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THE SEMILINEAR COUPLED SYSTEMS FOR THE EXTERNAL DAMPING MODELS WITH VARIABLE COEFFICIENTS

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Abstract. We present in this article some results on the global solvability with arbitrarily small data of the Cauchy problem for the following semilinear coupled system with a variable coefficient

$$\begin{cases} u_{tt} + a(x)(-\Delta)^\sigma u + u_t = F(|D|^\alpha v, v_t), \\ v_{tt} + a(x)(-\Delta)^\sigma v + v_t = G(|D|^\alpha u, u_t), \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \\ v(0,x) = v_0(x),\ v_t(0,x) = v_1(x). \end{cases}$$

The nonlinearities are of the form $(F,G) = (||D|^\alpha v|^p, ||D|^\alpha u|^q)$, or $(F,G) = (|v_t|^p, |u_t|^q)$, and the parameter σ satisfies $\sigma \in (0,1)$. We will show that the "critical exponents" p, q for the small data global solvability have a close relation to the established exponents of the corresponding semilinear problems for the external damping equations.
Keywords: external damping, coupled systems, global (in time) solvability, decay estimates, small data solutions.

1. Introduction

In [1] Nishihara and Wakasugi studied the Cauchy problem of the weakly coupled system of the damped wave equation

$$\begin{cases} u_{tt} - \Delta u + u_t = |v|^p,\ t \geq 0, x \in \mathbb{R}^n, \\ v_{tt} - \Delta v + v_t = |u|^q,\ t \geq 0, x \in \mathbb{R}^n, \\ (u, u_t, v, v_t)(0,x) = (u_0, u_1, v_0, v_1)(x),\ x \in \mathbb{R}^n \end{cases} \tag{1.1}$$

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They have proved that global existence of small data solutions to (1.1) holds in any space dimension $n \geq 1$, provided that

$$\alpha := \frac{\max\{p,q\}+1}{pq-1} < \frac{n}{2}. \tag{1.2}$$

It can be checked directly that condition (1.2) is equivalent to

$$\max\{p,q\}(\min\{p,q\}+1-p_F(n)) > p_F(n), \tag{1.3}$$

where $p_F(n) = 1 + 2/n$ is the Fujita critical exponent for the damped wave equation.

The aim of our current study is to obtain results analogous to that of [1] in the case of fractional damping models using the decay estimates for linear external damping models. Let us denote $p_E = p_E(\sigma,\alpha) := \dfrac{n+2\sigma}{n+\alpha}$. For the Cauchy problem

$$\begin{cases} u_{tt} + a(-\Delta)^\sigma u + u_t = ||D|^\alpha v|^p, \ t \geq 0, x \in \mathbb{R}^n, \\ v_{tt} + a(-\Delta)^\sigma v + v_t = ||D|^\alpha u|^q, \ t \geq 0, x \in \mathbb{R}^n, \\ (u,u_t,v,v_t)(0,x) = (u_0,u_1,v_0,v_1)(x), \ x \in \mathbb{R}^n, \end{cases} \tag{1.4}$$

where $0 < a = \text{const}$, $|D| := (-\Delta)^{\frac{1}{2}}$, we will show that the global existence of the small data solution holds if

$$\max\{p,q\}(\min\{p,q\}+1-p_E) > p_E \tag{1.5}$$

for all $n \geq 2$.

For the Cauchy problem with nonlinearities $(F,G) = (|v_t|^p, |u_t|^q)$

$$\begin{cases} u_{tt} + a(x)(-\Delta)^\sigma u + u_t = |v_t|^p, \ t \geq 0, x \in \mathbb{R}^n, \\ v_{tt} + a(x)(-\Delta)^\sigma v + v_t = |u_t|^q, \ t \geq 0, x \in \mathbb{R}^n, \\ (u,u_t,v,v_t)(0,x) = (u_0,u_1,v_0,v_1)(x), \ x \in \mathbb{R}^n, \end{cases} \tag{1.6}$$

we will prove the global small data solvability in the class H^s (for sufficiently large s) under the following condition

$$p, q > p_S, \tag{1.7}$$

where the exponent p_S is linked to the global solvability of the following Cauchy problem

$$u_{tt} + a(x)(-\Delta)^\sigma u + u_t = |u_t|^p, \ \ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x). \tag{1.8}$$

We recall that the general semi-linear structurally damping model is a family of following problems containing the fractional Laplacians with parameters δ and σ

$$\begin{aligned} u_{tt} + (-\Delta)^\sigma u + (-\Delta)^\delta u_t = F(u, u_t, |D|^\alpha u), \\ u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x), \end{aligned} \tag{1.9}$$

(see [2, 3, 4, 5] for the studied damping models).

In general, without any restriction on the parameter σ, we can define the fractional Laplacian by the Fourier transform

$$\mathcal{F}((-\Delta)^\sigma f(\xi)) = |\xi|^{2\sigma}\mathcal{F}(f)(\xi)$$

for all $\sigma > 0$, where $\mathcal{F}(f)$ denotes the Fourier transform of the function f with respect to the x variable.

For $\sigma \in (0,1)$ we can use the integral representation of the fractional Laplacian:

$$(-\Delta)^\sigma u(x) = c_{n,\sigma}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2\sigma}}dy,$$

for sufficiently smooth u, with a normalization positive constant $c_{n,\sigma} = \frac{2^{2\sigma}\sigma\Gamma(n/2+\sigma)}{\pi^{n/2}\Gamma(1-\sigma)}$ depending on n and σ.

