HNUE JOURNAL OF SCIENCE DOI: 10.18173/2354-1059.2022-0019
Natural Science, 2022, Volume 67, Issue 2, pp. 19-24
This paper is available online at http://stdb.hnue.edu.vn

SOME GEOMETRIC CHARACTERIZATIONS OF EXTREMAL SETS IN HILBERT SPACES

Nguyen Van Khiem
Faculty of Mathematics, Hanoi National University of Education

Abstract. Based on our previous result and by using the technique on α-minimal and χ-minimal sets with respect to the Kuratowski and Hausdorff measures of noncompactness, we give some new geometric characterizations of extremal sets in Hilbert spaces.
Keywords: Extremal sets, Jung's constant, Kuratowski and Hausdorff measures of noncompactness.

1. Introduction

In [1] the authors introduced the notion of extremal sets of a Banach space with respect to Jung constant. Given a Banach space $(X, \|\cdot\|)$, the Jung constant of X is defined by

$$J(X) = \sup\left\{\frac{r_X(A)}{d(A)} : A \text{ is a bounded subset of } X \text{ with diameter } d(A) > 0\right\},$$

where $d(A) = \sup\{\|x-y\| : x, y \in A\}$ and $r_X(A) = \inf\limits_{x\in X}\sup\limits_{y\in A}\|y-x\|$ denote the diameter and the absolute Chebyshev radius of A, respectively. A point $c \in X$ is called a Chebyshev center of A, if $r_X(A) = r_c(A) = \sup\limits_{y\in A}\|y-c\|$.

Recall that a bounded subset A of a Banach space X consisting of at least two points is extremal, if $r_X(A) = J(X)d(A)$. For given an n-dimensional Euclidean space E^n, the Jung's theorem asserts that $J(E^n) = \sqrt{\dfrac{n}{2(n+1)}}$. Furthermore, a bounded subset A of E^n is extremal if and only if A contains all vertices of a regular n-simplex with edges of length $d(A)$ (see [2]). For H is an infinite-dimensional Hilbert space, it is well known that $J(H) = \dfrac{1}{\sqrt{2}}$ (see [3]). Therefore, a bounded subset A

Received May 15, 2022. Revised: June, 20 2022. Accepted June 28, 2022.
Contact Nguyen Van Khiem, e-mail address: nvkhiem@hnue.edu.vn

of H is extremal if and only if $r_H(A) = \dfrac{1}{\sqrt{2}}\, d(A)$. We now recall the main result of [1] involving geometrically characterizing extremal sets in a Hilbert space, which is an infinite-dimensional generalization of classical Jung's theorem.

Theorem 1.1. *([1]) A bounded subset A of a Hilbert space H with $d(A) = d > 0$ is extremal if and only if for any $\varepsilon \in (0, d)$ and positive integer m, there exists an m–simplex $\Delta(\varepsilon, m)$ with vertices in A and edges of length not less than $d - \varepsilon$. Furthermore, for such a subset A, we have $\alpha(A) = d$ and $\chi(A) = r_H(A)$, where $\alpha(A)$ and $\chi(A)$ denote the Kuratowski and Hausdorff measures of noncompactness of A which are defined as inf $\{\varepsilon > 0 :$ A can be covered by finitely many sets of diameter $\leqslant \varepsilon\}$ and inf $\{\varepsilon > 0 :$ A can be covered by finitely many balls of radius $\leqslant \varepsilon\}$, respectively.*

In [4], Domingez-Benavides introduced the notions of α-minimal and χ-minimal sets. We say that an infinite set A of a metric space X is α-minimal (resp. χ-minimal) if for any infinite subset B of A one has $\alpha(B) = \alpha(A)$ (resp. $\chi(B) = \chi(A)$). A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be α-minimal (resp. χ-minimal) if the set $\{x_n\}_{n\in\mathbb{N}}$ is α-minimal (resp. χ-minimal). For the properties of α-minimal and χ-minimal sets we refer the reader to [4, 5, 6].

In this note we give three more geometric characterizations of extremal sets in Hilbert spaces.

