RANDOM VARIATIONAL INEQUALITIES FOR SEMI H-MONOTONE MAPPINGS

Nguyen Manh Hung, Nguyen Xuan Thuan

Received: 5 December 2017/ Accepted: 11 June 2019/ Published: June 2019

Abstract: *This paper is an extension of [2,4,6,7]. In this paper, one can solve some random variational inequalities for semi H-monotone and weakly semi H-monotone mappings.*

Keywords: *Random variational, semi H-momotone mapping.*

1. Notations and definitions

Let (Ω,Σ) be a measurable space, X and Z real Banach space, Z^* the dual of Z. We denote by $\langle z^*,z\rangle$ the dual pairing between $z^*\in Z^*, z\in Z$ and 2^X the set of the nonempty subsets of $X, cl(M)$ and $wcl(M)$, the respective closure and weak closure of $M\subset X$. Let $S_r=\{x\in X \mid \|x\|\le r, r>0\}$, ∂S be the boundary of S. The notations "$\to$" and "$\rightharpoonup$" mean the strong and weak convergence respectively, $WK(D)$ is the set of weakly compact subsets of $D\subset X$. A mapping $T:\Omega\to 2^X$ is said to be measurable (weakly measurable) if for each closed measurable (weakly closed) subset $C\subset X$, the set $T^{-1}(C)=\{\omega\in\Omega \mid T(\omega)\cap C=\phi\}\in\Sigma$. A mapping $\xi:\Omega\to X$ is called measurable (weakly measurable) selector of a measurable (weakly measurable) mapping T if ξ is measurable and $\xi(\omega)\in T(\omega), \forall\omega\in\Omega$. A mapping $F:X\to X^*$ is said to be monotone if $\langle Fx-Fy, x-y\rangle\ge 0, \forall x,y\in X$. A mapping $K:X\to X$ is said to be J-monotone if $\langle J(x-y), Kx-Ky\rangle\ge 0, \forall x,y\in X$. Where mapping $J:X\to X^*$ is dual mapping, that is $\langle Jx,x\rangle=\|x\|^2, \|Jx\|=\|x\|, \forall x\in X$. A mapping $B:X\to Z$ is said to be weakly continuous if $\{x_n\}\subset X, x_n\rightharpoonup x$ then $Bx_n\rightharpoonup Bx$, completely continuous if $x_n\rightharpoonup x$ then $Bx_n\rightharpoonup Bx$, hemicontinuous if the mapping: $t\in[0,1], t\mapsto\langle B(tx+(1-t)y), z\rangle$ is continuous for all $x,y,z\in X$. A mapping $A:\Omega\times X\to Z$ is called a random mapping if for each fixed $x\in X$,

Nguyen Manh Hung, Nguyen Xuan Thuan
Faculty of Natural Sciences, Hong Duc University
Email: Nguyenmanhhung@hdu.edu.vn (✉)

the mapping $A(.,x):\Omega\to Z$ is measurable. A random mapping A is said to be continuous (weakly continuous, monotone,...) if for each $\omega\in\Omega$, the mapping $A(\omega,.):X\to Z$ has respective property. We use also $A(\omega)x$ for $A(\omega,x)$. We denote by $\mathcal{M}(\Omega,X)$ the set of measurable mappings $\xi:\Omega\to X$ such that $\sup\{\|\xi(\omega)\| \mid \omega\in\Omega\}<+\infty$.

Definition 1.1. (def. 2.1 in [6]). *Let X,Z be Banach spaces, Z^* the dual space of Z, $H:X\to Z^*$ a mapping satisfying $H(0)=0, Hx\neq 0, \forall x\neq 0$. A mapping $A:X\to Z$ is said to be H-monotone if $\langle H(x-y), Ax-Ay\rangle\geq 0, \forall x,y\in X$.*

Theorem 1.1. (theorem 2.3 in [6]) *Let X,Z be finite dimensional Banach spaces, $H:X\to Z^*$ a mapping satisfying $H(0)=0, H(x)\neq 0, \forall x\neq 0, A:X\to Z$ a continuous random mapping. Assume, moreover, there exists $r=constant>0$ such that for each $\omega\in\Omega, \langle Hx, A(\omega)x\rangle\geq 0, \forall x\in\partial S_r$. Then there exists $\xi\in\mathcal{M}(\Omega,S_r)$ such that $A(\omega)\xi(\omega)=0, \forall\omega\in\Omega$.*

2. Semi H-monotone mappings

Let X,Z be real Banach spaces. Consider the mappings $H:X\to Z^*, A:\Omega\times X\to Z$. Let $\{X_n\},\{Z_n\}$ be inereasing sequences of finite dimensional subspaces of X and Z respectively, $\dim X_n=\dim Z_n$, and $P_n:X\to X_n, Q_n:Z\to Z_n, Q_n^*:Z^*\to Z_n^*$ linear projectors such that $P_nx\to x, Q_nz\to z$. Set $A_n=Q_nA|x_n, H_n=Q_nH|x_n$.