For the external damping model, where $\delta = 0$, using the diffusion phenomenon and the Markov property of the semigroup generated by non-negative self-adjoint operators (see [6]), we have recently obtained in [7, 8, 9] the following decay estimates for the solution of the linear external damping problem. Let $a(x)$ be a continuous function satisfying

$$a_1 \le a(x) \le a_2, \text{ for all } x \in \mathbb{R}^n, \tag{1.10}$$

with positive constants a_1, a_2. Then the solution $v(t,x)$ of the linear Cauchy problem for external damping model

$$\begin{cases} v_{tt} + a(x)(-\Delta)^\sigma v + v_t = 0, \ t \ge 0, x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), \ v_t(0,x) = v_1(x), \ x \in \mathbb{R}^n, \end{cases} \tag{1.11}$$

and its derivatives satisfy the following $(L^1 \cap L^2) - L^2$ estimates

$$\|v(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4\sigma}}\|v_0\|_{L^1\cap L^2} + (1+t)^{-\frac{n}{4\sigma}}\|v_1\|_{L^1\cap H^{-\sigma}}, \tag{1.12}$$

$$\|v(t,\cdot)\|_{\dot{H}^\sigma} \lesssim (1+t)^{-\frac{n}{4\sigma}-\frac{1}{2}}\|v_0\|_{L^1\cap \dot{H}^\sigma} + (1+t)^{-\frac{n}{4\sigma}-\frac{1}{2}}\|v_1\|_{L^1\cap L^2}, \tag{1.13}$$

$$\|v_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4\sigma}-1}\|v_0\|_{L^1\cap \dot{H}^\sigma} + (1+t)^{-\frac{n}{4\sigma}-1}\|v_1\|_{L^1\cap L^2}, \tag{1.14}$$

$$\|v(t,\cdot)\|_{\dot{H}^k} \lesssim (1+t)^{-\frac{n}{4\sigma}-\frac{k}{2\sigma}}\|v_0\|_{L^1\cap \dot{H}^k} + (1+t)^{-\frac{n}{4\sigma}-\frac{k}{2\sigma}}\|v_1\|_{L^1\cap \dot{H}^{k-\sigma}}, \tag{1.15}$$

$$\|v_t(t,\cdot)\|_{\dot{H}^k} \lesssim (1+t)^{-\frac{n}{4\sigma}-1-\frac{k}{2\sigma}}\|v_0\|_{L^1\cap \dot{H}^{k+\sigma}} + (1+t)^{-\frac{n}{4\sigma}-1-\frac{k}{2\sigma}}\|v_1\|_{L^1\cap \dot{H}^k}. \tag{1.16}$$

In the case when $a = \text{const} > 0$ we have also the following additional $L^2 - L^2$ estimates

$$\begin{aligned} \|v(t,.)\|_{L^2} &\lesssim \|v_0\|_{L^2} + (1+t)\|v_1\|_{L^2}, \\ \|v_t(t,.)\|_{L^2} &\lesssim (1+t)^{-1}\|v_0\|_{H^\sigma} + \|v_1\|_{L^2}, \\ \||D|^\sigma v(t,.)\|_{L^2} &\lesssim (1+t)^{-\frac{1}{2}}\|v_0\|_{H^\sigma} + (1+t)^{-\frac{1}{2}}\|v_1\|_{L^2}. \end{aligned}$$

The above linear estimates are essential in the study of the global solvability of various nonlinear Cauchy problems for damping models with arbitrarily small initial data.

2. Main results

In this section we will state our main results. First we consider the case when the nonlinearities have the form $|u_t|^p$ and $|v_t|^q$ with $p, q > 2$. We introduce the exponent

$$p_S := s + 1 - \sigma$$

for $s > \sigma + \frac{n}{2}$.

Let us consider the Cauchy problem for the following system of the nonlinear external damped model

$$\begin{cases} u_{tt} + a(x)(-\Delta)^\sigma u + u_t = |v_t|^p, \ t \geq 0, x \in \mathbb{R}^n, \\ v_{tt} + a(x)(-\Delta)^\sigma v + v_t = |u_t|^q, \ t \geq 0, x \in \mathbb{R}^n, \\ (u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x), \ x \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

with σ satisfies $\sigma \in (0, 1)$ and $n > 4\sigma$. The coefficient $a(x)$ is supposed to be a continuous function satisfying condition (1.10). The data (u_0, u_1, v_0, v_1) are assumed to belong to the function space $\Big((L^1 \cap H^s) \times (L^1 \cap H^{s-\sigma})\Big)^2$ with $s > \sigma + \frac{n}{2}$. Then, for any $p, q > p_S$, there exists a uniquely determined global (in time) small data energy solution from $\Big(C\big([0,\infty), H^s\big) \cap C^1\big([0,\infty), H^{s-\sigma}\big)\Big)^2$. Next, let us consider the case $(F, G) = (||D|^\alpha v|^p, ||D|^\alpha u|^q)$, where $\alpha \in [0, \sigma)$ and $0 < a =$ const. For $p_E = \frac{n+2\sigma}{n+\alpha}$, we put

$$\gamma(p) := \frac{(n+a)(p_E - p)_+}{2\sigma}, \tag{2.2}$$

$$\gamma(q) := \frac{(n+a)(p_E - q)_+}{2\sigma} \tag{2.3}$$

- the parameters that represent the possible loss of decay with respect to the corresponding linear estimates for u and v. We use here the notation $(k)_+ := \max\{k, 0\}$.

In the following study, we note that p, q can be strictly smaller than p_E, which means that $\gamma(p)$ or $\gamma(q)$ can be positive. Then we have the following global solvability result for problem (1.4). Let $p, q \geq p_E - 1$ and $p, q \in [2, \frac{n}{n+2(\alpha-\sigma)})$ such that the condition (1.5) holds for $\sigma \in (0, 1)$. Moreover, suppose that the data (u_0, u_1, v_0, v_1) are chosen from the space $A := \big((L^1 \cap H^\sigma) \times (L^1 \cap L^2)\big)^2$. Then there exists $\varepsilon > 0$ such that for all $n \geq 2$ and for any small data (u_0, u_1, v_0, v_1) with

$$\mathcal{A} := \|(u_0, v_0)\|_{L^1 \cap H^\sigma} + \|(u_1, v_1)\|_{L^1 \cap L^2} < \varepsilon \tag{2.4}$$

there exists the (global) solution

$$(u(t,x), v(t,x)) \in \big(\mathcal{C}([0,\infty), H^\sigma) \cap \mathcal{C}^1([0,\infty), L^2)\big)^2$$

to the Cauchy problem (1.4) with $\alpha \in [0, \sigma)$. Moreover, the following estimates are satisfied

