HNUE JOURNAL OF SCIENCE DOI: 10.18173/2354-1059.2022-0020
Natural Science, 2022, Volume 67, Issue 2, pp. 25-39
This paper is available online at http://stdb.hnue.edu.vn

GRADED VERSION OF EXEL'S EFFROS-HAHN CONJECTURE FOR LEAVITT PATH ALGEBRAS

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Abstract. In this paper, we prove a graded version of Exel's Effros-Hahn conjecture for Leavitt path algebras. More concretely, we show that any graded primitive ideal of the Leavitt path algebra is the annihilator of a module induced from a graded simple module over an isotropy group algebra. A graded version of Steinberg's results towards Exel's conjecture in (B. Steinberg, Ideals of étale groupoid algebras and Exel's Effros-Hahn conjecture, *J. Noncommut. Geom.*, Vol. 15, 2021, pp. 829-839) is also obtained for graded ample groupoid algebras.
Keywords: Chen simple module, Exel's Effros-Hahn conjecture, graded ample groupoid, primitive ideal, Steinberg algebra.

1. Introduction

A primitive ideal is the annihilator of a simple module. The original Effros-Hahn conjecture [1, 2] states that every primitive ideal of a crossed product of an amenable locally compact group with a commutative C^*-algebra is induced from a primitive ideal of an isotropy group. This conjecture was verified for discrete groups in [3]. A more general result than the original conjecture as well as analogues in the groupoid setting was also achieved, see [4, 5, 6].

In [7], B. Steinberg associated to each ample groupoid G an algebra $A_R(G)$ over a commutative ring R, which is now called the ample groupoid algebra or the Steinberg algebra. At the PARS conference in Gramado in 2014 (see also [8]), R. Exel conjectured that an analogue of the Effros-Hahn conjecture should hold for ample groupoids. This means that each primitive ideal of a Steinberg algebra should be the annihilator of an irreducible representation (equivalently, a simple module) induced from an isotropy group.

Received May 15, 2022. Revised: June, 20 2022. Accepted June 28, 2022.
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Towards Exel's conjecture, Steinberg [9] proved that each ideal of $A_R(G)$ is an intersection of annihilators of induced representations from isotropy groups, extending a result of Demeneghi for Steinberg algebras over a field in [10]. Consequently, he obtained that every primitive ideal is the annihilator of a single representation induced from an isotropy group, and established Exel's conjecture in the case that all isotropy group rings are left max rings. Steinberg's proof utilizes his Disintegration Theorem [11], which interprets modules over $A_R(G)$ as sheaves of R-modules over G.

Leavitt path algebras, firstly introduced in [12, 13], are prominent examples of Steinberg algebras. Moreover, they are naturally $\mathbb{Z}$-graded. It is known that Exel's conjecture holds for Leavitt path algebras (see Theorem 4.1). In a recent paper [14], L. Vaš presented a new class of graded simple modules over Leavitt path algebras, thereby classifying all the graded simple modules up to the equality of their annihilators.

In this paper, we apply the graded version of the Disintegration Theorem in [15] and of the induction functor in [16] to obtain the corresponding results to those in [9] for graded Steinberg algebras. Moreover, we prove that the graded simple modules defined by Vaš are induced from graded simple modules over isotropy group algebras. Consequently, combining with [14, Theorem 3.8] and the results in [16], we deduce that the graded version of Exel's conjecture also holds for Leavitt path algebras.

Our paper is structured as follows. In the next section, we recall some of the needed basic concepts and results. In Section 3, we verify the graded version of the main results in [9] for graded Steinberg algebras. The graded version of Exel's Effros-Hahn conjecture for Leavitt path algebras is proved in Section 4, with the main results being Theorem 4.2, Corollary 4.1 and Theorem 4.3.

2. Preliminaries

Throughout the paper, R denotes a commutative ring with identity. All modules are assumed to be left modules unless otherwise stated.

2.1. Graded algebras and graded modules

Let Γ be a group. An R-algebra A is called Γ-*graded* if $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$, where each A_γ is an R-submodule of A and $A_\gamma A_\delta \subseteq A_{\gamma\delta}$ for all $\gamma, \delta \in \Gamma$. Then A_γ is called the γ-*homogeneous component* of A, and its nonzero elements are called *homogeneous of degree* γ. If the algebra A is Γ-graded and Γ is clear from context, we say simply that A is a graded algebra. A *graded homomorphism* of Γ-graded algebras is an algebra homomorphism $f : A \to B$ such that $f(A_\gamma) \subseteq B_\gamma$ for all $\gamma \in \Gamma$. If such a homomorphism is bijective, then we say that A and B are *graded isomorphic*.

Let M be a (left) A-module. Then M is said to be *unital* if $AM = M$. If A is a Γ-graded R-algebra, M is called a *graded A-module* if $M = \bigoplus_{\gamma\in\Gamma} M_\gamma$, where each M_γ is an R-submodule of M and $A_\delta M_\gamma \subseteq M_{\delta\gamma}$ for all $\gamma, \delta \in \Gamma$. A graded homomorphism between graded A-modules is an A-module homomorphism $f : M \to N$ such that $f(M_\gamma) \subseteq N_\gamma$ for all $\gamma \in \Gamma$. We denote by $A\text{-}\mathrm{Mod}$ the category of unital A-modules and by $A\text{-}\mathrm{Gr}$ the category of unital graded A-modules with graded homomorphisms. The notions such as graded submodule, graded simple module, etc. are counterparts in $A\text{-}\mathrm{Gr}$ of the familiar concepts in $A\text{-}\mathrm{Mod}$. In particular, graded isomorphic modules M, N will be denoted as $M \cong_{gr} N$. A *graded primitive ideal* of A is the annihilator of a graded simple A-module.

