omega gudx)ds \leq \frac{\rho_0}{2}. \tag{3.5}$$

for all $t \geq T_1$. Putting $F(u) = \int_0^u f(s)ds$. Due to (1.2) and (1.3), it fulfills the bounds for some positive constants c_4, c_5 such that $c_4 |u|^q - c_5 \leq F(u) \leq f(u)u + \frac{c_3}{2}|u|^2$. (3.6)

$$\text{Therefore } c_4 \| u \|^q_{L^q(\Omega)} - c_5 |\Omega| \leq \int_\Omega F(u)dx \leq \int_\Omega f(u)udx + \frac{c_3\rho_0}{2}, \tag{3.7}$$

for all $t \geq T_1$. We deduce from (3.5) and (3.7) that

$$\int_t^{t+1} \| u \|^p_{\mathcal{D}_0^{1,p}(\Omega,a)} + \int_\Omega F(u)dx - \int_\Omega gudx]ds \leq \frac{\rho_0(c_3+1)}{2}. \tag{3.8}$$

On the other hand, multiplying (1.1) by u_t, we obtain

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u.\nabla u_t dx + \int_\Omega f(u)u_t dx - \int_\Omega g u_t dx \leq - \| u_t \|^2_{L^2(\Omega)} \leq 0. \tag{3.9}$$

$$\text{Therefore } \frac{d}{dt}\left(\frac{1}{p}\| u \|^p_{\mathcal{D}_0^{1,p}(\Omega,a)} + \int_\Omega F(u)dx - \int_\Omega gudx\right) \leq - \| u_t \|^2_{L^2(\Omega)} \leq 0.$$

Combining with (3.8) and (3.9), by virtue of the uniform Gronwall inequality, we get

$$\frac{1}{p}\| u \|^p_{\mathcal{D}_0^{1,p}(\Omega,a)} + \int_\Omega F(u)dx - \int_\Omega gudx \leq \frac{\rho_0(c_3+1)}{2}, \tag{3.10}$$

for all $t \geq T_2 = T_1 + 1$. Thanks to (3.7) and Lemma 2.2, we infer from (3.10) that

$$\int_\Omega a(x)|\nabla u|^p dx + \int_\Omega |u|^q dx \leq C(\| g \|_{L^s(\Omega)}, |\Omega|, \rho_0, p, c_3, c_4, c_5). \tag{3.11}$$

Thus, taking $\rho = C(\| g \|_{L^s(\Omega)}, \| \Omega \|, \rho_0, p, c_3, c_4, c_5)$ and $T = T_2$.

We complete the proof.

Proposition 3.2. *The semigroup $\{S(t)\}_{t\geq 0}$ is norm-to-weak continuous on $S(B_0)$, where B_0 is the $(L^2(\Omega), \mathcal{D}_0^{1,p}(\Omega,a) \cap L^q(\Omega))$ - bounded absorbing set obtained in Proposition 3.1.*

Proof. Choosing $Y = L^2(\Omega)$, $X = \mathcal{D}_0^{1,p}(\Omega, a) \cap L^q(\Omega)$, the conclusion follows imme-diately from Theorem 3.2 in [22].

Theorem 3.1. *$((L^2(\Omega), L^2(\Omega))$- global attractor) Suppose that the hypotheses* **(H1),(H2)** *and* **(H3)** *hold. Then the semigroup S(t) generated by problem (1.1) has an $(L^2(\Omega), L^2(\Omega))$- global attractor $\mathcal{A}_2$.*

Proof. We distinguish two cases to deal with our problem since it is subcritical if $\alpha \in (0, p + \frac{n(p-2)}{2})$, and supcritical if $\alpha \in [p + \frac{n(p-2)}{2}, n(p-1)]$.

Case 1. If $\alpha \in (0, p + \frac{n(p-2)}{2})$, then $p_\alpha^* \in [\frac{n}{n-1}, 2]$ and $\mathcal{D}_0^{1,p}(\Omega, a) \subset\subset L^2(\Omega)$.