$$\|u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4\sigma}+\gamma(p)}\mathcal{A}, \tag{2.5}$$

$$\|v(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4\sigma}+\gamma(q)}\mathcal{A}, \tag{2.6}$$

$$\|u(t,\cdot)\|_{\dot{H}^\sigma} \lesssim (1+t)^{-\frac{n}{4\sigma}-\frac{1}{2}+\gamma(p)}\mathcal{A}, \tag{2.7}$$

$$\|v(t,\cdot)\|_{\dot{H}^\sigma} \lesssim (1+t)^{-\frac{n}{4\sigma}-\frac{1}{2}+\gamma(q)}\mathcal{A}, \tag{2.8}$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4\sigma}-1+\gamma(p)}\mathcal{A}, \tag{2.9}$$

$$\|v_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4\sigma}-1+\gamma(q)}\mathcal{A}, \tag{2.10}$$

in the case when $p, q \neq p_E$. If $p = p_E$ or $q = q_E$ then the corresponding loss of decay $(1+t)^{\gamma(p)}$ or, respectively, $(1+t)^{\gamma(q)}$, is replaced by $\log(e+t)$.

3. The proof of main results

Proof. (of Theorem 2).

Let us denote

$$\mathcal{A} := \|(u_0, v_0)\|_{L^1\cap H^s} + \|(u_1, v_1)\|_{L^1\cap H^{s-\sigma}}. \tag{3.1}$$

We introduce the data space $A := \Big((L^2\cap H^s)\times(L^1\cap H^{s-\sigma})\Big)^2$ and the solution space $X(t) = \Big(C([0,t], H^s)\cap C^1([0,t], H^{s-\sigma})\Big)^2$ with the norm

$$\|(u,v)\|_{X(t)} = \sup_{0\le\tau\le t}\Big(W(u) + W(v)\Big), \tag{3.2}$$

where

$$\begin{aligned} W(u) = \big(&f_0(\tau)^{-1}\|u(\tau,\cdot)\|_{L^2} + f_s(\tau)^{-1}\||D|^s u(\tau,\cdot)\|_{L^2} + \\ &g(\tau)^{-1}\|u_t(\tau,\cdot)\|_{L^2} + g_{s-\sigma}(\tau)^{-1}\||D|^{s-\sigma}u_t(\tau,\cdot)\|_{L^2}\big), \end{aligned} \tag{3.3}$$

and similarly for v.

From the estimates of Proposition 1., we can choose

$$f_0(\tau) := (1+\tau)^{-\frac{n}{4\sigma}},\ \ f_s(\tau) := (1+\tau)^{-\frac{n}{4\sigma}-\frac{s}{2\sigma}},\ \ g(\tau) := (1+\tau)^{-\frac{n}{4\sigma}-1},\ g_{s-\sigma}(\tau) := (1+\tau)^{-\frac{n}{4\sigma}-\frac{s+\sigma}{2\sigma}}$$

We define the integral operator $N:\ (u,v)\in X(t)\mapsto N[u,v]\in X(t)$ by

$$N[u,v] = L + (Fv, Gu),$$

where

$$\begin{aligned} L &= E_0(t,x) *_x (u_0,v_0)(x) + E_1(t,x) *_x (u_1,v_1)(x), \\ F(v) &= \int_0^t E_1(t-\tau,x) *_x \big|v_t(\tau,x)\big|^p d\tau, \\ G(u) &= \int_0^t E_1(t-\tau,x) *_x \big|u_t(\tau,x)\big|^q d\tau. \end{aligned}$$

The local and global existence of small data solution to (2.1) will follow by the standard contraction argument if we can show the estimates

$$\|N[u,v]\|_{X(t)} \lesssim \mathcal{A} + \|(u,v)\|^p_{X(t)} + \|(u,v)\|^q_{X(t)}, \tag{3.4}$$

and the Lipschitz property

$$\begin{aligned} &\|N[u,v] - N[\tilde{u},\tilde{v}]\|_{X(t)} \lesssim \\ &\big\|(u,v)-(\tilde{u},\tilde{v})\big\|_{X(t)}\Big(\big\|(u,v)\big\|^{p-1}_{X(t)} + \big\|(\tilde{u},\tilde{v})\big\|^{p-1}_{X(t)} + \big\|(u,v)\big\|^{q-1}_{X(t)} + \big\|(\tilde{u},\tilde{v})\big\|^{q-1}_{X(t)}\Big), \end{aligned} \tag{3.5}$$

for the exponents p,q satisfying given conditions.

By the linear estimates (1.12)-(1.14) in Prop. 1., it follows immediately that

$$\|L\|_{X(t)} \lesssim \mathcal{A}.$$

Next, we will estimate the L^2-norm of Fv itself. To do that we apply the $(L^1 \cap L^2) - L^2$ estimates on the interval $\big[0,t\big]$ to conclude

$$\|Fv(t,\cdot)\|_{L^2} \lesssim \int_0^t (1+t-\tau)^{-\frac{n}{4\sigma}} \||v_t(\tau,\cdot)|^p\|_{L^1\cap L^2} d\tau. \tag{3.6}$$

From the definition of L_p-norms we get directly that

$$\||v_t(\tau,\cdot)|^p\|_{L^1\cap L^2} \lesssim \|v_t(\tau,\cdot)\|^p_{L^p} + \|v_t(\tau,\cdot)\|^p_{L^{2p}}.$$

To estimate the norm $\|v_t(\tau,\cdot)\|_{L^{kp}}, k=1,\ 2$, we apply the fractional Gagliardo-Nirenberg inequality from Proposition 5.1. in the following form

$$\|w(\tau,\cdot)\|_{L^q} \lesssim \||D|^{s-\sigma} w(\tau,\cdot)\|^{\theta_{0,s-\sigma}(q,2)}_{L^2} \|w(\tau,\cdot)\|^{1-\theta_{0,s-\sigma}(q,2)}_{L^2} \tag{3.7}$$

with $w(\tau,\cdot) = u_t(\tau,\cdot)$, where for $q \geq 2$ we need

$$\theta_{0,s-\sigma}(q,2) = \frac{n}{s-\sigma}\Big(\frac{1}{2} - \frac{1}{q}\Big) \in \big[0,1\big),$$

that is, $2 \leq q$ if $\frac{n}{2(s-\sigma)} < 1$.