For a graded A-module M and $\alpha \in \Gamma$, the α-*shifted* graded A-module $M(\alpha)$ is

$$M(\alpha) = \bigoplus_{\gamma \in \Gamma} M(\alpha)_\gamma,$$

where $M(\alpha)_\gamma = M_{\gamma\alpha}$. This defines the *shifted functor*: $A\text{-}\mathrm{Gr} \to A\text{-}\mathrm{Gr}, M \mapsto M(\alpha)$.

2.2. Groupoids and Steinberg algebras

A *groupoid* is a small category in which every morphism has an inverse. Let G be a groupoid. The set of objects of G is denoted by $G^{(0)}$ and called the *unit space*, where we identify objects with their identity morphisms. If $x \in G$, then $d(x) = x^{-1}x$ is the *domain* and $c(x) = xx^{-1}$ is the *codomain* of x. Thus we have maps $d, c : G \to G^{(0)}$ such that $xd(x) = x$ and $c(x)x = x$ for all $x \in G$. Moreover, a pair $(x, y) \in G \times G$ is composable (with product written as xy) if and only if $d(x) = c(y)$; we write $G^{(2)}$ for the set of all composable pairs.

Let $x \in G^{(0)}$. We denote by $L_x = d^{-1}(x)$ the set of all morphisms whose domain is x. The set

$$G_x = \{y \in G \mid d(y) = x = c(y)\}$$

is a group with identity x, called the *isotropy group* of G at x. The *orbit* of x is defined to be

$$\mathcal{O}_x = \{y \in G^{(0)} \mid \text{ there exists } z \in G \text{ with } d(z) = x, c(z) = y\}.$$

It is clear that if x and y are in the same orbit, then the isotropy groups G_x and G_y are isomorphic.

A *topological groupoid* is a groupoid endowed with a topology such that the inversion map $G \to G$ and the composition map $G^{(2)} \to G$ are continuous, where $G^{(2)}$ has the relative product topology. In addition, if the map d is a local homeomorphism, then G is called an *étale groupoid*; in this case, c is also a local homeomorphism. An *open bisection* of G is an open subset $U \subseteq G$ such that $d|_U$ and $c|_U$ are homeomorphisms onto an open subset of $G^{(0)}$. An étale groupoid G is called *ample* if it has a topological basis consisting of compact open bisections and $G^{(0)}$ is Hausdorff. Let $B^{\mathrm{co}}(G) = \{U \subseteq G \mid U \text{ is a compact open bisection}\}$.

Definition 2.1 (see [7]). *Let G be an ample groupoid. Define $A_R(G)$ to be the R-submodule of R^G spanned by the set $\{\mathbf{1}_U \mid U \in B^{\mathrm{co}}(G)\}$, where $\mathbf{1}_U$ denotes the* *characteristic function of U. The convolution product of $f, g \in A_R(G)$ is defined by*

$$f * g(x) = \sum_{d(y)=d(x)} f(xy^{-1})g(y) \quad \textit{for all } x \in G.$$

The R-module $A_R(G)$, with the convolution product, is called the Steinberg algebra of G over R.

2.3. Graded groupoids and graded Steinberg algebras

Let Γ be a discrete abelian group with identity ε and G be a topological groupoid. The groupoid G is said to be Γ-*graded* if there is a continuous map $\kappa : G \to \Gamma$ such that $\kappa(xy) = \kappa(x)\kappa(y)$ for all $(x, y) \in G^{(2)}$. Equivalently, G is Γ-*graded* if it decomposes as a disjoint union $\bigsqcup_{\gamma\in\Gamma} G_\gamma$, where G_γ's are clopen subsets of G such that $G_\gamma G_\delta \subseteq G_{\gamma\delta}$ for all $\gamma, \delta \in \Gamma$ (to see the equivalence, one takes $G_\gamma = \kappa^{-1}(\gamma)$). The set G_γ is called the γ-*homogeneous component* of G.

If G is a Γ-graded ample groupoid, then we denote by $B^{\mathrm{co}}_\gamma(G)$ the set of all γ-homogeneous compact open bisections of G. An important observation is that the set of all homogeneous compact open bisections is a basis for the topology on G.

Lemma 2.1. *[17, Lemma 3.1] Let G be a Γ-graded ample groupoid. Then*

$$A_R(G) = \bigoplus_{\gamma\in\Gamma} A_R(G)_\gamma$$

is a Γ-graded algebra, where $A_R(G)_\gamma$ is the R-submodule spanned by the set $\{\mathbf{1}_U \mid U \in B^{\mathrm{co}}_\gamma(G)\}$.

Note that any ample groupoid G admits a trivial grading from the trivial group $\{\varepsilon\}$, which gives rise to a trivial grading on $A_R(G)$.

2.4. Leavitt path algebras

Leavitt path algebras [12, 13] are $\mathbb{Z}$-graded algebras presented by generators and relations that are determined by a directed graph. We review the description of a Leavitt path algebra as a (graded) Steinberg algebra of some graph groupoid. For a comprehensive account of the theory of Leavitt path algebras, we refer to [18].