Using the so-called uniformly compact method, the existence of the bounded abosorbing set in $\mathcal{D}_0^{1,p}(\Omega, a) \cap L^q(\Omega)$ yields the existence of a global attractor in $L^2(\Omega)$ immediately.

Case 2. If $\alpha \in [p + \frac{n(p-2)}{2}, n(p-1)]$, then $p_\alpha^* \in [\frac{n}{n-1}, 2]$ and $s \ge 2$. We infer from Lemma 2.1 that there exists some $r \in [1, p_\alpha^*)$ such that $\mathcal{D}_0^{1,p}(\Omega, a) \subset\subset L^r(\Omega)$\$.

Obviously, B_0 is also a closed bounded absorbing set in $\mathcal{D}_0^{1,p}(\Omega, a)$, we can consider our problem only in B_0 and B_0 has a finite ε - net in $L^r(\Omega)$.

Using the so-called uniformly compact method, we deduce that there exists a global attractor in $L^r(\Omega)$. From Corollary 5.7 in [22] and Proposition 3.1, we need only verify for any $\varepsilon > 0$ and any bounded subset $B \subset L^2(\Omega)$, there exists $T = T(\varepsilon, B)$ and $M = M(\varepsilon)$ such that $\varepsilon > 0$ and any bounded subset $B \subset L^2(\Omega)$, there exists $T = T(\varepsilon, B)$ and $M = M(\varepsilon)$ such that

$$\int_{\Omega(|u| \ge 2M)} |u(t)|^2 \, dx < C\varepsilon \quad for\ u_0 \in B, t \le T$$

where the positive constant C is independent of ε and B. It follows from Lemma 5.2 in [22] that for any fixed $\varepsilon > 0$, there exist $\delta > 0, T = T(B)$ and $M = M(\varepsilon)$ such that the Lebesgue measure $|\Omega(|S(t)u_0| \ge M)| \le \delta$ for all $u_0 \in B$ and $t \ge T$ and

$$\int_{\Omega(|S(t)u_0| \ge M)} |g|^2 < \varepsilon. \tag{3.12}$$

We multiply the first equation in (1.1) by $(u-M)_+$, one gets

$$u_t(u-M)_+ - div(a(x)|\nabla u|^{p-2} \nabla u)(u-M)_+ + f(u)(u-M)_+ = g(x)(u-M)_+$$

where $(u-M)_+$ denotes the positive part of $(u-M)$, that is,

$$(u\text{-}M)_+ = \begin{cases} u-M, & if \quad u \ge M, \\ 0, & if \quad u < M. \end{cases}$$

If M is a large enough positive constant, it follows from (1.2) that $f(u) \ge \tilde{c}|u|^{q-1}$ for all $u \ge M$.

Therefore, $f(u)(u-M)_+ \ge \tilde{c}|u|^{q-1}(u-M)_+ \ge \tilde{c}(u-M)_+^q.$ (3.13)

Hence,

$$\frac{1}{2}\frac{d}{dt}\|(u-M)_+\|_{L^2(\Omega)}^2 + \int_{\Omega(u\ge M)} a(x)|\nabla u|^p\,dx + \int_{\Omega(u\ge M)} f(u)(u-M)_+\,dx = \int_{\Omega(u\ge M)} g(u-M)_+\,dx.$$

Since $L^q(\Omega) \subset L^2(\Omega), (q \ge 2)$, we deduce from Lemma 2.2, (3.12) and (3.13) that

$$\frac{d}{dt}\|(u-M)_+\|_{L^2(\Omega)}^2 + C\|(u-M)_+\|_{L^2(\Omega)}^q \le C\varepsilon.$$

It follows from Lemma 3.2 in [4] that $\|(u-M)_+\|_{L^2(\Omega)}^2 \le C\varepsilon$ (3.14)

for M, T large enough and $t \ge T$. Replacing $(u-M)_+$ by $(u+M)_-$,

we obtain also similar assertion $\|(u+M)_-\|_{L^2(\Omega)}^2 \le C\varepsilon,$ (3.15)

where $(u\text{+}M)_- = \begin{cases} u+M, & if \ u \le -M, \\ 0, & if \ u > -M. \end{cases}$

It follows from (3.14) anh (3.15) that $\int_{\Omega(|u|\ge M)} (|u|-M)^2 dx < C\varepsilon$.