Since $\theta_{0,s-\sigma}(p,2) < \theta_{0,s-\sigma}(2p,2)$, we obtain on the interval $(0,t)$ the estimate

$$\int_0^t (1+t-\tau)^{-\frac{n}{4\sigma}} \| |u_t(\tau,\cdot)|^p \|_{L^1\cap L^2} d\tau \lesssim \\ \|(u,v)\|^p_{X(t)} \int_0^t (1+t-\tau)^{-\frac{n}{4\sigma}} (1+\tau)^{-p\left(\frac{n}{4\sigma}+1+\frac{n}{2\sigma}(\frac{1}{2}-\frac{1}{p})\right)} d\tau. \tag{3.8}$$

Similarly, we get the L^2 estimate for Gu.

Now we recall that, for $\max\{\alpha;\beta\} > 1$, the inequality

$$\int_0^t (1+t-\tau)^{-\alpha}(1+\tau)^{-\beta} d\tau \lesssim (1+t)^{-\min(\alpha,\beta)} \tag{3.9}$$

holds.

If $p, q > p_S$, then in that case,

$$p\left(\tfrac{n}{4\sigma}+1+\tfrac{n}{2\sigma}(\tfrac{1}{2}-\tfrac{1}{p})\right) > \frac{n}{4\sigma} > 1,$$

thus

$$\int_0^t (1+t-\tau)^{-\frac{n}{4\sigma}} (1+\tau)^{-p\left(\frac{n}{4\sigma}+1+\frac{n}{2\sigma}(\frac{1}{2}-\frac{1}{p})\right)} d\tau \lesssim (1+t)^{-\frac{n}{4\sigma}} \tag{3.10}$$

Therefore the L^2-norm of Fv is bounded by $C(1+t)^{-\frac{n}{4\sigma}}\|(u,v)\|^p_{X(t)}$. Similarly, the same argument is applied for Gu

By this approach we have proved the L^2-norm estimate in (3.4) for Fv and Gu. Differentiating the equations for $N[u,v]$ with respect to t we obtain

$$\partial_t N[u,v] = L_t + \int_0^t \partial_t\big(G_1(t-\tau,x) *_{(x)} [Fv, Gu]\big) d\tau.$$

Using the above technique to estimate Fv we arrive at

$$(1+\tau)^{\frac{n}{4\sigma}+1}\|\partial_t Fv(\tau,\cdot)\|_{L^2} \le C\|(u,v)\|^p_{X(t)} \text{ for all } \tau\in[0,t] \tag{3.11}$$

under the same assumption for p, q.

Now we turn to estimate $\|\partial_t |D|^{s-\sigma} N[u(t,\cdot), v(t,\cdot)]\|_{L^2}$. We use the following

$$\partial_t |D|^{s-\sigma} N[u,v] = |D|^{s-1} L_t(t,x) + \int_0^t \partial_t |D|^{s-\sigma}\big(E_1(t-\tau,x) *_{(x)} (|v_t(\tau,\cdot)|^p, |u_t(\tau,\cdot)|^p)\big) d\tau.$$

Taking account of the linear estimate (1.16) with $k = s - \sigma$ and using the $(L^1 \cap L^2) - L^2$ estimates on the interval $(0, t)$ we obtain

$$\|\partial_t |D|^{s-\sigma}(Fv)\|_{L^2} \lesssim \int_0^t (1+t-\tau)^{-\frac{n+2(s+\sigma)}{4\sigma}} \big(\||v_t(\tau,\cdot)|^p\|_{L^1 \cap L^2} + \||v_t(\tau,\cdot)|^p\|_{\dot{H}^{s-\sigma}} \big) d\tau. \tag{3.12}$$

The integral with $\||v_t(\tau,\cdot)|^p\|_{L^1 \cap L^2}$ will be estimated by the Gagliardo-Nirenberg inequality, as before, in the following manner

$$\int_0^t (1+t-\tau)^{-\frac{n+2(s+\sigma)}{4\sigma}} \||v_t(\tau,\cdot)|^p\|_{L^1 \cap L^2}\, d\tau$$
$$\lesssim \|(u,v)\|^p_{X(t)} \int_0^t (1+t-\tau)^{-\frac{n+2(s+\sigma)}{4\sigma}} (1+\tau)^{-p\left(\frac{n}{4\sigma}+1+\frac{n}{2\sigma}\left(\frac{1}{2}-\frac{1}{p}\right)\right)}\, d\tau.$$

In order to apply inequality (3.9) with $a := \frac{n+2(s+\sigma)}{4\sigma}$ and $b := p\big(\frac{n}{4\sigma} + 1 + \frac{n}{2\sigma}\big(\frac{1}{2} - \frac{1}{p}\big)\big)$ we need the condition $a \le b$, that is equivalent to the following condition for p:

$$p > 1 + \tfrac{n\sigma + 2(s-\sigma)}{2(n+2\sigma)}. \tag{3.13}$$

To estimate the integrals with $\||v_t(\tau,\cdot)|^p\|_{\dot{H}^{s-\sigma}}$ we apply the composition result (see Corollary 5.2. in Appendix) for $p > s$. Thus we can proceed further as follows:

$$\begin{aligned}
&\int_0^t (1+t-\tau)^{-\frac{n}{4\sigma}-\frac{s+\sigma}{2\sigma}} \||v_t(\tau,\cdot)|^p\|_{\dot{H}^{s-\sigma}} d\tau \\
&\lesssim \int_0^t (1+t-\tau)^{-\frac{n}{4\sigma}-\frac{s+\sigma}{2\sigma}} \||v_t(\tau,\cdot)|\|_{\dot{H}^{s-\sigma}} \||v_t(\tau,\cdot)|\|^{p-1}_{L^\infty} d\tau \\
&\lesssim \int_0^t (1+t-\tau)^{-\frac{n}{4\sigma}-\frac{s+\sigma}{2\sigma}} \||v_t(\tau,\cdot)|\|_{\dot{H}^{s-\sigma}} \||v_t(\tau,\cdot)|\|^{p-1}_{H^{s_0}} d\tau \qquad (3.14)\\
&\lesssim \int_0^t (1+t-\tau)^{-\frac{n}{4\sigma}-\frac{s+\sigma}{2\sigma}} \||v_t(\tau,\cdot)|\|_{\dot{H}^{s-\sigma}} \Big(\||v_t(\tau,\cdot)|\|_{L^2} + \||v_t(\tau,\cdot)|\|_{\dot{H}^{s-\sigma}} \Big)^{p-1} d\tau,
\end{aligned}$$

with $s - \sigma > s_0 > \frac{n}{2}$.

