

IDENTICAL SYNCHRONIZATION IN COMPLETE NETWORK OF ORDINARY DIFFERENTIAL EQUATIONS OF FITZHUGH-NAGUMO

Phan Van Long Em

Received: 10 January 2019/ Accepted: 11 June 2019/ Published: June 2019
 ©Hong Duc University (HDU) and Hong Duc University Journal of Science

Abstract: *Synchronization is a ubiquitous feature in many natural systems and nonlinear science. In this paper, the synchronization in complete network consisting of n nodes is studied. Each node is connected to all other nodes by linear coupling and it is represented by a system of ordinary differential equations of FitzHugh-Nagumo type which is obtained by simplifying the famous Hodgkin-Huxley model. From this complete network, the sufficient condition under the coupling strength is sought such that the synchronization phenomenon occurs. The result shows that the networks with bigger in-degrees of the nodes synchronize more easily. The paper also shows this theoretical result numerically and see that there is a compromise.*

Keywords: *Coupling strength, complete network, FitzHugh-Nagumo model, synchronization.*

1. Introduction

The FitzHugh-Nagumo model was introduced as a dimensional reduction of the well-known Hodgkin-Huxley model [6], [7], [9], [10], [11], [12]. It is more analytically tractable and it maintains a certain biophysical meaning. The model is constituted by two equations in two variables u and v . The first one is the fast variable called excitatory: it represents the transmembrane voltage. The second variable is the slow recovery variable: it describes the time dependence of several physical quantities, such as the electrical conductance of the ion currents across the membrane. The FitzHugh-Nagumo equations (FHN), using the notation in [1], [2], [3], are given by

$$\begin{cases} \varepsilon \frac{du}{dt} = u_t = f(u) - v \\ \frac{dv}{dt} = v_t = au - bv + c \end{cases} \quad (1)$$

where a, b and c are constants (a and b are strictly positive), $0 < \varepsilon \ll 1$ and $f(u) = -u^3 + 3u$, $t \in \mathbb{R}^+$ presents the time. The system (1) is the model of a neuron, then we consider a network of n coupled systems (1) based on FHN type as follows:

$$\begin{cases} \varepsilon u_{it} = \varepsilon \frac{du_i}{dt} = f(u_i) - v_i - h(u_i, v_i) \\ v_{it} = \frac{dv_i}{dt} = au_i - bv_i + c \end{cases} \quad i, j = 1, \dots, n, i \neq j, \quad (2)$$

where $(u_i, v_i), i = 1, 2, \dots, n$ is defined by (1).

Phan Van Long Em
 Faculty of Education, An Giang University
 Email: pvlem@agu.edu.vn (✉)

The function h is the coupling function that determines the type of connection between neurons i and j . Connections between neurons are essentially of two types: chemical which is much more abundant, and electrical. In the case where the connections are made by electrical synapse, the coupling is linear and given by the function

$$h(u_i, v_i) = g_{syn} \sum_{j=1}^n c_{ij} (u_i - u_j), \quad i = 1, 2, \dots, n. \quad (3)$$

The parameter g_{syn} represents the coupling strength. The coefficients c_{ij} are the elements of the connectivity matrix $C_n = (c_{ij})_{n \times n}$, defined by

$$\begin{cases} c_{ij} = 1 & \text{if } i \text{ and } j \text{ are coupled} \\ c_{ij} = 0 & \text{if } i \text{ and } j \text{ are not coupled} \end{cases} \quad i, j = 1, 2, \dots, n, i \neq j.$$

A neural network describes a population of physically interconnected nerve cells. Communication between cells is mainly due to electrochemical processes. This work focuses on analyzing the behavior of a set of neurons connected together with a given topology by electrical. Thus, the complex system based on a network of interactions between neurons is considered in which each network node is modeled by a ODE of FHN type.

The article is divided as follows: in section 1, there is the introduction; in section 2, the definition of synchronization is introduced, especially identical synchronization. And this paper looks forward to finding out the sufficient conditions such that there is a such type of synchronization in network; in section 3, this research focuses on the minimal value of coupling strength such that the synchronization in complete network occurs and numerical experiments to give an insight into the influence of the number of neurons on the minimal coupling strength needed to obtain synchronization in network. The numerical simulations show that when the number of nodes in graph grows, the network becomes easier to synchronize. And the conclusion is left to the last section.