Theorem 1.2. *Let A be a bounded subset of a Hilbert space H with diameter $d(A) > 0$. Then A is an extremal set of H if and only if for any $\varepsilon \in (0, d(A))$, there exists an infinite simplex $\Delta(\varepsilon, \infty)$ with vertices in A and edges of length not less than $d(A) - \varepsilon$.*

Theorem 1.3. *Let A be a bounded subset of a Hilbert H with diameter $d(A) > 0$. Then A is an extremal set of H if and only if A contains a sequence $\{x_n\}$ satisfying the following properties*

(i) *$\{x_n\}$ is both α–minimal and χ–minimal;*

(ii) *$\{x_n\}$ converges weakly to the Chebyshev center of A in H;*

(iii) *$\alpha(\{x_n\}) = d(A)$ and $\chi(\{x_n\}) = r_H(A)$.*

Definition 1.1. *We say that a sequence $\{x_n\}$ in a Hilbert space H is an asymptotically orthonormal sequence, if*

$$\lim_{n\to\infty} \|x_n\| = 1 \ \textit{and} \ \lim_{\substack{m,n\to\infty \\ m\neq n}} \langle x_m\ ,\ x_n\rangle = 0,$$

where $\langle\cdot,\cdot\rangle$ denote the inner product of H.

Theorem 1.4. *Let A be a subset of the closed unit ball of a Hilbert space H with diameter $d(A) = \sqrt{2}$. Then A is an extremal if and only if A contains an asymptotically orthonormal sequence.*

2. Two auxiliary lemmas

Lemma 2.1. *If A is an extremal subset of a Hilbert space H with diameter d, then there exists a separable Hilbert subspace H' of H such that $A_{H'} = A \cap H'$ is also an extremal subset of H'. Furthermore, $r_{H'}(A_{H'}) = r_H(A)$ and $d(A_{H'}) = d$.*

Proof. From the proof of Theorem 1 in [1], it follows that for every positive integer m there exists a subset $A_m = \{x_1,\ x_2, \ldots, x_m\} \subset A$ such that

$$\|x_i - x_j\| > d - \frac{1}{m} \quad \forall\, i \neq j;\ i,\ j\ = 1, 2, ..., m.$$

Take c arbitrarily in the convex hull coA_m of A_m, then there exist non-negative numbers $t_1,\ t_2,\ \ldots,\ t_m$ such that

$$\sum_{i=1}^{m} t_i x_i = c \quad \text{and} \quad \sum_{i=1}^{m} t_i = 1.$$

For each $j \in \{1,\ 2,\ \ldots,\ m\}$, we have

$$\begin{aligned}
(1 - t_j)\left(d - \frac{1}{m}\right)^2 &\leqslant \sum_{i=1}^{m} t_i \|x_i - x_j\|^2 = \sum_{i=1}^{m} t_i \|(x_i - c) - (x_j - c)\|^2 \\
&= \sum_{i=1}^{m} t_i \left(\|x_i - c\|^2 + \|x_j - c\|^2 - 2\left\langle x_i - c\ ,\ x_j - c\right\rangle\right) \\
&\leqslant 2r_c^2(A_m) + 2\left\langle \sum_{i=1}^{m} t_i x_i - c\ ,\ x_j - c \right\rangle \\
&= 2r_c^2(A_m).
\end{aligned}$$

Hence,

$$(m-1)\left(d - \frac{1}{m}\right)^2 = \sum_{j=1}^{m}(1 - t_j)\left(d - \frac{1}{m}\right)^2 \leqslant 2m.r_c^2(A_m),$$

or

$$r_c(A_m) \geqslant \sqrt{\frac{m-1}{2m}}\ \left(d - \frac{1}{m}\right).$$

By a result of Garkavi and Klee (see [7, 8]), the Chebyshev center of A_m lies in the convex hull coA_m of A_m. Since c is arbitrary in coA_m, one gets

$$r_H(A_m) \geqslant \sqrt{\frac{m-1}{2m}}\ \left(d - \frac{1}{m}\right).$$

Cleary $A_\infty = \bigcup\limits_{m=1}^{\infty} A_m$ is a countable subset of A. From the estimates above we get

$$d(A_\infty) = d(A) = d,$$

$$r_H(A_\infty) = \frac{1}{\sqrt{2}}\, d = r_H(A).$$

Now consider $H' = \overline{\text{span}}\, A_\infty$ the closed subspace of H which is generated by A_∞, and $A_{H'} = A \cap H'$. Clearly H' is a separable subspace of H. Since $A_\infty \subset A_{H'} \subset A$ one gets

$$r_H(A_\infty) \leqslant r_H(A_{H'}) \leqslant r_H(A),$$

$$d(A_\infty) \leqslant d(A_{H'}) \leqslant d(A).$$

Hence, $r_H(A_{H'}) = r_H(A)$ and $d(A_{H'}) = d(A)$, so $A_{H'}$ is an extremal set in H. By using the orthogonal projection of H on the closed subspace H', it is easy to see that the Chebyshev center of $A_{H'}$ lies in H'. So $r_H(A_{H'}) = r_{H'}(A_{H'})$. Hence $A_{H'}$ also is extremal in H'. The proof of Lemma 2.1 is completed. $\square$

Lemma 2.2. *Let A be an α-minimal and χ-minimal subset of a Hilbert space. Then* $\chi(A) = \dfrac{1}{\sqrt{2}}\alpha(A)$.