Definition 2.2. (def. 3.1 in [6]). *Let X,Z be Banach spaces, Z^* the dual space of Z, $H:X\to Z^*$ a mapping satisfying $H(0)=0, H(x)\neq 0, \forall x\neq 0$. A mapping $A:X\to Z$ is said to be semi H-monotone if there exists a mapping $S:X\times X\to Z$ such that*

(i) $Ax=S(x,x), \ \forall x\in X$,

(ii) for each fixed $y\in Y$, the mapping S(., y) is H-monotone and hemicontinuous,

(iii) for each fixed $x\in X$, the mapping S(x, .) is completely continuous.

Theorem 2.2. *Let D be nonempty, convex, closed subset of a separable reflexive Banach space X, Z a separable reflexive Banach space, $H:X\to Z^*$ a weakly continuous mapping satisfying $H(0)=0, Hx\neq 0, \forall x\neq 0$ and for each $t>0, H(tx)=tHx$, $A:\Omega\times X\to Z$ a semi H-monotone random mapping. Supose, moreover, $Q_n^*Hx=Hx, \forall x\in X_n$ and for each finite dimensional subspace E of X, in $D_E=D\cap E$ there exists $\xi\in\mathcal{M}(\Omega,S)$ such that*

$$\langle H(\xi(\omega)-y), A(\omega)\xi(\omega)\rangle \le 0, \forall y \in D, \omega \in \Omega.$$

Proof. Let $D_n = D \cap X_n$. The sequence $\{D_n\}$ is increasing. Let us define mappings $H_n = Q_n H, A_n = Q_n A, H : X \to Z^*, A_n : \Omega \times D_n \to Z_n$. Obvionsly, $Q_n A$ is a continuous random mapping in D_n. For each $\omega \in \Omega$, we have

$$\langle Hx, Q_n A(\omega)x\rangle = \langle Q_n^* Hx, A(\omega)x\rangle = \langle Hx, A(\omega)x\rangle, \forall x \in \partial S_r.$$

The mapping $Q_n A$ satisfies all conditions of Theorem 1.1. So there exists $\xi \in \mathcal{M}(\Omega, S)$ such that $Q_n A(\omega)\xi(\omega) = 0, \forall \omega \in \Omega$. By the reflexivity of X the ball S is weakly compact. Let us consider mappings $B_n, B : \Omega \to WK(S)$ as follows:

$$B_n(\omega) = wcl\{\xi_n(\omega)\}, B(\omega) = \bigcap_{n=1}^{\infty} B_n(\omega).$$

As in the proof from [[9], p, 135] it is clear that B is weakly measurable and B has a measurable selector $\xi : \Omega \to S, \xi(\omega) \in B(\omega), \forall \omega \in \Omega$. Consequently, for each $\omega \in \Omega$, the sequence $\{\xi_n(\omega)\}$ has a subsequence denoted by $\{\xi_k(\omega)\}$ (for the simplicity of notations) weakly converging to $\xi(\omega)$. Moreover, for each $x \in S$ that is $x \in M_m$ for some m, and by the sequence D_n is increasing, obviously $x \in D_k, \forall k \ge m$. The semi H-monotonicity of the mapping A provides us a mapping $S : D \times D \to Z, A(\omega)x = S(\omega, x, x), \forall x \in D.$

Since the mapping $x \mapsto S(\omega, x, y)$ is H-monotone, we obtain

$$\langle H(\xi_k(\omega)-x), A(\omega)\xi_k(\omega) - S(\omega, x, \xi_k(\omega))\rangle \ge 0 \tag{2.1}$$

But $\langle H(\xi_k(\omega)-x), A(\omega)\xi_k(\omega)\rangle = \langle H(\xi_k(\omega)-x), Q_k A(\omega)\xi_k(\omega)\rangle = 0$

It follows from inequality (2.1) that

$$\langle H(\xi_k(\omega)-x), S(\omega, x, \xi_k(\omega))\rangle \le 0 \tag{2.2}$$

By $H(\xi_k(\omega)-x) \rightharpoonup H(\xi(\omega)-x)$ and $S(\omega, x, \xi_k(\omega)) \to S(\omega, x, \xi(\omega))$ as $\xi_k(\omega) \to \xi(\omega)$ from inequality (2.2) we get

$$\langle H(\xi(\omega)-x), S(\omega, x, \xi(\omega))\rangle \le 0 \tag{2.3}$$

The hemicontinuity of the mapping $S(\omega, ., \xi(\omega))$ and inequality (2.3) yield

$$\langle H(\xi(\omega)-x), S(\omega, \xi(\omega), \xi(\omega))\rangle \le 0$$

Or $$\langle H(\xi(\omega)-y), A(\omega)\xi(\omega)\rangle \le 0, \forall y \in D, \omega \in \Omega.$$