Therefore,

$$\int_{\Omega(|u|\ge 2M)} |u|^2\,dx = \int_{\Omega(|u|\ge 2M)} (|u|-M+M)^2 dx \le 4\int_{\Omega(|u|\ge 2M)} (|u|-M)^2 dx + 4\int_{\Omega(|u|\ge 2M)} M^2 dx < C\varepsilon,$$

for M large enough and C is independent of ε and B.

As a consequence, the semigroup $S(t)$ has an $(L^2(\Omega), L^2(\Omega))$- global attractor $\mathcal{A}_2$.

We now show that the existence of an $(L^2(\Omega), L^q(\Omega))$- global attractor.

Theorem 3.2. *($(L^2(\Omega), L^q(\Omega))$- global attractor) Assume that the hypotheses* (**H1**), (**H2**) *and* (**H3**) *are satisfied. Then the semigroup $S(t)$ associated to (1.1) has an $(L^2(\Omega), L^q(\Omega))$- global attractor $\mathcal{A}_q$.*

Proof. Thanks to Corollary 5.7 in [22], Proposition 3.1 and Theorem 3.1, it is sufficient to prove that for any $\varepsilon > 0$ and any bounded subset $B \subset L^2(\Omega)$, there exist two positive constants $T = T(\varepsilon, B)$ and $M = M(\varepsilon)$ such that $\int_{\Omega(|u|\ge 2M)} |u|^q < C\varepsilon,$ for all $u_0 \in B$ and $t \ge T$, where the constant C is independent of ε and B.

Obviously, we also deduce from Lemma 5.2 in [22] that for any fixed $\varepsilon > 0$, there exist $\delta > 0, T = T(B)$ and $M = M(\varepsilon)$ such that the Lebesgue measure $|\Omega(|S(t)u_0| \geq M)| \leq \delta$ for all $u_0 \in B$ and $t \geq T$ and $\int_{\Omega(|S(t)u_0| \geq M)} |g|^s < \varepsilon.$ (3.16)

We now multiply the first equation in (1.1) by $(u-M)_+^{q-1}$, for M large enough, one gets $u_t(u-M)_+^{q-1} - div(a(x)|\nabla u|^{p-2}\nabla u)(u-M)_+^{q-1} + f(u)(u-M)_+^{q-1} = g(x)(u-M)_+^{q-1}.$ (3.17)

Since $f(u) \geq \tilde{c}|u|^{q-1}$ with $u \geq M$. Thus,

$$f(u)(u-M)_+^{q-1} \geq \tilde{c}|u|^{q-1}(u-M)_+^{q-1} = \frac{\tilde{c}}{2}|u|^{q-1}(u-M)_+^{q-1} + \frac{\tilde{c}}{2}|u|^{q-1}(u-M)_+^{q-1} \tag{3.18}$$

$$\geq \frac{\tilde{c}}{2}|u|^{q-1-\frac{q-1}{s-1}}(u-M)_+^{(q-1)\frac{s}{s-1}} + \frac{\tilde{c}}{2}|u|^{q-2}(u-M)_+^q \geq \frac{\tilde{c}}{2}M^{q-1-\frac{q-1}{s-1}}(u-M)_+^{(q-1)\frac{s}{s-1}} + \frac{\tilde{c}}{2}M^{q-2}(u-M)_+^q.$$

We also have

$$g(u-M)_+^{q-1} \leq \frac{\tilde{c}}{2}M^{q-1-\frac{q-1}{s-1}}(u-M)_+^{(q-1)\frac{s}{s-1}} + C|g|^s. \tag{3.19}$$

