HNUE JOURNAL OF SCIENCE DOI: 10.18173/2354-1059.2021-0027
Natural Science, 2021, Volume 66, Issue 2, pp. 48-56
This paper is available online at http://stdb.hnue.edu.vn

REMARKS ON DECAYING SOLUTIONS TO DISTRIBUTED ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we present results on the asymptotic behavior of the solutions to distributed order differential equations in Euclidean spaces. The paper develops some recent results in [1] on fractional differential equations to the case of more general nonlocal derivatives. We utilize the detailed behavior of the relaxation functions instead of the Mittag-Leffler functions. The typical decaying rate is logarithmic due to the ultra-slow phenomenon.
Keywords: distributed order differential equation; mild solution; asymptotic behavior, logarithmic decaying rate.

1. Introduction

Consider a distributed order equation of the following form

$$\begin{cases} \mathbb{D}_t^{(\mu)} u = f(t, u(t)), t > 0 \\ u(0) = u_0 \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

where the function $u = (u_1, u_2, \dots, u_d) \colon \mathbb{R}_+ \to \mathbb{R}^d$, $\mathbb{D}_t^{(\mu)}$ is the distributed order differential operator with weight $\mu(t)$ which acts componentwise on u

$$\mathbb{D}_t^{(\mu)} u_i(t) = \int_0^t k(t-\tau) u_i'(\tau) d\tau,\ 1 \le i \le d, \text{ where } k(s) = \int_0^1 \frac{s^{-\alpha}}{\Gamma(1-\alpha)} \mu(\alpha) d\alpha, s > 0;$$

the nonlinearity $f \colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is global Lipschitz continuous and $f(t, 0) = 0$.

The study of nonlocal in time differential equations has been of great attraction in the last decades due to its intense connection with nonlocal transport phenomena, control

Received May 3, 2021. Revised June 7, 2021. Accepted June 14, 2021.
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of stochastic jump processes, description of anomalous diffusion in physics, and memory effects in parabolic equations, see [2] and the references therein. In particular, distributed fractional derivatives appear in various applications describing certain technical or physical processes: in the theory of viscoelasticity [3], in the kinetic theory [4], from a mathematical point of view, the investigation on existence, uniqueness and qualitative behavior of solutions is also of interest, let us mention the works of Pskhu [5], Kochubei [6], Umarov and Gorenflo [7].

In a special case $k(t) = g_{1-\alpha}(t) := t^{-\alpha}/\Gamma(1-\alpha), \alpha \in (0,1)$, the term $\frac{d}{dt}[k*(u-u_0)]$ represents the Caputo fractional derivative of order α, the corresponding equation

$$D_C^{\alpha}u(t) = f(t,u(t)), t > 0, u(0+) = u_0, \tag{1.2}$$

has been studied extensively. We refer to [1, 8] for fractional differential equations on finite dimensional case and [2] for results on existence, uniqueness, certain asymptotic behavior of the solutions to nonlocal differential equations on Hilbert spaces. Moreover, in [1], the author showed that the nontrivial solutions to a fractional differential equation cannot converge to the fixed points faster than $t^{-\alpha}$, where α is the order of the Caputo derivative.

In this paper, we aim at generalizing such an asympotic behavior to the case of distributed order differential equations. We assume through this work the following conditions:

(W) The weight $\mu(t) \in C^3[0,1]$ and either $\mu(1) \neq 0$ or $\mu(t) \sim at^{\nu}$ as $t \to 0^+$ for constants $a > 0$ and $\nu > 0$. By convention, we will write (W) as $\mu(t) \sim at^{\nu}$ as $t \to 0^+$ for constants $a > 0$ and $\nu \geq 0$.

(F1) The nonlinearity satisfies that $f\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and $f(t,0) = 0$.

(F2) f is global Lipschitz continuous with respect to the second variable, that is, there exists a constant $L > 0$ such that

$$\|f(t,u) - f(t,v)\| \leq L\|u - v\|, \text{ for all } t \geq 0, u, v \in \mathbb{R}^d. \tag{1.3}$$

Our main results are Theorem 3.1 and Theorem 3.2 about the existence and nonexistence of decaying solutions.