A (directed) *graph* $E = (E^0, E^1, s, r)$ consists of two sets E^0 and E^1 together with maps $s, r : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 are called *edges* of E. For an edge $e \in E^1$, its *source* and *range* are $s(e)$ and $r(e)$, respectively. A vertex $v \in E^0$ is called a *sink* if it emits no edges, i.e., if $s^{-1}(v) = \emptyset$, and v is called an *infinite emitter* if $s^{-1}(v)$ is an infinite set. A vertex which is a sink or an infinite emitter is said to be *singular*.

A *finite path* in E is a finite sequence of edges $\mu = e_1e_2 \cdots e_n$ with $r(e_i) = s(e_{i+1})$ for all $i = 1, \dots, n-1$. In this case, we set $s(\mu) = s(e_1)$, $r(\mu) = r(e_n)$ and call $|\mu| = n > 0$ the *length* of μ. An *exit* for μ is an edge e such that $s(e) = s(e_i)$ for some i but $e \neq e_i$. The finite path μ is called *closed* if $s(\mu) = r(\mu)$; then μ is said to be based at the vertex $s(\mu)$. Following [19], a closed path μ is called *simple* in case $\mu \neq c^n$ for any closed path c and integer $n \geq 2$. The closed path μ is called a *cycle* if it does not pass through any of its vertices twice. Following [14], a cycle μ is *exclusive* if "no exit returns", i.e., no vertex on μ is the base of a cycle distinct from μ. By convention, a vertex $v \in E^0$ is considered as a finite path of length 0 with v as the source and the range. We denote by $F(E)$ the set of all finite paths (including the paths of length 0) in the graph E.

An *infinite path* in E is an infinite sequence of edges $p = e_1e_2 \cdots$ with $r(e_i) = s(e_{i+1})$ for all i. Again, $s(p) = s(e_1)$ is called the source of p. For instance, if c is a closed path, then $c^\infty = ccc\cdots$ is an infinite path. We denote by E^∞ the set of all infinite paths in E. If $p \in F(E) \cup E^\infty$ and $\mu \in F(E)$ are such that $p = \mu q$ for some $q \in F(E) \cup E^\infty$, then we say that μ is an *initial subpath* of p.

For each $e \in E^1$, we introduce a symbol e^* and call it a ghost edge. We define $s(e^*) = r(e)$ and $r(e^*) = s(e)$. For $v \in E^0$ and $\mu = e_1e_2 \cdots e_n \in F(E)$, let $v^* = v$ and $\mu^* = e_n^* \cdots e_2^*e_1^*$. The *extended graph* $\widehat{E}$ of the graph E is the graph with vertices E^0 and edges $E^1 \cup \{e^* \mid e \in E^1\}$.

Let K be any field. Given an arbitrary graph E, the *path algebra* $P_K(E)$ of E over K is the free K-algebra generated by the set $E^0 \cup E^1$ subject to the relations:

(V) $vw = \delta_{v,w}v$ for all $v, w \in E^0$,

(E) $s(e)e = e = er(e)$ for all $e \in E^1$.

The *Leavitt path algebra* of E over K, denoted $L_K(E)$, is the path algebra of the extended graph $\widehat{E}$ over K subject to the relations:

(CK1) $e^*f = \delta_{e,f}r(e)$ for all $e, f \in E^1$,

(CK2) $v = \sum_{e \in s^{-1}(v)} ee^*$ for each non-singular vertex $v \in E^0$.

It is well-known that the algebra $L_K(E)$ is spanned as a K-vector space by the set $\{\mu\nu^* \mid \mu, \nu \in F(E) \text{ with } r(\mu) = r(\nu)\}$. Moreover, $L_K(E)$ has a canonical $\mathbb{Z}$-grading with the homogeneous component of degree k spanned by

$$\{\mu\nu^* \mid \mu, \nu \in F(E), r(\mu) = r(\nu), |\mu| - |\nu| = k\}.$$

We now describe $L_K(E)$ as a graded Steinberg algebra associated with a graded ample groupoid G_E. Let

$$\partial E = E^\infty \cup \{\mu \in F(E) \mid r(\mu) \text{ is singular}\}.$$

Let $x, y \in \partial E$ and $k \in \mathbb{Z}$. Then x is said to be *tail-equivalent* to y with lag k, denoted $x \sim_k y$, if there exist $\mu, \nu \in F(E)$ and $p \in \partial E$ with $r(\mu) = r(\nu) = s(p)$ such that $x = \mu p, y = \nu p$ and $|\mu| - |\nu| = k$. Thus two paths in ∂E are tail-equivalent if they differ only by initial subpaths, and the lag is the difference of lengths of these subpaths. Following [19], an infinite path is called *rational* if it is tail-equivalent to c^∞ for some closed path c, and called *irrational* otherwise.

It is easy to see that $\sim$ is an equivalence relation on ∂E that respects the partition between finite and infinite paths. For $x \in \partial E$, we denote by $[x]$ the equivalence class of all paths in ∂E which are tail-equivalent (with some lags) to x.