Using again the linear estimates we get

$$\int_0^t (1+t-\tau)^{-\frac{n}{4\sigma}-\frac{s+\sigma}{2\sigma}} \||v_t(\tau,\cdot)|^p\|_{\dot{H}^{s-\sigma}} d\tau \lesssim \|(u,v)\|^p_{X(t)} \cdot$$
$$\cdot \int_0^t (1+t-\tau)^{-\frac{n}{4\sigma}-\frac{s+\sigma}{2\sigma}} (1+\tau)^{-\frac{n}{4\sigma}-1} \Big((1+\tau)^{-\frac{n}{4\sigma}-1} + (1+\tau)^{-\frac{n}{4\sigma}-\frac{s+\sigma}{2\sigma}} \Big)^{p-1} d\tau. \tag{3.15}$$

It is obvious that for $p > 2$ and $s > 1$ the integral in the right-hand side of (3.15) can be estimated by application of inequality (3.9), therefore it implies

$$\int_0^t (1+\tau)^{-\frac{n}{4\sigma}-\frac{s+\sigma}{2\sigma}} \||v_t(\tau,\cdot)|^p\|_{\dot{H}^{s-1}} d\tau \lesssim (1+t)^{-\frac{n}{4\sigma}-\frac{s+\sigma}{2\sigma}} \|(u,v)\|^p_{X(t)}.$$

An analogous reasoning leads to the other estimates in (3.4) which are required in the definition of $X(t)$-norm.

The second inequality (3.5) is obtained by Hölder's and Kato - Ponce's inequalities. Namely, the $L^1 \cap L^2$-norm of $F(v_t(\tau,x)) - F(\tilde{v}_t(\tau,x))$ for $F(v_t) = |v_t|^p$ is estimated by using

$$|F(v_t) - F(\tilde{v}_t)| \lesssim |v_t - \tilde{v}_t|(|v_t|^{p-1} + |\tilde{v}_t|^{p-1}).$$

Applying Hölder's inequality we obtain

$$\|F(v_t(\tau,\cdot)) - F(\tilde{v}_t(\tau,\cdot))\|_{L^1} \lesssim \|v_t(\tau,\cdot) - \tilde{v}_t(\tau,\cdot)\|_{L^p}\big(\|v_t(\tau,\cdot)\|_{L^p}^{p-1} + \|\tilde{v}_t(\tau,\cdot)\|_{L^p}^{p-1}\big),$$
$$\|F(v_t(\tau,\cdot)) - F(\tilde{v}_t(\tau,\cdot))\|_{L^2} \lesssim \|v_t(\tau,\cdot) - \tilde{v}_t(\tau,\cdot)\|_{L^{2p}}\big(\|v_t(\tau,\cdot)\|_{L^{2p}}^{p-1} + \|\tilde{v}_t(\tau,\cdot)\|_{L^{2p}}^{p-1}\big).$$

The L^p and L^{2p} norms of the difference $v_t - \tilde{v}_t$ are estimated again by the fractional Gagliardo-Nirenberg inequality. Therefore, these norms can be bounded from above by the norms of $\|v_t - \tilde{v}_t\|_{L^2}$ and $\|v_t - \tilde{v}_t\|_{\dot{H}^{s-\sigma}}$ that are included in the $X(t)$- norm.

To obtain other Sobolev norms estimates for the difference $(u,v) - (\tilde{u},\tilde{v})$ we will use a version of Kato-Ponce inequality (see Proposition 5.2.) that is formulated in the homogeneous Sobolev scale. That inequality is also known as the fractional Leibniz rule. Let us denote $\gamma = s - \sigma$. We can estimate

$$\||v_t(s,\cdot)|^p - |\tilde{v}_t(s,\cdot)|^p\|_{\dot{H}^\gamma} \lesssim \int_0^1 \big\|(v_t(s,\cdot) - \tilde{v}_t(s,\cdot))f\big(\theta v_t(s,\cdot) + (1-\theta)\tilde{v}_t(s,\cdot)\big)\big\|_{\dot{H}^\gamma}\, d\theta \tag{3.16}$$

with $f(w) = w|w|^{p-2}$.

Now we apply the Kato-Ponce inequality from Proposition 5.2. with the following constants $p_1, q_1, p_2, q_2 > 0$:

$$r = p_2 = q_1 = 2, p_1 = q_2 = \infty,$$

to estimate the $\dot{H}^\gamma$-norm of the product fg, with $f := f(\theta v_t + (1-\theta)\tilde{v}_t)$ and $g := v_t - \tilde{v}_t$, in the right-hand side of (3.16).

$$\||D|^\gamma(fg)\|_{L^2} \lesssim \|f\|_{L^\infty}\||D|^\gamma g\|_{L^2} + \||D|^\gamma f\|_{L^2}\|g\|_{L^\infty}. \tag{3.17}$$

With the assumption $p-1 > \gamma = s - \sigma$ the norm in $\dot{H}^{\gamma,2} \equiv \dot{H}^\gamma$ of $f(\theta v_t + (1-\theta)\tilde{v}_t)$ can be estimated if we apply a general composition result for homogeneous Sobolev spaces (see Corollary 5.2.).

By this way we obtain the second inequality (3.5) for the operator $N[u,v]$, since $p - 1 > \gamma = s - \sigma$ is equivalent to our condition $p > s + 1 - \sigma$ and both of the norms $\|v_t(t,\cdot)\|_{L^2}$ and $\|v_t(t,\cdot)\|_{\dot{H}^\gamma}$ are included in the norm $\|(u,v)\|_{X(t)}$.