The rest of the present work is organized as follows: in Section 2, we summarize important asymptotic behavior of the kernels k, $\varkappa$ and relaxation function $u_\lambda(t)$; and existence theorem of mild solutions to Equation (1.1); in Section 3, we state and prove our main results (Theorem 3.1 and Theorem 3.2) and some auxiliary lemmas.

2. Preliminaries

2.1. Important asymptotic behavior

In this part, we summarize important facts about the asymptotic behavior of the kernel and the relaxation function, which will be used to investigate the behavior of the solutions to Problem (1.1).

Consider the eigenvalue problem

$$\begin{cases} \mathbb{D}_t^{(\mu)} u_\lambda(t) = \lambda u_\lambda(t), t > 0, \\ u_\lambda(0) = 1, \end{cases} \tag{2.1}$$

where $\lambda \in \mathbb{R}$.

Proposition 2.1. *[6, Proposition 2.1] If $\mu \in C^3[0,1]$, $\mu(1) \neq 0$, then as $s \to 0$*

$$\begin{aligned} k(s) &\sim s^{-1}(\log s)^{-2}\mu(1), \\ k'(s) &\sim s^{-2}(\log s)^{-2}\mu(1). \end{aligned} \tag{2.2}$$

It follows from (2.2) that $k \in L_1(0,T)$, but $k(t) \notin L_p(0,T)$ for any $p > 1$.

Applying the Laplace transform to (2.1), denote by $\mathcal{K}(p) = \mathcal{L}(k(t))(p)$ we have the Laplace transform of $u_\lambda(t)$:

$$\widetilde{u}_\lambda(t) = \frac{\mathcal{K}(p)}{p\mathcal{K}(p) - \lambda}.$$

Therefore,

$$u_\lambda(t) = \frac{d}{dt}\left(\frac{1}{2\pi \mathrm{i}}\int_{\Re p=\gamma} \frac{e^{pt}}{p}\frac{\mathcal{K}(p)}{p\mathcal{K}(p) - \lambda}dp\right). \tag{2.3}$$

Using these expressions, we recall the asymptotic behaviors of $\mathcal{K}(p)$ and $u_\lambda(t)$.

Proposition 2.2. *[6, Proposition 2.2]*

1. *If $\mu \in C^3[0,1]$ then if $p \in \mathbb{C} \setminus \mathbb{R}_-$ and $|p| \to \infty$, it holds*

$$\mathcal{K}(p) = \frac{\mu(1)}{\log p} - \frac{\mu'(1)}{(\log p)^2} + O(\frac{1}{(\log |p|)^3}).$$

2. *If $\mu(t)$ satisfies* (W) *then if $p \in \mathbb{C} \setminus \mathbb{R}_-$ and $p \to 0$, it holds*

$$\mathcal{K}(p) \sim a\Gamma(1+\nu)p^{-1}\left(\log\frac{1}{p}\right)^{-1-\nu}.$$

These asymptotic expansions and a version of the Karamata-Feller Tauberian theorem allow us to get the asymptotic behaviors of the function k and $u_\lambda(t)$.

Theorem 2.1. *[6, Theorem 2.3]*

1. *The function $u_\lambda(t)$ is continuous at the origin $t = 0$ and belongs to $C^\infty(0,\infty)$.*

2. *If $\lambda < 0$, then $u_\lambda(t)$ is completely monotone.*

3. *Let $\lambda < 0$. If $\mu(t)$ satisfies* (W), *then*
$$u_\lambda(t) \sim C(\log t)^{-1-\nu}, \text{ as } t \to \infty. \tag{2.4}$$

Consider the linear problem
$$\begin{cases} \mathbb{D}_t^{(\mu)} u(t) = g(t), t > 0, \\ u(0) = 0, \end{cases} \tag{2.5}$$

Applying again the Laplace transform, we arrive at the formula of the Laplace transform of the solution
$$\mathcal{L}(u)(p) = \frac{1}{p\mathcal{K}(p)} \mathcal{L}(g)(p).$$