Remark 2.1. *It follows from the definition that two finite paths $x, y \in \partial E$ are tail-equivalent if and only if $r(x) = r(y) = v$, which is either a sink or an infinite emitter. In this case, we have $[x] = [y] = [v]$ consists of all finite paths ending at v.*

The groupoid of the graph E is

$$\begin{aligned} G_E &= \{(x, k, y) \in \partial E \times \mathbb{Z} \times \partial E \mid x \sim_k y\} \\ &= \{(\mu p, |\mu| - |\nu|, \nu p) \mid \mu, \nu \in F(E), p \in \partial E, r(\mu) = r(\nu) = s(p)\}, \end{aligned}$$

with the multiplication, inversion, domain and codomain maps given by

$$\begin{aligned} &(x, k, y)(y, l, z) = (x, k + l, z), \quad (x, k, y)^{-1} = (y, -k, x), \\ &d(x, k, y) = (y, 0, y), \quad c(x, k, y) = (x, 0, x). \end{aligned}$$

The unit space of G_E is $G_E^{(0)} = \{(x, 0, x) \mid x \in \partial E\}$, which will be identified with ∂E via the map $(x, 0, x) \mapsto x$. It is clear that for $x \in \partial E$, the orbit $\mathcal{O}_x$ of x is the same as the equivalence class $[x]$.

For $\mu, \nu \in F(E)$ with $r(\mu) = r(\nu)$ and a finite set $F \subseteq s^{-1}(r(\mu))$, define

$$Z(\mu, \nu) = \{(\mu p, |\mu| - |\nu|, \nu p) \mid p \in \partial E, r(\mu) = s(p)\}$$

and

$$Z((\mu, \nu) \backslash F) = Z(\mu, \nu) \backslash \bigcup_{e \in F} Z(\mu e, \nu e).$$

The sets $Z((\mu, \nu) \backslash F)$ form a basis of compact open bisections for a topology under which G_E is a Hausdorff ample groupoid (see, e.g., [20, Theorem 2.4]). Moreover, the continuous groupoid homomorphism $\kappa : G_E \to \mathbb{Z}, (x, k, y) \mapsto k$ provides G_E with the structure of a $\mathbb{Z}$-graded groupoid, that is,

$$G_E = \bigsqcup_{k \in \mathbb{Z}} G_{E,k} \text{ with } G_{E,k} = \{(x, k, y) \mid x, y \in \partial E, x \sim_k y\}. \tag{2.1}$$

By Lemma 2.1, the graded Steinberg algebra of the $\mathbb{Z}$-graded ample groupoid G_E is

$$A_K(G_E) = \bigoplus_{k\in\mathbb{Z}} A_K(G_E)_k, \text{ where } A_K(G_E)_k = \text{span}_K\{\mathbf{1}_{Z((\mu,\nu)\setminus F)} \mid |\mu| - |\nu| = k\}.$$

By [17, Example 3.2], the Leavitt path algebra $L_K(E)$ is naturally graded isomorphic to $A_K(G_E)$ via the map $\pi : L_K(E) \to A_K(G_E)$ given by

$$\begin{aligned} &\pi(v) = \mathbf{1}_{Z(v,v)} \text{ for all } v \in E^0, \\ &\pi(e) = \mathbf{1}_{Z(e,r(e))}, \quad \pi(e^*) = \mathbf{1}_{Z(r(e),e)} \text{ for all } e \in E^1. \end{aligned} \tag{2.2}$$

In particular, we have $\pi(\mu\nu^*) = \mathbf{1}_{Z(\mu,\nu)}$, where $\mu, \nu \in F(E)$ with $r(\mu) = r(\nu)$. As a result, we may consider (graded) modules over $A_K(G_E)$ as (graded) modules over $L_K(E)$, or vice versa.

3. Graded version of Exel's Effros-Hahn conjecture

Let G be an ample groupoid and $x \in G^{(0)}$. In [7], the induction and the restriction functors between the category of $A_R(G)$-modules and the category of modules over the group algebra RG_x were constructed. It was also shown that the induction functor sends simple modules to simple modules. If G is a graded ample groupoid, then these functors are graded functors between the graded categories (see [16]). Let us recall some needed results from [16].

Let $L_x = d^{-1}(x)$ denotes the set of morphisms starting at x. Let RL_x be the free R-module with basis L_x. The isotropy group G_x acts freely on the right of L_x by composition of morphisms. Thus RL_x is a free right RG_x-module with basis being a transversal for L_x/G_x. Also, a left $A_R(G)$-module structure on RL_x is given by

$$ft = \sum_{d(y)=c(t)} f(y)yt,$$

for $f \in A_R(G)$ and $t \in L_x$, making RL_x an $A_R(G)$-RG_x-bimodule. By [7, Definition 7.9], the induction functor

$$\text{Ind}_x : RG_x\text{-}\text{Mod} \to A_R(G)\text{-}\text{Mod}$$

is given by

$$\text{Ind}_x(N) = RL_x \otimes_{RG_x} N.$$

Now let $G = \bigsqcup_{\gamma\in\Gamma} G_\gamma$ be a Γ-graded ample groupoid. Then $G_x = \bigsqcup_{\gamma\in\Gamma} G_{x,\gamma}$, where $G_{x,\gamma} = G_x \cap G_\gamma$. Thus RG_x is a Γ-graded R-algebra whose γ-homogeneous component is the free R-module with basis $G_{x,\gamma}$; in other words, $RG_x = \bigoplus_{\gamma\in\Gamma} RG_{x,\gamma}$.