2. Identical synchronization of a complete network of n systems of ordinary differential equations on FitzHugh-Nagumo type

Synchronization is a ubiquitous feature in many natural systems and nonlinear science. The word "*synchronization*" is from the Greek, *syn* (common) and *chronos* (time), and means having the same behavior at the same time. Therefore the synchronization of two dynamical systems usually means that one system copies the movement of the other. When the behavior of many systems is synchronized, these systems are called *synchronous*. It is known that a phenomenon of synchronization may appear in a network of many weak coupled oscillators [4], [5], [13]. A broad variety of applications have emerged, for example to increase the power of lasers, to synchronize the output of electric circuits, to control oscillations in chemical reactions or to encode electronic messages for secure communications. Here are some synchronization regimes:

Identical (or complete) synchronization, which is defined as the coincidence of states of interacting systems.

Generalized synchronization, which extends the identical synchronization phenomenon and implies the presence of some functional relation between two coupled systems; if this relationship is the identity, we recover the identical synchronization.

Phase synchronization, which means driving of phases of chaotic oscillators, whereas their amplitudes remain uncorrelated. Lag synchronization, which appears as a coincidence of shifted-in-time states of two systems.

In this article, the identical synchronization is investigated in a complete network which means that each node connects to all other nodes of network [2], [3]. For example, Figure 1 shows the complete graphs from 2 to 10 nodes and complete graphs of 40 nodes. In this study, each node represents a neuron modeled by a system of ordinary differential equations on FHN type and each edge represents a synaptical connection modeled by a coupling function.

Definition 1. Let $S_i = (u_i, v_i)$, $i = 1, 2, \dots, n$ and $S = (S_1, S_2, \dots, S_n)$ be a network. We say that S synchronizes identically if $\lim_{t \rightarrow +\infty} |u_j - u_i| = 0$ and $\lim_{t \rightarrow +\infty} |v_j - v_i| = 0$, for all $i, j = 1, 2, \dots, n$.

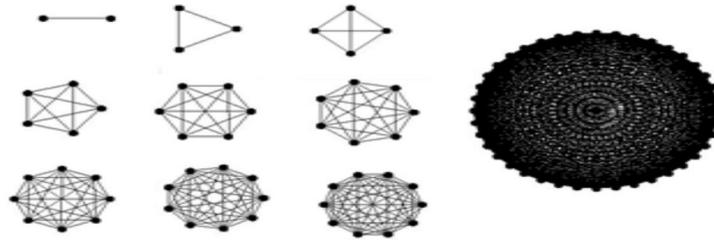


Figure1. Complete graphs from 2 to 10 nodes and complete graphs of 40 nodes. In our study, each node represents a neuron modeled by a system of ordinary differential equations of FHN type and each edge represents a synaptical connection modeled by a coupling function.

A system of n "neurons" (1) bi-directionally coupled by the electrical synapses, based on FHN, is given as follows:

$$\begin{cases} \varepsilon u_{it} = f(u_i) - v_i - g_n \sum_{j=1, j \neq i}^n (u_i - u_j) \\ v_{it} = au_i - bv_i + c \end{cases} \quad i = 1, 2, \dots, n, \quad (4)$$

where g_n is the coupling strength between u_i and u_j .

Theorem 1. Suppose that $g_n > \frac{M}{n}$

where $M = \sup_{u \in B, x \in \mathbb{R}} \sum_{k=1}^3 \frac{f^{(k)}(u)}{k!} x^{k-1}$, B is a compact interval including u and

$f^{(k)}(u)$ is the k -th derivative of f with respect to u . Then the network (4) synchronizes in the sense of Definition 1.

Proof. Let $\Phi(t) = \frac{1}{2} \left[\sum_{i=2}^n \left(a\varepsilon(u_i - u_1)^2 + (v_i - v_1)^2 \right) \right]$.

By deriving the function $\Phi(t)$, there is the following:

$$\frac{d\Phi(t)}{dt} = \sum_{i=2}^n \left[a\varepsilon(u_i - u_1)(u_{it} - u_{1t}) + (v_i - v_1)(v_{it} - v_{1t}) \right]$$

$$\begin{aligned}
 &= \sum_{i=2}^n \left[a(u_i - u_1) \left(f(u_i) - v_i - g_n \sum_{k=1, k \neq i}^n (u_i - u_k) - f(u_1) + v_1 + g_n \sum_{l=2}^n (u_1 - u_l) \right) \right. \\
 &\quad \left. + (v_i - v_1) (a(u_i - u_1) - b(v_i - v_1)) \right] \\
 &\leq \sum_{i=2}^n \left[a(u_i - u_1) (f(u_i) - f(u_1) - ng_n(u_i - u_1)) - b(v_i - v_1)^2 \right] \\
 &\leq \sum_{i=2}^n \left[a(u_i - u_1)^2 \left(f'(u_1) - ng_n + \sum_{k=2}^3 \frac{f^{(k)}(u_1)}{k!} (u_i - u_1)^{k-1} \right) - b(v_i - v_1)^2 \right] \\
 &\leq \sum_{i=2}^n \left[a(u_i - u_1)^2 (M - ng_n) - b(v_i - v_1)^2 \right].
 \end{aligned}$$

If $g_n > \frac{M}{n}$, then $\frac{d\Phi(t)}{dt} \leq -\beta\Phi(t) \Rightarrow \Phi(t) \leq \Phi(0)e^{-\beta t}$,

where $\beta = \min\left(2\frac{ng_n - M}{\varepsilon}, 2b\right)$. Thus, the synchronization occurs if the coupling strength verifies $g_n > \frac{M}{n}$. In the case where f is cubic, there is the following corollary.