From Proposition 2.2, the asymptotic behavior of $\dfrac{1}{p\mathcal{K}(p)}$ guarantees that it is the Laplace transform of a function. This function is determined by the following expression:
$$\varkappa(t) = \frac{d}{dt}\left(\frac{1}{2\pi \mathrm{i}} \int_{\Re p = \gamma} \frac{e^{pt}}{p} \frac{1}{p\mathcal{K}(p)} dp\right), \gamma > 0. \tag{2.6}$$

By definition, it is obvious that $k * \varkappa = 1$. We recall the asymptotic behavior of $\varkappa(t)$.

Proposition 2.3. *[6, Proposition 3.1] Let $\mu(t)$ satisfy the condition* (W) *. Then*

1. *$\varkappa(t) \in C^\infty(0, \infty)$ and $\varkappa$ is completely monotone.*
2. *As $t \to 0^+$*
$$\varkappa(t) \le C \log \frac{1}{t}, \text{ and } |\varkappa'(t)| \le C t^{-1} \log \frac{1}{t}. \tag{2.7}$$

By (2.7), $\varkappa \in L_{1,\mathrm{loc}}(0, \infty)$. We can rewrite Problem (2.1) in the integral form
$$u_\lambda(t) = 1 + \lambda \int_0^t \varkappa(t - \tau) u_\lambda(\tau) d\tau. \tag{2.8}$$

2.2. Gronwall type inequality

We first recall the definitions of relaxation functions $s(t, \mu), r(t, \mu)$ for general pair of kernel (k, ℓ) such that $k * l(t) = 1$ for $t > 0$ and k is a nonnegative, nondecreasing function on $\mathbb{R}_+$. For a given $\mu \in \mathbb{R}$, denote by $s(t, \mu), r(t, \mu)$ the unique solution of the following Volterra integral equations, respectively:
$$s(t, \mu) + \mu \int_0^t l(t - \tau) s(\tau, \mu) d\tau = 1, t > 0$$
$$r(t, \mu) + \mu \int_0^t l(t - \tau) r(\tau, \mu) d\tau = l(t), t > 0.$$

Note that in our special case of the distributed order, $l(t) = \varkappa(t)$ and $s(t, \lambda) = u_{-\lambda}(t)$.

Proposition 2.4. *[2, Lemma 2.2] Let v be a nonnegative function satisfying*

$$v(t) \le s(t,\mu)v_0 + \int_0^t r(t-\tau,\mu)[\alpha v(\tau) + \beta(\tau)]d\tau,\ t \ge 0,$$

for $\mu > 0, v_0 \ge 0, \alpha > 0$ and $\beta \in L^1_{loc}(\mathbb{R}^+)$. Then

$$v(t) \le s(t,\mu-\alpha)v_0 + \int_0^t r(t-\tau,\mu-\alpha)\beta(\tau)d\tau.$$

Particularly, if β is constant then

$$v(t) \le s(t,\mu-\alpha)v_0 + \frac{\beta}{\mu-\alpha}(1 - s(t,\mu-\alpha)).$$

2.3. Existence result

Definition 2.1. *A function $u \in C([0,T];\mathbb{R}^d)$ is said to be a mild solution to* (1.1) *on $[0,T]$ if and only if it satisfies the Volterra integral equation*

$$u(t) = u_0 + \int_0^t \varkappa(t-\tau)f(\tau,u(\tau))d\tau.$$

Let us define an operator $\Phi\colon C([0,T];\mathbb{R}^d) \to C([0,T];\mathbb{R}^d)$ as follows:

$$\Phi(u)(t) = u_0 + \int_0^t \varkappa(t-\tau)f(\tau,u(\tau))d\tau.$$

It is obvious that a mild solution to (1.1) if and only if u is a fixed point of Φ. We obtain a global existence theorem by fixed point argument.