Analogously, there is a decomposition $L_x = \bigsqcup_{\gamma\in\Gamma} L_{x,\gamma}$ with $L_{x,\gamma} = L_x \cap G_\gamma$. Then $RL_x = \bigoplus_{\gamma\in\Gamma} RL_{x,\gamma}$ is a graded $A_R(G)$-RG_x-bimodule. Consequently, by [16, Proposition 3.2],

$$\mathrm{Ind}_x : RG_x\text{ - Gr} \to A_R(G)\text{ - Gr}$$

is a functor between the categories of graded modules, and it commutes with the shifted functors. Moreover, Ind_x maps graded simple RG_x-modules to graded simple $A_R(G)$-modules [16, Proposition 3.6].

Theorem 3.1 (see Theorem 3.1 in [16])**.** *Let G be a Γ-graded ample groupoid. Then each spectral graded simple $A_R(G)$-module is of the form $\mathrm{Ind}_x(N)$ for a pair (x, N), where $x \in G^{(0)}$ and N is a graded simple RG_x-module. Two pairs (x, N) and (y, N') give rise to isomorphic graded $A_R(G)$-modules, i.e., $\mathrm{Ind}_x(N) \cong_{gr} \mathrm{Ind}_y(N')$, if and only if x, y are in the same orbit and $N \cong_{gr} N'(\gamma)$ for some $\gamma \in \Gamma$ such that there exists $z \in G_\gamma$ with $d(z) = x, c(z) = y$.*

For an ample groupoid G, Steinberg proved that the category of $A_R(G)$-modules is equivalent to the category of G-sheaves of R-modules [11]. This result (which is called the Disintegration Theorem) was used, for instance, to study Exel's Effros-Hahn conjecture for Steinberg algebras in [9]. A graded version of the Disintegration Theorem was proved in [15]. We cite the following result; for a detailed description of the involved functors and the category $\mathcal{B}_R^{gr}\, G$ of graded G-sheaves of R-modules, we refer to [15, 11].

Proposition 3.1. *[15, Proposition 3.14] Let G be a Γ-graded ample groupoid. Then $\Gamma_c : \mathcal{B}_R^{gr}\, G \to A_R(G)\text{ - Gr}$ and $\mathrm{Sh} : A_R(G)\text{ - Gr} \to \mathcal{B}_R^{gr}\, G$ are mutually inverse equivalences of categories.*

By using Proposition 3.1 and analyzing the proofs of Theorem 5, Theorem 7 and Theorem 8 in [9], we see that they still hold in the case G is a graded ample groupoid and the involved ideals are graded (with minor changes in the proofs). Namely, we have the graded version of these results as follows:

Proposition 3.2 (cf. Theorem 5 in [9])**.** *Let G be a Γ-graded ample groupoid and $\mathcal{E} = (E, p)$ be a graded G-sheaf of R-modules. Then the equality*

$$\mathrm{Ann}(\Gamma_c(\mathcal{E})) = \bigcap_{x\in G^{(0)}} \mathrm{Ann}(\mathrm{Ind}_x(E_x))$$

holds. Consequently, every graded ideal I of $A_R(G)$ is an intersection of annihilators of induced graded modules.

The proof of Proposition 3.2 is the same as that of [9, Theorem 5]. Note that E_x is a graded RG_x-module (see [15, Definition 3.4]).

Theorem 3.2 (cf. Theorem 7 in [9])**.** *Let G be a Γ-graded ample groupoid and let I be a graded primitive ideal of $A_R(G)$. Then $I = \operatorname{Ann}(\operatorname{Ind}_x(M))$ for some $x \in G^{(0)}$ and graded RG_x-module M.*

Proof. The proof is almost the same as that of [9, Theorem 7], where a section s therein is replaced by a homogeneous section as in the sense of [15, Lemma 3.13]. ☐

A Γ-graded ring S is called a graded left max ring if each nonzero graded S-module has a maximal graded (proper) submodule.

Theorem 3.3 (cf. Theorem 8 in [9])**.** *Let G be a Γ-graded ample groupoid such that RG_x is a Γ-graded left max ring for all $x \in G^{(0)}$. Then the graded primitive ideals of RG are exactly the ideals of the form $\operatorname{Ann}(\operatorname{Ind}_x(M))$, where M is a graded simple RG_x-module.*

Proof. The proof is almost the same as that of [9, Theorem 8], where the submodule N therein is taken to be graded. ☐

4. Graded simple modules over Leavitt path algebras

Let E be an arbitrary graph and K an arbitrary field. Let $x \in \partial E$. Recall that

$$L_x = d^{-1}(x) = \{(y,k,x) \mid y \in \partial E, y \sim_k x \text{ for some } k \in \mathbb{Z}\}.$$

By (2.2) and the bimodule structure of KL_x, we obtain immediately as follows:

Lemma 4.1. *The $L_K(E)$-module structure on KL_x is given by*

$$(\mu\nu^*).(y,k,x) = \mathbf{1}_{Z(\mu,\nu)}(y,k,x) = \begin{cases} (\mu p, |\mu| - |\nu| + k, x) & \textit{if } y = \nu p \ (p \in \partial E), \\ 0 & \textit{else.} \end{cases}$$

In particular, if $(y,k,x) = (\mu p, k, \nu p) \in L_x$, then $(y,k,x) = \mu\nu^*(x,0,x)$.