Corollary 1. Suppose that f is a cubic function, $f(u) = m_3u^3 + m_2u^2 + m_1u + m_0$,

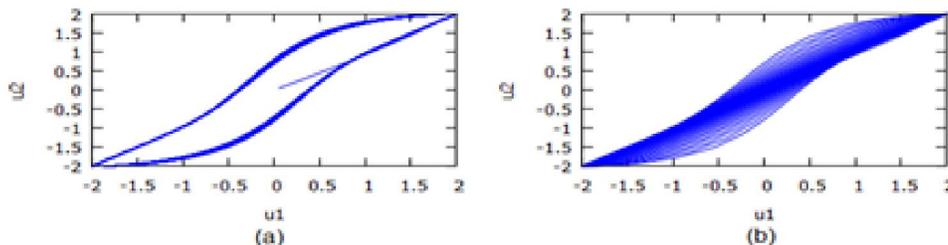
where m_3, m_2, m_1, m_0 are constants with $m_3 < 0$ and if $g_n > \frac{1}{n} \left(m_1 - \frac{m_2^2}{3m_3} \right)$,

the network $S = ((u_1, v_1), (u_2, v_2), \dots, (u_n, v_n))$ synchronizes in the sense of Definition 1.

3. Numerical simulations

In the following part, the paper shows the numerical results obtained by integrating the system (4) where $n = 2, f(u) = -u^3 + 3u$, and with the following parameter values: $a = 1, b = 0.001, c = 0, \varepsilon = 0.1$. The integration of system was realized by using C++.

Figure 2 illustrates the phenomenon of synchronization. The simulations show that the system synchronizes from the value $g_2 = 1.4$. The figures (a), (b), (c), (d) represent the phase portraits (u_1, u_2) corresponding to the different values of coupling strength. It is easy to see that the synchronization occurs in figure (d) for $g_2 = 1.4, u_1 \approx u_2$.



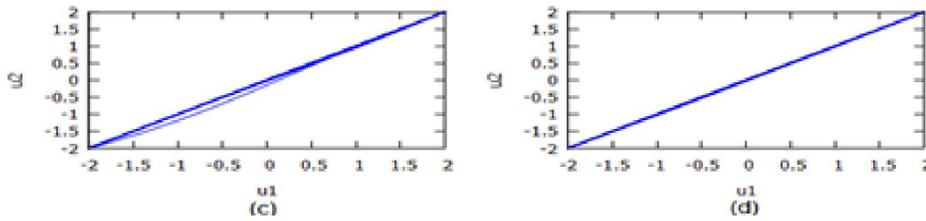


Figure 2. Synchronization of a complete network of two linearly coupled "neurons" with $f(u) = -u^3 + 3u$, $a = 1, b = 0.001, c = 0, \varepsilon = 0.1$. The synchronization occurs for $g_2 = 1.4$. Before synchronization, for $g_2 = 0.0001$, the figure (a) represents the temporal dynamic of u_2 with respect to u_1 ; the figure (b) represents the temporal dynamic of u_2 with respect to u_1 for $g_2 = 0.01$; the figure (c) represents the temporal dynamic of u_2 with respect to u_1 for $g_2 = 0.5$. For the value $g_2 = 1.4$, the synchronization of two "neurons" occurs: $u_1 \approx u_2$

The following research focuses on the minimal values of coupling strength g_n to observe a phenomenon of synchronization between n subsystems modeling the function of neuron.

From the above result, in the case of two linearly coupled neurons, for the coupling strength which is bigger than or equal to $g_2 = 1.4$, these neurons have a synchronous behavior (figure 2). By doing similarly for the complete networks of n linearly identical coupled neurons, the values of coupling strength are reported in table 1.