Putting (3.18), (3.19) together with (3.17) and integrating over $\Omega(u \geq M)$, we obtain

$$\frac{1}{q}\frac{d}{dt}\int_{\Omega(u\geq M)}(u-M)_+^q dx + (q-1)\int_{\Omega(u\geq M)} a(x)|\nabla u|^p (u-M)_+^{q-2}dx + \frac{\tilde{c}}{2}M^{q-2}\int_{\Omega(u\geq M)}(u-M)_+^q dx \leq C\int_{\Omega(u\geq M)}|g|^s dx$$

and then $\dfrac{d}{dt}\int_{\Omega(u\geq M)}(u-M)_+^q dx + \dfrac{\tilde{c}}{2}qM^{q-2}\int_{\Omega(u\geq M)}(u-M)_+^q dx \leq Cq\int_{\Omega(u\geq M)}|g|^s dx.$

By the Gronwall inequality, the following estimate follows from taking M large enough and using (3.16), we have $\int_{\Omega(u\geq M)}(u-M)_+^q dx < \varepsilon,$ (3.20)
where C is independent of ε and M. Repeating the same steps above, just taking $(u+M)_-$ instead of $(u-M)_+$ we also obtain

$$\int_{\Omega(u\leq -M)}|(u+M)_-^q dx < \varepsilon. \tag{3.21}$$

In both cases, we imply from (3.20) and (3.21) that $\int_{\Omega(|u|\geq M)}(|u|-M)^q dx < \varepsilon,$ (3.22)
for M large enough. Therefore,

$$\int_{\Omega(|u|\geq 2M)}|u|^q dx = \int_{\Omega(|u|\geq 2M)}|u-M+M|^q dx \leq 2^q\int_{\Omega(|u|\geq 2M)}(|u|-M)^q dx + 2^q\int_{\Omega(|u|\geq 2M)}M^q dx < C\varepsilon,$$

for M large enough. As a consequence, the semigroup $S(t)$ also has an $(L^2(\Omega), L^q(\Omega))$-global attractor $\mathcal{A}_q$.

It is possible to show that the regularity of the attractor increases. First, we will give a *priori* estimate for u_t endowed with L^2- norm.

Lemma 3.1. *Assume that the hypotheses* $(\mathbf{H1}),(\mathbf{H2})$ *and* $(\mathbf{H3})$ *hold. Then for any bounded subset* B *in* $L^2(\Omega)$, *there exists a positive constant* $T=T(B)$ *such that* $\|u_t(s)\|^2_{L^2(\Omega)}\le \rho_1$, *for all* $u_0\in B$, *and* $s\ge T$, *where* $u_t(s)=\frac{d}{dt}(S(t)u_0)|_{t=s}$ *and* ρ_1 *is a positive constant independent of* B.

Proof. By differentiating the first equation in (1.1) in time and denoting $v=u_t$, we get

$$v_t-div(a(x)|\nabla u|^{p-2}\nabla v)-(p-2)div(a(x)|\nabla u|^{p-4}(\nabla u.\nabla v)\nabla u)+f'(u)v=0. \tag{3.23}$$

Multiplying the above equality by v, integrating over Ω and using (1.3), we have

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(\Omega)}+\int_\Omega a(x)|\nabla u|^{p-2}|\nabla v|^2dx+(p-2)\int_\Omega a(x)|\nabla u|^{p-4}(\nabla u.\nabla v)^2dx\le c_3\|v\|^2_{L^2(\Omega)}. \tag{3.24}$$

Hence
$$\frac{d}{dt}\|v\|^2_{L^2(\Omega)}\le 2c_3\|v\|^2_{L^2(\Omega)} \tag{3.25}$$

On the other hand, integrating (3.9) from t to $t+1$ and combining (3.10), we get

$$\int_t^{t+1}\|u_t\|^2_{L^2(\Omega)}dx\le C. \tag{3.26}$$

as t large enough. Combining (3.25) with (3.26) and using the uniform Gronwall inequality, we have $\|u_t\|^2_{L^2(\Omega)}\le\rho_1$, as t large enough, and ρ_1 is a some positive constant. The proof is complete.