Theorem 2.2. *Assume that f satisfies (F1) and (F2). Then Problem* (1.1) *has a unique solution $U(\cdot,u_0) \in C([0,T];\mathbb{R}^d)$.*

Proof. Because $f\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous, Φ is well-defined on $C([0,T];\mathbb{R}^d)$. Fix a positive $\omega > 0$ such that

$$L\int_0^T \varkappa(t)e^{-\omega t}dt < 1/2. \tag{2.9}$$

We furnish the space $C([0,T];\mathbb{R}^d)$ with the following equivalent norm:

$$\|u\|_\omega = \sup_{t\in[0,T]} \|u(t)\|e^{-\omega t}, u \in C([0,T];\mathbb{R}^d). \tag{2.10}$$

Using the global Lipschitz continuity of f, we can estimate

$$\begin{aligned} e^{-\omega t}\|\Phi(u)(t) - \Phi(v)(t)\| &= e^{-\omega t}\left\|\int_0^t \varkappa(t-\tau)\big[f(\tau,u) - f(\tau,v)\big]d\tau\right\| \\ &\le L\sup_{s\in[0,t]}\|u(s)-v(s)\|e^{-\omega s}\int_0^t \|\varkappa(t-\tau)\|e^{-\omega(t-\tau)}d\tau \\ &\le L\|u-v\|_\omega \int_0^T \varkappa(s)e^{-\omega s}ds. \end{aligned}$$

Therefore, thanks to (2.9), $\|\Phi(u)-\Phi(v)\|_\omega \leq \frac{1}{2}\|u-v\|_\omega$. So Φ is a contraction mapping, it has a unique solution $U(t,u_0) \in C([0,T];\mathbb{R}^d)$. $\square$

Remark 2.1. *By this result, Problem* (1.1) *has a unique solution in each* $[0,T]$. *In particular, if all the assumptions hold in* $\mathbb{R}_+$ *then the solution is defined globally in* $\mathbb{R}_+$. *In the next section, we investigate the asymptotic behavior of this global solutions.*

3. Asymptotic behavior of the solutions

In this part, let us consider first a special form of the nonlinearity

$$f(t,u) = -bu + h(t,u), \tag{3.1}$$

where b is positive constant and the function $h\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous with respect to the second variable with the Lipschitz constant $L_h < b$ and $h(t,0)=0$.

Theorem 3.1. *The zero solution of Problem* (1.1) *is globally asymptotically stable with the logarithmic decaying rate. More precisely, for an arbitrary* $u_0 \in \mathbb{R}^d$, *there exists a constant* $C > 0$ *such that*

$$\|U(t,u_0)\| \leq C(\log t)^{-1-\nu}. \tag{3.2}$$

Furthermore, this decaying rate is optimal.

Proof. By Definition 2.1 of mild solution and fundamental properties of eigenfunction, the global solution $U(t,u_0)$ fulfills the following integral identity:

$$U(t,u_0) = u_{-b}(t)u_0 + \int_0^t r_b(t-s)h(s,U(s,u_0))ds, \tag{3.3}$$

where $r_b(t)$ is given by

$$r_b(t) = -\frac{1}{b}\frac{du_{-b}(t)}{dt}. \tag{3.4}$$

Moreover, $r_b(t) \geq 0$ due to Theorem 2.1. Hence, we get

$$\|U(t,u_0)\| \leq u_{-b}(t)\|u_0\| + L_h \int_0^t r_b(t-s)\|U(s,u_0)\|ds. \tag{3.5}$$

Set $w(t) = \|U(t,u_0)\| \geq 0$ then w satisies the integral inequality

$$w(t) \leq u_{-b}(t) + L_h \int_0^t r_b(t-s)w(s)ds, t > 0.$$

Applying Gronwall type inequality (Lemma 2.4), we finally gain

$$w(t) \leq u_{-b+L_h}(t)\|u_0\|, \text{ for all } t > 0.$$

Using the asymptotic expansion of $u_{-b+L_h}(t)$ with $-b+L_h<0$, we conclude that there exists $C>0$ such that

$$\|U(t,u_0)\| \leq C(\log t)^{-1-\nu}, \text{ for all } t>0. \tag{3.6}$$

In the case $f(t,u) = -bu + \tilde{b}u$ with $b > \tilde{b} > 0$ then $U(t,u_0) = u_{-b+\tilde{b}}(t)u_0$. Since $u_{-b+\tilde{b}}(t) \sim C(\log t)^{-1-\nu}$ as $t \to +\infty$, the decaying rate $C(\log t)^{-1-\nu}$ is optimal.