Table 1. The minimal coupling strength necessary to observe a phenomenon of synchronization of n linearly coupled neurons

n		2	3	4	5	6	7	8	9	10
g_n		1.4	0.933	0.7	0.56	0.467	0.4	0.35	0.311	0.28
n	11	12	13	14	15	16	17	18	19	20
g_n	0.255	0.233	0.215	0.2	0.187	0.175	0.165	0.156	0.147	0.14
n	21	22	23	24	25	26	27	28	29	30
g_n	0.133	0.127	0.122	0.117	0.112	0.108	0.104	0.1	0.097	0.093
n	31	32	33	34	35	36	37	38	39	40
g_n	0.09	0.088	0.085	0.082	0.08	0.079	0.076	0.074	0.072	0.07

Following these numerical experiments, it is easy to see that the coupling strength required to observe the synchronization of n neurons depends on the number of neurons.

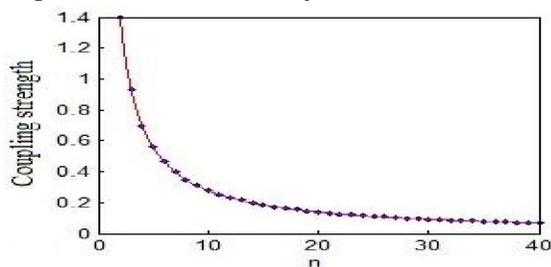


Figure 3. The evolution of the coupling strength g_n for which the synchronization of n neurons takes place according to the number n linearly coupled neurons in complete network, and it follows the law $g_n = \frac{2g_2}{n-1}$

Indeed, the points in figure 3 represent the coupling strength of synchronization according to the number of neurons in complete network, and the red curve represents the representative one: $g_n = \frac{2g_2}{n-1}$, where n is the number of neurons in network and g_2 is the

coupling strength that permits to have the synchronization of two neurons coupled. Thus, the coupling strength necessary to obtain the synchronization of n neurons follows this law.

4. Conclusion

The paper shows a phenomenon of synchronization in complete network of n coupled systems of ordinary differential equations on Fitzhugh-Nagumo type. From Theorem 1, there is the result $g_n > \frac{M}{n}$ which shows that g_n becomes smaller when n takes the big values.

Numerically, it is seen that the synchronization is stable when the coupling strength is exceeded to certain threshold and depends on the number of "neurons" in graphs. The bigger the number of "neurons" is, the easier we obtain the phenomenon of synchronization. Then, there is a compromise between the theoretical and numerical results. For future works, it is interesting to study about the identical synchronization in complete network coupled by chemical synapse.

References

- [1] Ambrosio, B. (2009), *Propagation d'ondes dans un milieu excitable: simulations numériques et approche analytique*, Thesis of University Pierre and Marie Curie-Paris 6.
- [2] Ambrosio, B., & Aziz-Alaoui, M. A. (2012), *Synchronization and control of coupled reaction-diffusion systems of the FitzHugh-Nagumo-type*, Computers and Mathematics with Applications 64, p. 934-943.
- [3] Ambrosio, B., & Aziz-Alaoui, M. A. (March 2013), *Synchronization and control of a network of coupled reaction-diffusion systems of generalized FitzHugh-Nagumo type*, ESAIM: Proceedings, Vol. 39, p. 15-24.
- [4] Aziz-Alaoui, M. A. (2006), *Synchronization of Chaos*, Encyclopedia of Mathematical Physics, Elsevier, Vol. 5, p. 213-226.
- [5] Corson, N. (2009), *Dynamique d'un modèle neuronal, synchronisation et complexité*, Thèse de l'Université du Havre. Corson N., Aziz-Alaoui M.A. (2009), *Complex emergent properties in synchronized neuronal oscillations*, in M.A.A. and C. Bertelle (eds.): From System Complexity to Emergent Properties, Springer, p. 243-259.
- [6] Ermentrout, G. B., & Terman, D.H. (2009), *Mathematical Foundations of Neurosciences*, Springer.
- [7] Hodgkin, A. L., & Huxley, A. F. (1952), *A quantitative description of membrane current and its application to conduction and excitation in nerve*, J. Physiol. 117, p. 500-544.
- [8] Izhikevich, E. M. (2005), *Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting*, The MIT Press, Cambridge, Massachusetts, London, England.
- [9] Izhikevich, E. M. (2007), *Dynamical Systems in Neuroscience*, The MIT Press.
- [10] Keener, J. P., & Sneyd, J. (2009), *Mathematical Physiology*, Springer.
- [11] Murray, J. D. (2010), *Mathematical Biology*, Springer.
- [12] Nagumo, J., Arimoto, S., & Yoshizawa, S. (1962), *An active pulse transmission line simulating nerve axon*, Proc. IRE. 50, p. 2061-2070.
- [13] Phan, V.L.E. (2018), *Identical synchronization in complete networks of reaction-diffusion equations of FitzHugh-Nagumo*, Journal of Science Ho Chi Minh City Open University, 8(2), p. 36-42.