Theorem 2.3. (H-monotone perturbation). *Let* D, X, Z, H, A *be as in Theorem . Let* $K:\Omega\times D\to Z$ *be a H-monotone, completely continuous random mapping. Assume, furthermore,* $Q_n^* Hx = Hx, \forall x \in X_n$ *and for each finite dimensional subspace* E *of* X, *in* $D_E = D\cap E$, *there exists a ball* S *such that*

$$\langle Hx, QA(\omega)x\rangle \ge 0 \text{ and } \langle Hx, K(\omega)x\rangle \ge 0, \forall y \in D, \omega\in\Omega.$$

Then there exists $\xi \in \mathcal{M}(\Omega, S)$ *such that*

$$\langle H(\xi(\omega)-y), A(\omega)\xi(\omega)+K(\omega)\xi(\omega)\rangle \le 0, \forall y\in D, \omega\in\Omega.$$

Proof. Let us use the notations $D_n, Q_n^* H, Q_n A, Q_n K:\Omega\times D_n \to Z_n$ as in the proof of Theorem 2.2. The mapping $Q_n A, Q_n K$ are continuous in D_n. So they satisfy all conditions in Theorem 1.1. Consequently there exists $\xi\in\mathcal{M}(\Omega, S)$ such that $Q_n A(\omega)\xi(\omega)=0, Q_n K(\omega)\xi(\omega)=0.$ Let us use the mappings B_n, B in the proof of Theorem 2.2. It is clear that B is weakly measurable and B possesses a measurable selector ξ. Hence the sequence $\{\xi_n(\omega)\}$ weakly converging to $\{\xi(\omega)\}$. The semi H-monotonicity of K yield $\langle H(\xi_k(\omega)-x), S(\omega,x,)\xi_k(\omega)+K(\omega)\xi_k(\omega)\rangle \le 0$

whence $$\langle H(\xi(\omega)-x), S(\omega,x,)\xi(\omega)+K(\omega)\xi(\omega)\rangle \le 0 \tag{2.4}$$

or $$\langle H(\xi(\omega)-y), A(\omega)\xi(\omega)+K(\omega)\xi(\omega)\rangle \le 0, \forall y\in D, \omega\in\Omega.$$

3. Weakly semi H-monotone mappings

Definition 3.3. (def. 4.1 in [6]). *Let* X, Z, Z^* *be as in Definition 1.1. A mapping* $A: X\to Z$ *is said to be weakly semi H-monotone if there exists a mapping* $R: X\times X\to Z$ *$R: X\times X\rightarrow Z$ such that*

(i) $Ax = R(x,x), \forall x\in X,$

(ii) for each fixed $y\in X$, *the mapping* $R(.,y)$ *is H-monotone and hemicontinuous.*

(iii) for each fixed $x\in X$, *the mapping* $R(x,.)$ *is weakly continuous.*

Obvionsly the semi H-monotonicity implies the weak semi H-monotonicity and in finitely dimensional space in which those concepts coinside.

Theorem 3.4. *Let* D, X, Z *be as in Theorem 2.2,* $H: X\to Z^*$ *be a completely continuous mapping*

$H(0)=0, Hx\ne 0, \forall x\ne 0$ *and for each* $t>0, H(tx)=tHx, A:\Omega\times D\to Z$ *a weakly semi H-monotone random mapping.*

Suppose, furthermore, $Q_n^* Hx = Hx, \forall x \in X_n$ *and for each finite dimensional subspaces* E *of* X, *in* $D_E = D \cap E$ *there exists a ball* S *such that* $\langle Hx, A(\omega)x \rangle \geq 0, \forall x \in \partial S$. *Then there exists* $\xi \in \mathcal{M}(\Omega, S)$ *such that* $\langle H(\xi(\omega) - y), A(\omega)\xi(\omega) \rangle \leq 0, \forall y \in D, \omega \in \Omega.$.

Proof. Let us use the notations D_n, A_n, H_n in the proof Theorem 2.2. The mapping $Q_n A$ is continuous in D_n. Moreover $\langle H_n x, Q_n A(\omega)x \rangle = \langle Hx, A(\omega)x \rangle \geq 0, \forall x \in \partial D$.

Hence the random mapping $Q_n A$ satisfies all conditions of Theorem 1.1. So there exists $\xi \in \mathcal{M}(\Omega, S)$ such that $Q_n^* A(\omega)\xi_n(\omega) = 0$.