We can summarize now all conditions for the exponents p and q that have been derived from the above considerations as follows:

- From the application of the Gagliardo - Nirenberg inequality: $p, q > 2$.
- From (3.13): $p, q > 1 + \frac{n\sigma + 2(s-\sigma)}{2(n+2\sigma)}$.
- From the above application of composition result: $p, q > s + 1 - \sigma$.

It is clear that these conditions are reduced to the last one

$$p, q > s + 1 - \sigma,$$

since $s + 1 - \sigma > 2$ and $s + 1 - \sigma > 1 + \frac{n\sigma + 2(s-\sigma)}{2(n+2\sigma)}$ for $s > \sigma + \frac{n}{2}$.

Thus we have verified the required sufficient condition $p, q > s + 1 - \sigma$ for the existence of global (in time) small data solution to the Cauchy problem for system (2.1).

Theorem 2. has been proved completely. □

Proof. (of Theorem 2.) We introduce for all $t > 0$ the function space

$$X(t) := \Big(\mathcal{C}\big([0,t], H^\sigma\big) \cap \mathcal{C}^1\big([0,t], L^2\big)\Big)^2$$

with the norm

$$\|(u,v)\|_{X(t)} = \sup_{0 \le \tau \le t} \Big((1+\tau)^{-\gamma(p)} W(u) + (1+t)^{-\gamma(q)} W(v)\Big), \tag{3.18}$$

where

$$W(u) = \big(f_0(\tau)^{-1} \|u(\tau,\cdot)\|_{L^2} + f_\sigma(\tau)^{-1} \||D|^\sigma u(\tau,\cdot)\|_{L^2} + g(\tau)^{-1} \|u_t(\tau,\cdot)\|_{L^2}\big), \tag{3.19}$$

and similarly for v.

From the linear estimates in Proposition 1. we can choose

$$f_0(\tau) := (1+\tau)^{-\frac{n}{4\sigma}}, \;\; f_\sigma(\tau) := (1+\tau)^{-\frac{n}{4\sigma} - 1/2}, \;\; g(\tau) := (1+\tau)^{-\frac{n}{4\sigma} - 1}.$$

In the case $p = p_E$ (or $q = p_E$) then we will replace $(1+\tau)^{-\gamma(p)}$ (respectively, $(1+\tau)^{-\gamma(q)}$) by $(\log(e+t))^{-1}$.

In the following we denote by $E_0(t,x)$ and $E_1(t;x)$ the fundamental solutions to the linear equation, corresponding to the two initial data, namely the solution for the linear problem (1.11) with the Cauchy data (u_0, u_1) is given by

$$u = E_0(t,x) *_x u_0(x) + E_1(t,x) *_x u_1(x).$$

We define the integral operator $N : (u,v) \in X(t) \mapsto N[u,v] \in X(t)$ by:

$$N[u,v] = L + (Fv, Gu),$$

where

$$\begin{aligned} L\ &= E_0(t,x) *_x (u_0,v_0)(x) + E_1(t,x) *_x (u_1,v_1)(x), \\ F(v) &= \int_0^t E_1(t-\tau,x) *_x \big||D|^\alpha v(\tau,x)\big|^p d\tau, \\ G(u) &= \int_0^t E_1(t-\tau,x) *_x \big||D|^\alpha u(\tau,x)\big|^q d\tau. \end{aligned}$$

The local and global existence of small data solutions to (1.4) will follow by the standard contraction argument if we can show that the estimates

$$\|N[u,v]\|_{X(t)} \lesssim \mathcal{A} + \|(u,v)\|^p_{X(t)} + \|(u,v)\|^q_{X(t)}, \tag{3.20}$$

and the Lipschitz property

$$\begin{aligned} &\|N[u,v]-N[\tilde{u},\tilde{v}]\|_{X(t)} \lesssim \\ &\big\|(u,v)-(\tilde{u},\tilde{v})\big\|_{X(t)}\Big(\big\|(u,v)\big\|^{p-1}_{X(t)} + \big\|(\tilde{u},\tilde{v})\big\|^{p-1}_{X(t)} + \big\|(u,v)\big\|^{q-1}_{X(t)} + \big\|(\tilde{u},\tilde{v})\big\|^{q-1}_{X(t)}\Big) \end{aligned} \tag{3.21}$$

are valid for the exponents p,q satisfying the given conditions.

First, we will prove the inequality (3.20).

By the linear estimates (1.12)-(1.14) in Prop. 1., it immediately follows that

$$\|L\|_{X(t)} \lesssim \mathcal{A}.$$

Now we will estimate the L^2 and $\dot{H}^\sigma$ norms of Fv and Gu.

We use the $L^1\cap L^2 - L^2$ estimates if $\tau\in[0,t/2]$ and the L^2-L^2 estimates if $\tau\in[t/2,t]$. Then

$$\begin{aligned} \|\partial_t^j|D|^{k\sigma}Fv\|_{L^2} &\lesssim \int_0^{t/2}(1+t-\tau)^{-\frac{n}{4\sigma}-(k/2+j)}\big\|\big||D|^\alpha v(\tau,\cdot)\big|^p\big\|_{L^1\cap L^2}d\tau \\ &+\int_{t/2}^t (1+t-\tau)^{1-(3k/2+j)}\big\|\big||D|^\alpha v(\tau,\cdot)\big|^p\big\|_{L^2}d\tau, \end{aligned}$$

where $j,k=0,1$ and $(j,k)\neq(1,1)$. We will estimate $\big||D|^\alpha v(\tau,\cdot)\big|^p$ in $L^1\cap L^2$ and in L^2.

Obviously $\big\|\big||D|^\alpha v(\tau,\cdot)\big|^p\big\|_{L^1\cap L^2} \lesssim \big\||D|^\alpha v(\tau,\cdot)\big\|^p_{L^p} + \big\||D|^\alpha v\big\|^p_{L^{2p}}$, and $\big\|\big||D|^\alpha v\big|^p\big\|_{L^2} = \big\||D|^\alpha v\big\|^p_{L^{2p}}$.