Proof. From Lemma 3.5 of [4], let us omit it here. $\square$

3. Proofs of the main results

Proof of Theorem 1.2. First, assume that A is an extremal set of a Hilbert space H with diameter $d(A) = d > 0$. By Lemma 2.1, we can assume that H is a separable Hilbert space. By Theorem 1.1, one has $\alpha(A) = d$ and $\chi(A) = r_H(A) = \dfrac{1}{\sqrt{2}}d$. From [4] (Propositions 3.2, 3.3) it follows that there exists a subset B of A which is both α-minimal and χ-minimal with $\chi(B) = \chi(A)$. In view of Lemma 2.2 one obtains

$$\frac{1}{\sqrt{2}} = \frac{\chi(B)}{\alpha(B)} \geqslant \frac{\chi(A)}{\alpha(A)} = \frac{1}{\sqrt{2}}.$$

Hence $\alpha(B) = \alpha(A) = d$. By using Ramsey's argument of (see [4], Lemma 3.4) there exists an infinite subset $B_\varepsilon \subset B$, $\forall \varepsilon \in (0, d)$, such that $\|x - y\| > d - \varepsilon$ $\forall x, y \in B_\varepsilon$, $x \neq y$. So we can choose an infinite simplex $\Delta(\varepsilon, \infty)$ with vertices in B_ε and its edges have length not less than $d(A) - \varepsilon$.

Conversely, for given $\varepsilon \in (0, d)$ if A contains an infinite simplex $\Delta(\varepsilon, \infty)$ with vertices in A and edges of length not less than $d(A) - \varepsilon$. Consequently by Theorem 1.1, A is an extremal set. The proof of Theorem 1.2 is completed. $\square$

Proof of Theorem 1.3. If A is an extremal set of a Hilbert space H, then from the proof of Theorem 1.2 it follows that there exists a subset $B \subset A$ which is both α–minimal and χ–minimal and such that $\chi(B) = \chi(A) = r_H(A)$, $\alpha(B) = \alpha(A) = d(A)$. Let $\{x_n\}$ be a sequence in B, then $\{x_n\}$ also is both α–minimal and χ–minimal. Furthermore $\alpha(\{x_n\}) = d(A)$ and $\chi(\{x_n\}) = r_H(A)$. Since H is reflexive, we may assume that $\{x_n\}$

converges weakly to a point, say c. It is known that the function $\Phi : H \to \mathbb{R}$ defined by $\Phi(z) = \limsup_{n\to\infty} \|x_n - z\|$ attains its unique minimum at c and $\Phi(c) = \chi(\{x_n\}) = r_H(A)$ (see [9], cf. [4]). Thus c is the Chebyshev center of A. Hence the sequence $\{x_n\}$ satisfies the conditions (i)–(iii).

Conversely if A contains a sequence $\{x_n\}$ satisfying the conditions (i)–(iii), then by Lemma 2.2 one has

$$\frac{r_H(A)}{d(A)} = \frac{\chi(\{x_n\})}{\alpha(\{x_n\})} = \frac{1}{\sqrt{2}}.$$

Hence A is an extremal set in H. The proof of Theorem 1.3 is completed. □

Proof of Theorem 1.4. Assume that A is a subset of closed unit ball $\overline{B}(O, 1)$ of H with diameter $d(A) = \sqrt{2}$. If A is an extremal, then $r_H(A) = 1$ and O is a unique Chebyshev center of A. By Theorem 1.3 there exists a sequence $\{x_n\} \subset A$ satisfying the properties (i)–(iii). Hence we have $\limsup_{n\to\infty} \|x_n - O\| = \chi(\{x_n\}) = r_H(A)$. By proceeding to a subsequence if necessary, one may assume that $\lim_{n\to\infty} \|x_n\| = r_H(A) = 1$. Since $\{x_n\}$ is an α–minimal sequence and by Lemma 3.4 in [4], there exists a decreasing chain of subsequences of $\{x_n\}$:

$$\{x_n\} \supset \{x_{n_{1,1}}\,,\ x_{n_{1,2}}\,, \ldots\} \supset \{x_{n_{2,1}}\,,\ x_{n_{2,2}}\,, \ldots\} \supset \cdots \supset \{x_{n_{k,1}}\,,\ x_{n_{k,2}}\,, \ldots\} \supset \cdots$$

satisfying

$$\sqrt{2} - \frac{1}{k} \leqslant \|x_{n_{k,i}} - x_{n_{k,j}}\| \leqslant \sqrt{2}\,,\ \forall k \geqslant 1,\ \forall i \neq j.$$

Taking the diagonal sequence $\{z_k\}$, defined by $z_k = x_{n_{k,k}}$ one sees that

$$\sqrt{2} - \frac{1}{k} \leqslant \|z_p - z_k\| \leqslant \sqrt{2}\,,\ \forall p > k \geqslant 1.$$

Since $\lim_{k\to\infty} \|z_k\| = 1$ and $\|z_p - z_k\|^2 = \|z_p\|^2 + \|z_k\|^2 - 2\,\langle z_p\,,\ z_k\rangle$, one gets

$$\lim_{\substack{p,k\to\infty \\ p\neq k}} \langle z_p\,,\ z_k\rangle = 0,$$

i.e. $\{z_k\}$ is an asymptotically orthornormal sequence.

Conversely, if $d(A) = \sqrt{2}$ and A contains an asymptotically orthonormal sequence $\{z_k\}$, then we have

$$\lim_{\substack{p,k\to\infty \\ p\neq k}} \|z_p - z_k\| = \sqrt{2} = d(A).$$

Hence for every $\varepsilon \in (0, d(A))$ there exists a positive integer n_0 such that

$$\|z_p - z_k\| > d(A) - \varepsilon,\ \forall p > k \geqslant n_0.$$

It follows that the infinite simplex $\Delta(\varepsilon, \infty)$ with vertices $z_{n_0},\ z_{n_0+1},\ \ z_{n_0+2}, \ldots$ has all edges of length not less than $d(A) - \varepsilon$. Therefore, by Theorem 1.1, A is an extremal set. The proof of Theorem 1.4 is completed. □

4. Conclusions

In this paper, we study the geometrical properties of extremal sets in Hilbert spaces. Based on our results in [1], and by using the technique on α-minimal and χ-minimal sets with respect to the Kuratowski and Hausdorff measures of noncompactness, we obtain three new geometric characterizations of extremal sets in Hilbert spaces.

REFERENCES

[1] V. Nguyen-Khac, K. Nguyen-Van, 2006. An infinite-dimensional generalization of the Jung theorem, *Mat. Zametki*, Vol. 80, No. 2, pp. 231-239 (in Russian), (English transl.: Mathematical Notes, Vol. 80, No. 2, pp. 224-232).

[2] H. W. E. Jung, 1901. Über die kleinste Kugel, die eine räumliche Figur einschliesst. *J. Reine Angew. Math.*, Vol. 123, pp. 241-257.

[3] N. A. Routledge, 1952. A result in Hilbert space, *Quart. J. Math., Oxford*, (2) Vol. 3, No. 9, pp. 12-18.

[4] T. Dominguez-Benavides, 1986. Some properties of set and ball measures of non-compactness and applications. *J. London Math. Soc.*, Vol. 34, No. 2, pp. 120-128.

[5] T. Dominguez-Benavides, 1988. Set-Contractions and Ball-Contractions in some classes of spaces. *J. Math. Anal. Appl.*, Vol. 136, pp. 131-140.

[6] T. Dominguez-Benavides, J. M. Ayerbe, 1991. Set-Contractions and Ball-Contractions in L^p-spaces. *J. Math. Anal. Appl.*, Vol. 159, pp. 500-506.

[7] A. L. Garkavi, 1964. On the Chebyshev center and convex hull of a set. *Uspekhi Mat. Nauk*, Vol. 19, No. 6, pp. 139-145 (in Russian).

[8] V. Klee, 1960. Circumspheres and inner product. *Math. Scand.*, Vol. 8, No. 2, pp. 275-295.

[9] J. R. L. Webb, 1982. Approximation solvability of nonlinear equations. *Nonlinear Analysis Function Spaces and Application*, Vol. 2, (Taubner Texte zur Mathematik, Leipzig), pp. 234-257.