By using the mapping $\omega \mapsto B(\omega) = \bigcap_{n=1}^{\infty} B_n(\omega)$ as in the proof of Theorem 2.2, it is clear that B has a measurable selector $\xi, \xi(\omega) \in B(\omega), \forall \omega \in \Omega$. Consequently for each $\omega \in \Omega$, the sequence $\{\xi_n(\omega)\}$ provides us a subsequence, say $\{\xi_n(\omega)\}$ weakly converging to $\{\xi(\omega)\}$ and for each $x \in D$, we see $x \in D_k, \forall k \geq m$ for some m. By the H-monotonicity of the mapping $R(\omega, ., y)$, where $R(\omega, x, x) = A(\omega)x$, we obtain $\langle H(\xi_k(\omega) - x), A(\omega)\xi_k(\omega) - R(\omega, x, \xi_k(\omega)) \rangle \geq 0,$

which implies $\langle H(\xi_k(\omega) - x), R(\omega, x, \xi_k(\omega)) \rangle \leq 0,$ (3.5)

But $H(\xi_k(\omega) - x) \to H(\xi(\omega) - x)$, $R(\omega, x, \xi_k(\omega)) \rightharpoonup R(\omega, x, \xi(\omega))$ as $\xi_k(\omega) \rightharpoonup \xi(\omega)$. Therefore from inequality (3.5) it follows that

$$\langle H(\xi(\omega) - x), R(\omega, x, \xi(\omega)) \rangle \leq 0,$$

The hemicontinuity of $R(\omega, ., \xi(\omega))$ and inequality (3.6) yield

$$\langle H(\xi(\omega) - y), A(\omega)\xi(\omega) \rangle \leq 0, \forall y \in D, \omega \in \Omega. \quad (3.6)$$

It is not difficult to prove.

Theorem 3.5. *Let* D, X, Z, H, A *be as in Theorem 3.4,* $K : \Omega \times D \to Z$ *be a H-monotone, weakly continuous random mapping. Assume, moreover,* $Q_n^* Hx = Hx, \forall x \in X_n$ *and for each finite dimensional subspace* E *of* X, *in* $D_E = D \cap E$, *there exists a ball* S *such that* $\langle Hx, A(\omega)x \rangle \geq 0, \langle Hx, K(\omega)x \rangle \geq 0, \forall x \in \partial S$. *Then there exists* $\xi \in \mathcal{M}(\Omega, S)$ *such that*

$$\langle H(\xi(\omega) - y), A(\omega)\xi(\omega) + K(\omega)\xi(\omega) \rangle \leq 0, \forall y \in D, \omega \in \Omega.$$

4. Conclusion

The theorems 3.4 and 3.5 solve some random variational inequalities for semi H-monotone and weakly semi H-monotone mappings. These are good results in solving random variational inequalities for semi H-monotone and weakly semi H-monotone mappings.

References

[1] Nguyen Minh Chuong and Nguyen Xuan Thuan (2001), *Random fixed point theorems for multivalued nonlinear mappings*, Random Oper and Stoch. Equa. 9(3) 345-355.

[2] Nguyen Minh Chuong and Nguyen Xuan Thuan (2001), *Nonlinear variational inequalities for random weakly semi-monotone operators*, Random Oper and Stoch. Equa. 9(4),1-10.

[3] Nguyen Minh Chuong and Nguyen Xuan Thuan (2002), *Surjectivity of random semiregular maximal monotone mappings*, Random Oper and Stoch. Equa. 10(1), 135-144.

[4] Nguyen Minh Chuong and Nguyen Xuan Thuan (2002), *Random equations for weakly semimonotone operators of type (S) and J-semimonotone operators of type (J-S),* Random Oper and Stoch. Equa. 10(2),345-354.

[5] Nguyen Minh Chuong and Nguyen Xuan Thuan (2002), *Some new fixed point theorems for nonlinear set-valued mappings*, *Sumited to Math. Ann.*

[6] Nguyen Minh Chuong and Nguyen Xuan Thuan (2002), *Random equations for semi H-monotone operators and weakly semi Semi H-monotone operators*, Random Oper and Stoch. Equa. 10(4), 1-8.

[7] Nguyen Minh Chuong and Nguyen Xuan Thuan (2006), *Random nonlinear variational inequalities for mappings of monotone type in Banach spaces*, Stoch Analysis and Appl. 24(3), 489-499.

[8] C. J. Himmelberg (1978), *Nonlinear random equations with monotone operators in Banach spaces*, Math. Anal. 236 133-146.

[9] S. Itoh (1975), *Measurable relations*, Funct. Math. 87, 53-72.