We apply the fractional Gagliardo-Nirenberg inequality with the interpolation exponents $\theta_{\alpha,\sigma}(p,2)$ and $\theta_{\alpha,\sigma}(2p,2)$ from the interval $[0,1)$ (see [3, 5, 10] and Appendix for the formulation, proof and notations). This gives the condition $2 \leq p < \dfrac{n}{n+2(\alpha-\sigma)}$. Accordingly

$$\big\| \big| |D|^\alpha v(\tau,\cdot)\big|^p \big\|_{L^1\cap L^2} \lesssim (1+\tau)^{\frac{-p(n+\alpha)+n}{2\sigma}+p\gamma(q)} \|(u,v)\|^p_{X(\tau)},$$

because of $\theta_{\alpha,\sigma}(p,2) < \theta_{\alpha,\sigma}(2p,2)$, meanwhile

$$\begin{aligned}\big\| \big| v(\tau,\cdot)\big|^p \big\|_{L^2} &\lesssim (1+\tau)^{p\left(-\frac{n}{4\sigma}-\frac{\theta_{\alpha,\sigma}(2p,2)}{2}\right)+p\gamma(q)} \|(u,v)\|^p_{X(\tau)} \\ &= (1+\tau)^{-\frac{p(n+\alpha)-n/2}{2\sigma}+p\gamma(q)} \|(u,v)\|^p_{X(\tau)}.\end{aligned}$$

Combining the last estimates we conclude

$$\begin{aligned}&\|\partial_t^j |D|^{k\sigma} Fv\|_{L^2} \lesssim (1+t)^{-\frac{n}{4\sigma}-(k/2+j)} \|(u,v)\|^p_{X(t)} \int_0^{t/2} (1+\tau)^{\frac{-p(n+\alpha)+n}{2\sigma}+p\gamma(q)} d\tau \\ &+ (1+t)^{-\frac{p(n+\alpha)-n/2}{2\sigma}} \|(u,v)\|^p_{X(t)} \int_{t/2}^t (1+t-\tau)^{1-(3k/2+j)+p\gamma(q)} d\tau.\end{aligned}$$

and similarly for Gu

$$\begin{aligned}&\|\partial_t^j |D|^{k\sigma} Gu\|_{L^2} \lesssim (1+t)^{-\frac{n}{4\sigma}-(k/2+j)} \|(u,v)\|^p_{X(t)} \int_0^{t/2} (1+\tau)^{\frac{-q(n+\alpha)+n}{2\sigma}+q\gamma(p)} d\tau \\ &+ (1+t)^{-\frac{q(n+\alpha)-n/2}{2\sigma}} \|(u,v)\|^p_{X(t)} \int_{t/2}^t (1+t-\tau)^{1-(3k/2+j)+q\gamma(p)} d\tau.\end{aligned}$$

We will proceed with the integrals over $(0,t/2)$ first.

If $p,q > \dfrac{n+2\sigma}{n+\alpha} = p_E$, then $\gamma(p)=\gamma(q)=0$, and the terms $(1+\tau)^{\frac{-p(n+\alpha)+n}{2\sigma}}$, $(1+\tau)^{\frac{-q(n+\alpha)+n}{2\sigma}}$ are integrable.

Moreover, we have

$$\begin{aligned}&(1+t)^{-\frac{p(n+\alpha)-n/2}{2\sigma}} \|(u,v)\|^p_{X(t)} \int_{t/2}^t (1+t-\tau)^{1-(3k/2+j)} d\tau \\ =\ &(1+t)^{-\frac{p(n+\alpha)-n/2}{2\sigma}} \|(u,v)\|^p_{X(t)} \int_0^{t/2} (1+\tau)^{1-(3k/2+j)} d\tau \lesssim (1+t)^{-\frac{n}{4\sigma}-(k/2+j)} \|(u,v)\|^p_{X_0(t)}.\end{aligned}$$

Now consider the more interesting case, when $\min\{p,q\} \leq p_E$. Suppose that $q \geq p$. Then our condition implies that $q > p_E$ and $\gamma(q)=0$. In that case

$$\int_0^{t/2} (1+\tau)^{\frac{-p(n+\alpha)+n}{2\sigma}} d\tau \lesssim \begin{cases}(1+t)^{\gamma(p)} \text{ if } p < p_E, \\ \log(e+t) \text{ if } p = p_E.\end{cases} \tag{3.22}$$

On the other hand, it is easy to verify that condition (1.5) is equivalent to

$$\frac{-q(n+\alpha)+n}{2\sigma}+q\gamma(p)<-1,$$

therefore the term $(1+\tau)^{\frac{-q(n+\alpha)+n}{2\sigma}+q\gamma(p)}$ again is integrable over $(0,t/2)$.

The second integrals over $(t/2,t)$ are easier to be estimated thanks to the condition $p,q\geq p_E-1$.

Summarizing the above we arrive at estimates of the L^2 and $\dot{H}^\sigma$ of Fv and Gu. The L_2 norms of $(Fv)_t$ and $(Gu)_t$ can be estimated in the similar way if we apply again the linear estimates from Prop. 1.

Hence the first inequality (3.20) has been verified. The second inequality (3.21) can be proved by the same arguments as in the proof of Theorem 2, with the application of Kato - Ponce's inequality in the homogeneous Sobolev spaces.

Theorem 2. thus has been proved completely. □

It is obvious, thanks to the linear estimates, that the statement of Theorem 2. remains valid for the following model

$$\begin{cases} u_{tt}+a_1(x)(-\Delta)^\sigma u+b_1u_t=|v_t|^p,\ t\geq 0,x\in\mathbb{R}^n,\\ v_{tt}+a_2(x)(-\Delta)^\sigma v+b_2v_t=|u_t|^q,\ t\geq 0,x\in\mathbb{R}^n,\\ (u,u_t,v,v_t)(0,x)=(u_0,u_1,v_0,v_1)(x),\ x\in\mathbb{R}^n,\end{cases} \tag{3.23}$$

where $a_1(x),a_2(x)$ are continuous functions, b_1,b_2 are positive constants, and

$$0<C_1<a_i(x)<C_2,$$

for $i=1,2$ with some positive constants C_1,C_2.