Moreover, the $\mathbb{Z}$-grading on G_E (2.1) induces a $\mathbb{Z}$-grading on KL_x, namely

$$KL_x = \bigoplus_{k \in \mathbb{Z}} KL_{x,k} \text{ with } L_{x,k} = L_x \cap G_{E,k} = \{(y,k,x) \in G_E\}.$$

We now review the construction of Chen simple modules over the Leavitt path algebra $L_K(E)$ in [19], [21] and [22] under the groupoid approach. Let $x \in \partial E$. Let $V_{[x]}$ be the K-vector space having the equivalence class $[x]$ as a basis. In view of Remark 2.1, we may assume that x is a singular vertex or an infinite path in E. Then $V_{[x]}$ is a left $L_K(E)$-module by defining, for all $p \in [x]$ and $v \in E^0, e \in E^1$,

$$\begin{aligned} v.p &= \delta_{v,s(p)} p\,, \\ e.p &= \delta_{r(e),s(p)} ep\,, \\ e^*.p &= \begin{cases} p', & \text{if } p = ep' \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

In addition, if p is a sink or an infinite emitter, we define $e^*.p = 0$. By [23, Proposition 3.6] (see also [16, Corollary 4.6]), the module $V_{[x]}$ is graded (so graded simple) if and only if x is not a rational path.

More concretely, there are four types of Chen simple modules (with the notations as in [19, 21, 22]) over $L_K(E)$:

(1) The infinite-path type $V_{[x]}$, where x is an infinite path in E.

(2) The twisted type $V_{[x]}^f$, where x is an infinite rational path. This is the module which is twisted from $V_{[x]}$ by an irreducible polynomial f in the Laurent polynomial ring $K[t, t^{-1}]$. This module is simple but not graded simple. We refer to [21] for details.

(3) The sink type $N_v = V_{[x]}$, where $x = v$ is a sink in E.

(4) The infinite-emitter type $N_v = V_{[x]}$, where $x = v$ is an infinite emitter in E.

By [21, Theorem 5.9], any primitive ideal of $L_K(E)$ is the annihilator of some Chen simple module. On the other hand, these Chen simple modules were shown to be modules induced from simple modules over isotropy group algebras (see [16, Propositions 4.1, 4.2, Corollaries 4.3, 4.5]). Therefore, we deduce that Excel's Effros-Hahn conjecture holds for Leavitt path algebras, that is:

Theorem 4.1. *Let E be an arbitrary graph and K an arbitrary field. Then any primitive ideal of the Leavitt path algebra $L_K(E)$ is the annihilator of a module induced from a simple module over an isotropy group algebra.*

Among the four types of Chen simple modules above, the types (3), (4), and type (1) for x being an irrational path are graded simple modules. By [16, Proposition 4.1], these modules are induced from graded simple modules over isotropy group algebras. In a recent paper [14], L. Vaš introduced a new type of graded simple $L_K(E)$-modules, denoted N_c^v, by using the notion of graded branching systems. Then it was proved that any graded primitive ideal of $L_K(E)$ is the annihilator of some graded Chen module or the graded module of type N_c^v. We now show that the module N_c^v is also induced from a graded simple module over an isotropy group.

Assume c is an exclusive cycle in the graph E and v is a vertex of c. Let Y be the set of the basis elements of the path algebra $P_K(\widehat{E})$ of the extended graph $\widehat{E}$ which have the form pq^*, where p, q are finite paths in E with $s(q) = v$ and $r(p) = r(q)$ being a vertex of c. Note that c is exclusive implies all edges of q must be in c. An element $pq^* \in Y$ is said to be *not reduced* if p and q have positive lengths and they end with the same edge e. There is a *reduction function* red on Y defined by

$$\mathrm{red}(ee^*q^*) = q^* \text{ and } \mathrm{red}(pee^*q^*) = \mathrm{red}(pq^*)$$

for an edge e in c (see [14]). An element $pq^* \in Y$ is said to be *reduced* if $\mathrm{red}(pq^*) = pq^*$. Then N_c^v is the K-vector space with basis

$$\{pq^* \in Y \mid pq^* \text{ is reduced}\}$$

and the $L_K(E)$-module structure on N_c^v is given by

$$w \cdot pq^* = \delta_{w,s(p)} pq^*;$$
$$e \cdot pq^* = \begin{cases} \mathrm{red}(epq^*) & \text{if } r(e) = s(p), \\ 0 & \text{otherwise;} \end{cases}$$
$$e^* \cdot pq^* = \begin{cases} \mathrm{red}(\mu\nu^*) & \text{if } pq^* = \mathrm{red}(e\mu\nu^*) \text{ for some } e\mu\nu^* \in Y, \\ 0 & \text{otherwise} \end{cases}$$

for $w \in E^0$ and $e \in E^1$. A basis element pq^* is defined to be homogeneous of degree $|p| - |q|$.

Using the above notations, we have the following auxiliary results.