For $(L^2(\Omega),\mathcal{D}_0^{1,p}(\Omega,a)\cap L^q(\Omega))$- global attractor, we need the following lemma ([5, Theorem 2.7])

Lemma 3.2. Let X be a Banach space and Z be a metric space. Let $\{S(t)\}_{t\ge0}$ be a semigroup on X such that: *(i)* $\{S(t)\}_{t\ge0}$ has an (X,Z)- bounded absorbing set B_0;

(ii) $\{S(t)\}_{t\ge0}$ is (X,Z)- asymptotically compact;

(iii) $\{S(t)\}_{t\ge0}$ is norm-to-weak continuous on $S(B_0)$.

Then $\{S(t)\}_{t\ge0}$ has an (X,Z)- global attractor.

Theorem 3.3. Assume that the hypotheses $(\mathbf{H1}),(\mathbf{H2})$ and $(\mathbf{H3})$ are satisfied. Then the semigroup $\{S(t)\}_{t\ge0}$ associated to (1.1) has an $(L^2(\Omega),\mathcal{D}_0^{1,p}(\Omega,a)\cap L^q(\Omega))$- global attractor $\mathcal{A}$.

Proof. By Lemma 3.2 and Propositions 3.1-3.2, we only need to show that the semigroup $\{S(t)\}_{t\ge0}$ is $(L^2(\Omega),\mathcal{D}_0^{1,p}(\Omega,a)\cap L^q(\Omega))$- asymptotically compact.

This means that we take B a bounded subset of $L^2(\Omega)$, we will show that for any $\{u_{0n}\}\subset B$ and $t_n\to+\infty$, $\{u_n(t_n)\}_{n=1}^\infty$ is precompact in $\mathcal{D}_0^{1,p}(\Omega,a)\cap L^q(\Omega)$, where $u_n(t_n)=S(t_n)u_{0n}$.

By Theorem 3.2, it is sufficient to verify that $\{u_n(t_n)\}_{n=1}^{\infty}$ is precompact in $\mathcal{D}_0^{1,p}(\Omega,a)$. To do this, we will prove that $\{u_n(t_n)\}$ is a Cauchy sequence in $\mathcal{D}_0^{1,p}(\Omega,a)$. Thanks to Theorems 3.1-3.2, one can assume that $\{u_n(t_n)\}$ is a Cauchy sequence in $L^2(\Omega)$ and in $L^q(\Omega)$. Since $L_{p,a}$ is strongly monotone when $p \geq 2$. We have

$$\delta\| u_n(t_n)-u_m(t_m)\|^p_{\mathcal{D}_0^{1,p}(\Omega,a)} \leq \langle L_{p,a}u_n(t_n)-L_{p,a}u_m(t_m), u_n(t_n)-u_m(t_m)\rangle$$

$$=\langle -\frac{d}{dt}u_n(t_n)-f(u_n(t_n))+\frac{d}{dt}u_m(t_m)+f(u_m(t_m)), u_n(t_n)-u_m(t_m)\rangle$$

$$\leq \int_\Omega \left|\frac{d}{dt}u_n(t_n)-\frac{d}{dt}u_m(t_m)\right|\left|u_n(t_n)-u_m(t_m)\right| dx$$

$$+\int_\Omega \left|f(u_n(t_n))-f(u_m(t_m))\right|\left|u_n(t_n)-u_m(t_m)\right| dx$$

$$\leq \left\|\frac{d}{dt}u_n(t_n)-\frac{d}{dt}u_m(t_m)\right\|_{L^2(\Omega)}\left\|u_n(t_n)-u_m(t_m)\right\|_{L^2(\Omega)}$$

$$+\left\|f(u_n(t_n))-f(u_m(t_m))\right\|_{L^{q'}(\Omega)}\left\|u_n(t_n)-u_m(t_m)\right\|_{L^q(\Omega)}$$

Because $f(u_n(t))$ is bounded in $L^{q'}(\Omega)$, in addition, from Lemma 3.1, we deduce that $\{u_n(t_n)\}$ is the Cauchy sequence in $\mathcal{D}_0^{1,p}(\Omega,a)$.

4. Conclusion

In this paper, we study an initial boundary value problem for a class of quasilinear degenerate parabolic equations involving weighted p-Laplacian operators by using the compactness method and weak convergence techniques in Orlicz spaces. The long-time behavior of solutions to the problem is considered via the concepts of global attractors in various bi-spaces.

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