4. Conclusion

We have obtained two results on the global solvability of the Cauchy problems for nonlinear coupled systems for external damping models. Our results generalize previous works in [1] for the classical damping model and in [2] for the coupled system of structurally damped waves with constant coefficients, where the authors considered the nonlinearity of the form $(F,G)=(|v|^p,|u|^q)$. We have developed a new approach to using the diffusion phenomenon to deal with the difficulties that arose in the external damping models with the variable coefficients. In order to apply the contraction arguments with new nonlinearities containing the derivatives of solutions, such as $(|v_t|^p,|u_t|^q)$ or $(||D|^\alpha v|^p,||D|^\alpha u|^q)$, we need some advanced technical results from the harmonic analysis: the composition results, the fractional Gagliardo-Nirenberg inequalities, the Kato-Ponce's inequality, that allow us to obtain estimates for the nonlinearities in Sobolev spaces with non-integer orders. In the limit case $\sigma = 1$ and $\alpha = 0$ the exponent p_E for solvability obtained in Theorem 2. becomes $p_E(1,0) = \frac{n+2}{n} = p_F$, and thus our result coincides with the result in [1] for the coupled system of classical damping wave equations.

5. Appendix

5.1. The fractional Gagliardo - Nirenberg inequality

We will present here the formulation of the fractional Gagliardo - Nirenberg inequality in homogeneous Sobolev spaces (see [10]). [Fractional Gagliardo - Nirenberg inequality for the homogeneous Sobolev spaces] Let $a \in (0,\sigma)$. Then for all $m \in (1,\infty)$

$$\||D|^a u\|_{L^q} \lesssim \||D|^\sigma u\|_{L^m}^{\theta_{a,\sigma}(q,m)} \|u(\tau,\cdot)\|_{L^m}^{1-\theta_{a,\sigma}(q,m)} \tag{5.1}$$

for all $u \in \dot{H}^{\sigma,m}$, whenever the condition

$$\theta_{a,\sigma}(q,m) := \frac{n}{\sigma}\Big(\frac{1}{m} - \frac{1}{q} + \frac{a}{n}\Big) \in \big[\frac{a}{\sigma}, 1\big).$$

is satisfied. We note that the condition $\theta_{a,\sigma}(q,m) \in \big[\frac{a}{\sigma}, 1\big)$ is equivalent to

$$m \le q < \frac{mn}{n + m(a-\sigma)}.$$

5.2. The composition results

We introduce the class $Lip\,\mu$ as follows (see [11]). Let $\mu > 0, N \in \mathbb{N}_0$ and $0 < \alpha \le 1$ such that $\mu = N + \alpha$. We define

$$\text{Lip}\ \mu = \Big\{ f \in C^{N,loc}(\mathbb{R}) : f^{(j)}(0) = 0, j = 0, ..., N,\ and\ \sup_{t_0 \neq t_1} \frac{|f^{(N)}(t_0) - f^{(N)}(t_1)|}{|t_0 - t_1|^\alpha} < \infty \Big\}. \tag{5.2}$$

Further, we put

$$\|f\|_{Lip\ \mu} = \sum_{j=0}^{N-1} \frac{|f^{(j)}(t)|}{|t|^{\mu-j}} + \sup_{t_0 \neq t_1} \frac{|f^{(N)}(t_0) - f^{(N)}(t_1)|}{|t_0 - t_1|^\alpha}. \tag{5.3}$$

It is clear that $|t|^\mu \in$ Lip μ, $t|t|^{\mu-2} \in$ Lip $(\mu - 1)$ for $\mu > 1$. The following useful general composition result for the class $Lip\ \mu$ was obtained in [11]. Let us denote $\sigma_p = n\max\{0; \frac{1}{p} - 1\}$. [Theorem 6.3.4 (i) in [11]] Let $\sigma_p < s < \mu$ and $\mu > 1$.

Then there exists some constant c such that

$$\|G(f)\|_{F^s_{p,q}} \le c\|G\|_{Lip\ \mu} \|f\|_{F^s_{p,q}} \|f\|_{L^\infty}^{\mu-1}$$

holds for all $f \in F^s_{p,q} \cap L_\infty$ and all $G \in Lip\ \mu$. Proposition 5.2. together with the Sobolev embedding imply immediately the following result in the supercritical case $s > \frac{n}{2}$. [Composition result]Let $s \in (\frac{n}{2}, p)$. Denote either $G(u) = |u|^p$ or $G = \pm u|u|^{p-1}$ with $p > 1$. Then for all $u \in H^s$ the following composition estimate holds:

$$\|G(u)\|_{H^s} \lesssim \|u\|^p_{H^s}.$$

In the homogeneous spaces, we can obtain the composition result by the following estimate Let $p > 1$, $1 < r < \infty$ and $u \in H^{s,r}$, where $s \in \left(\frac{n}{r}, p\right)$. Let us denote by $F(u)$ one of the functions $|u|^p$, $\pm|u|^{p-1}u$ with $p > 1$. Then the following estimate holds:

$$\|F(u)\|_{H^{s,r}} \lesssim \|u\|_{H^{s,r}} \|u\|^{p-1}_{L^\infty}.$$

Under the assumptions of Proposition 5.2, it holds

$$\|F(u)\|_{\dot{H}^{s,r}} \lesssim \|u\|_{\dot{H}^{s,r}} \|u\|^{p-1}_{L^\infty}.$$

The proof can be found in [5]. [The Kato-Ponce inequality for homogeneous Sobolev spaces] For all functions $f \in \dot{H}^{s,p_2} \cap L^{q_1}$ and $g \in \dot{H}^{s,q_2} \cap L^{p_1}$ it holds

$$\||D|^s(fg)\|_{L^r} \lesssim \|f\|_{L^{p_1}} \||D|^s g\|_{L^{q_1}} + \||D|^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}, \tag{5.4}$$

where $s > 0$ and $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$ for $1 < r < \infty,\ 1 < p_1, q_2 \le \infty, 1 < p_2, q_1 < \infty$. The proof of this harmonic analysis result can be found in [12].

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