Lemma 4.2. *[14, Lemma 3.4] If $pq^* \in Y$ and $s(p)$ is a vertex of c, then $p^* \cdot \mathrm{red}(pq^*) = q^*$.*

Lemma 4.3. *If $pq^* \in Y$, then $p \cdot q^* = \mathrm{red}(pq^*)$.*

Proof. We use induction on the length $|p|$. If $|p| = 0$, then $p = r(q)$ is a vertex of c and $p \cdot q^* = q^* = \mathrm{red}(pq^*)$. Assuming the induction hypothesis for $pq^* \in Y$, we prove the claim for $epq^* \in Y$ with $e \in E^1$ and $r(e) = s(p)$. Indeed, if pq^* is reduced, then

$$(ep) \cdot q^* = e \cdot (p \cdot q^*) = e \cdot (\mathrm{red}(pq^*)) = e \cdot (pq^*) = \mathrm{red}(epq^*).$$

If pq^* is not reduced, then $p = p_1 e_1 \cdots e_t$ and $q = q_1 e_1 \cdots e_t$ for some edges $e_1, \ldots, e_t$ in c, so that $\mathrm{red}(pq^*) = p_1 q_1^*$ is reduced. Hence

$$(ep) \cdot q^* = e \cdot (p \cdot q^*) = e \cdot (\mathrm{red}(pq^*)) = e \cdot (p_1 q_1^*) = \mathrm{red}(ep_1 q_1^*)) = \mathrm{red}(epq^*).$$

□

When c is an exclusive cycle, by [24, Proposition 4.2], the isotropy group of G_E at c^∞ is

$$(G_E)_{c^\infty} = \{(c^\infty, kn, c^\infty) \mid k \in \mathbb{Z}\}$$

with $n = |c|$. Consequently, $K(G_E)_{c^\infty}$ is $\mathbb{Z}$-graded isomorphic to $K[t^n, t^{-n}]$, where $K[t^n, t^{-n}]$ is the graded subalgebra of the Laurent polynomial algebra $K[t, t^{-1}]$ with support $n\mathbb{Z}$ (i.e., its m-homogeneous components is Kt^m if $m \in n\mathbb{Z}$, and is zero otherwise). The algebra $K[t^n, t^{-n}]$ is graded simple, so the only nonzero graded ideal of $K(G_E)_{c^\infty}$ is $K(G_E)_{c^\infty}$ itself. It follows that up to graded isomorphisms, all the graded simple $K(G_E)_{c^\infty}$-modules are the shifted modules $K[t^n, t^{-n}](m)$ for $m \in \mathbb{Z}$. Moreover, it suffices to take $m \in \{0, 1, \ldots, n-1\}$, as $K[t^n, t^{-n}]$ and $K[t^n, t^{-n}](k)$ are graded isomorphic for all $k \in n\mathbb{Z}$. Since the induction functor sends graded simple modules to graded simple modules by [16, Proposition 3.6], the modules $\mathrm{Ind}_{c^\infty}(K[t^n, t^{-n}](m))$ are graded simple $L_K(E)$-modules. These modules are not simple, as $K[t^n, t^{-n}]$ is not a simple $K[t^n, t^{-n}]$-module.

Theorem 4.2. *Let c be an exclusive cycle based at a vertex v in E. Then the graded $L_K(E)$-modules $\operatorname{Ind}_{c^\infty}(K(G_E)_{c^\infty})$ and N_c^v are graded isomorphic.*

Proof. Since

$$\operatorname{Ind}_{c^\infty}(K(G_E)_{c^\infty}) = KL_{c^\infty} \otimes_{K(G_E)_{c^\infty}} K(G_E)_{c^\infty} \cong KL_{c^\infty},$$

we need to show that $KL_{c^\infty} \cong N_c^v$ as $\mathbb{Z}$-graded modules.

Consider the K-linear map $\varphi : KL_{c^\infty} \longrightarrow N_c^v$ given by

$$\varphi(y, k, c^\infty) = \operatorname{red}(pq^*),$$

where $p, q \in F(E)$ and $z \in \partial E$ are such that $y = pz, c^\infty = qz, |p| - |q| = k$. Note that $z = c_i^\infty$ for some i since q is a subpath of c^m for some $m \geq 1$. We claim that φ is well-defined. Indeed, let $p_1, p_2, q_1, q_2 \in F(E)$ and $z_1, z_2 \in \partial E$ be such that $y = p_1 z_1 = p_2 z_2, c^\infty = q_1 z_1 = q_2 z_2, |p_1| - |q_1| = |p_2| - |q_2| = k$. We may assume $|p_1| \geq |p_2|$, so $p_1 = p_2 q$ for some $q \in F(E)$. Then $z_2 = q z_1$, thus $q_1 z_1 = q_2 q z_1$. Moreover, $|p_1| - |q_1| = |p_2| - |q_2|$ implies $|q_1| - |q_2| = |q|$. Hence $q_1 = q_2 q$. Therefore,

$$\operatorname{red}(p_1 q_1^*) = \operatorname{red}(p_2 q q^* q_2^*) = \operatorname{red}(p_2 q_2^*),$$

so φ is well-defined. We also define the K-linear map $\psi : N_c^v \longrightarrow KL_{c^\infty}$ given by

$$\psi(pq^*) = (pc_i^\infty, |p| - |q|, c^\infty),$$

where $c^\infty = qc_i^\infty$. Now we check that φ and ψ are inverses of eath other. For $(y, k, c^\infty) = (pc_i^\infty, |p| - |q|, qc_i^\infty) \in L_{c^\infty}$, we may write $p = \mu e_1 e_2 \cdots e_t$ and $q = \nu e_1 \cdots e_t$, where no edges of μ are in c and $r(\mu) = r(\nu) = s(e_1) = j, r(e_t) = i$ are vertices of c. Then $pq^* = (\mu e_1 \cdots e_t)(e_t^* \cdots e_1^* \nu^*)$, so that $\operatorname{red}(pq^*) = \mu\nu^*$. Thus

$$\psi\varphi(y, k, c^\infty) = \psi(\operatorname{red}(pq^*)) = \psi(\mu\nu^*) = (\mu c_j^\infty, |\mu| - |\nu|, c^\infty) = (\mu c_j^\infty, |p| - |q|, c^\infty).$$

Observe that

$$\mu c_j^\infty = \mu e_1 e_2 \cdots e_t c_i^\infty = pc_i^\infty = y.$$

Hence $\psi\varphi$ is the identity map. On the other hand, for $pq^* \in N_c^v$ and $c^\infty = qc_i^\infty$, we have

$$\varphi\psi(pq^*) = \varphi(pc_i^\infty, |p| - |q|, c^\infty) = \operatorname{red}(pq^*) = pq^*.$$

Thus φ and ψ are inverses of each other; moreover, it is clear that they preserve the gradings on KL_{c^∞} and N_c^v. Since KL_{c^∞} is a cyclic module generated by $(c^\infty, 0, c^\infty)$ [7, Corollary 7.11], to show that φ is an $L_K(E)$-homomorphism it suffices to check that

$$\varphi(\mu\nu^*.(c^\infty, 0, c^\infty)) = \mu\nu^* \cdot \varphi((c^\infty, 0, c^\infty)) \tag{4.1}$$

for all $\mu, \nu \in F(E)$ with $r(\mu) = r(\nu)$. Indeed, by Lemma 4.1, we have

$$\mu\nu^*.(c^\infty, 0, c^\infty) = \begin{cases} (\mu z, |\mu| - |\nu|, c^\infty) & \text{if } c^\infty = \nu z \\ 0 & \text{else.} \end{cases}$$

Consequently, if ν is a subpath of c^∞, then the left-hand side of (4.1) equals $\mathrm{red}(\mu\nu^*)$. Meanwhile, its right-hand side equals

$$\mu\nu^* \cdot \mathrm{red}(\nu\nu^*) = \mu \cdot (\nu^* \cdot \mathrm{red}(\nu\nu^*)) = \mu \cdot \nu^* = \mathrm{red}(\mu\nu^*),$$

by Lemmas 4.2 and 4.3. If ν is not a subpath of c^∞, then both sides of (4.1) are zero. Hence we conclude that $\mathrm{Ind}_{c^\infty}(K(G_E)_{c^\infty})$ and N_c^v are graded isomorphic. □

As a consequence of Theorems 4.2 and 3.1, we recover [14, Proposition 3.6(2)–(5)] with a different proof.

Corollary 4.1. *Let c be an exclusive cycle based at a vertex v in the graph E. Then we have the following:*

1. *The $L_K(E)$-module N_c^v is graded simple and not simple.*
2. *If c has more than one vertex and if $w \neq v$ are vertices of c, then N_c^v and $N_c^w(m)$ are graded isomorphic, where m is the length of the path from v to w in c.*
3. *If d is an exclusive cycle and w is a vertex of d, the modules N_c^v and N_d^w are graded isomorphic if and only if $c = d$ and $v = w$. The modules N_c^v and N_d^w are isomorphic if and only if $c = d$.*
4. *The module N_c^v is not isomorphic to any of the Chen simple modules.*

Proof. (1) $\mathrm{Ind}_{c^\infty}(K(G_E)_{c^\infty}) \cong_{gr} N_c^v$ and $\mathrm{Ind}_{c^\infty}(K(G_E)_{c^\infty})$ is a graded simple, non-simple $L_K(E)$-module.

(2) and (3) Let d be the rotation of c which is based at w. By Theorem 4.2, we have

$$N_c^w(m) = N_d^w(m) \cong_{gr} \mathrm{Ind}_{d^\infty}(K(G_E)_{d^\infty})(m) \cong_{gr} \mathrm{Ind}_{d^\infty}(K(G_E)_{d^\infty}(m)).$$

On the other hand, $(d^\infty, -m, c^\infty) \in G_{E,-m}$ and $K(G_E)_{c^\infty} \cong_{gr} K(G_E)_{d^\infty}$ as graded $K(G_E)_{c^\infty}$-modules. Therefore, it follows from Theorem 3.1 that

$$N_c^v \cong_{gr} \mathrm{Ind}_{c^\infty}(K(G_E)_{c^\infty}) \cong_{gr} \mathrm{Ind}_{d^\infty}(K(G_E)_{d^\infty}(m)) \cong_{gr} N_c^w(m).$$

Now (3) is clear.

(4) The Chen simple modules are induced from different orbits of $x \in \partial E$ by [16, Proposition 4.1] (the twisted Chen modules are not graded).

□

Finally, it follows from [16, Proposition 4.1], [14, Theorem 3.8] and Theorem 4.2 that the graded version of Excel's Effros-Hahn conjecture holds for Leavitt path algebras.

Theorem 4.3. *Let E be an arbitrary graph and K an arbitrary field. Then any graded primitive ideal of the Leavitt path algebra $L_K(E)$ is the annihilator of a module induced from a graded simple module over an isotropy group algebra.*

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