□

Lemma 3.1. *For a given* $\rho>0$, $\lim\limits_{t\to+\infty}\displaystyle\int_{t-\rho}^{t} \varkappa(s)ds = 0.$

Proof. We have $\displaystyle\int_{t-\rho}^{t}\varkappa(s)ds = 1*\varkappa(t) - 1*\varkappa(t-\rho)$. By [6, Theorem 4.3 (i)], we obtain

$$1*\varkappa(t) = \frac{m(t)}{2n(2\pi)^n} \sim C(\log t)^{\nu+1} \text{ as } t\to+\infty. \tag{3.7}$$

Therefore, we conclude that as $t\to+\infty$

$$\begin{aligned}
\int_{t-\rho}^{t}\varkappa(s)ds &= C\left[(\log t)^{1+\nu} - (\log(t-\rho)^{1+\nu}\right] \\
&\leq C(\nu+1)(\log t)^{\nu}[\log t - \log(t-\rho)] \\
&= C(\nu+1)(\log t)^{\nu}\left[-\log\left(1-\frac{\rho}{t}\right)\right] \\
&\sim C(\nu+1)(\log t)^{\nu}\frac{\rho}{t}.
\end{aligned}$$

Consequently,

$$\lim_{t\to+\infty}\int_{t-\rho}^{t}\varkappa(s)ds \leq \lim_{t\to+\infty} C(1+\nu)(\log t)^{\nu}\frac{\rho}{t} = 0.$$

□

Lemma 3.2. *Every nontrivial solution of* (1.1) *does not converge to the equilibrium with exponential rate.*

Proof. Assume that the nontrivial solution $U(t,u_0)$ converges to the origin with an exponential rate, namely, there exist positive constants λ_0 and T_1 such that

$$\|U(t,u_0)\| < e^{-\lambda_0 t}, \text{ for all } t\geq T_1. \tag{3.8}$$

Let K be a constant such that $K\|u_0\|>1$. Moreover, according to (2.4), there exists a positive constant T_2 such that

$$e^{-\lambda_0 t} < \frac{u_{-L}(t)}{K}, \text{ for all } t\geq T_2. \tag{3.9}$$

Set $T_0 = \max\{T_1, T_2\}$. By Definition 2.1 of a mild solution, we have

$$u_0 = U(t, u_0) - \int_0^t \varkappa(t-\tau) f(\tau, U(\tau, u_0)) d\tau.$$

Combining with the Lipschitz continuity of f: $\|f(\tau, U(\tau, u_0)) - f(t, 0)\| \le L\|U(\tau, u_0)\|$, and (3.8), (3.9), we obtain

$$\begin{aligned}\|u_0\| &\le \|U(t, u_0)\| + L \int_0^t \varkappa(t-\tau)\|U(\tau, u_0)\| d\tau \\ &\le \|U(t, u_0)\| + L \left(\int_0^{T_0} \varkappa(t-\tau)\|U(\tau, u_0)\| d\tau + \int_{T_0}^t \varkappa(t-\tau)\|U(\tau, u_0)\| d\tau \right) \\ &\le \|U(t, u_0)\| + L \sup_{[0,T_0]} \|U(\tau, u_0)\| \int_0^{T_0} \varkappa(t-\tau) d\tau + \frac{L}{K} \int_0^t \varkappa(t-\tau) u_{-L}(t) d\tau.\end{aligned} \tag{3.10}$$

Moreover, by (2.8)

$$\int_0^t \varkappa(t-\tau) u_{-L}(t) d\tau = \frac{1 - u_{-L}(t)}{L}.$$

Therefore, passing to the limit as $t \to \infty$ in (3.10), thanks to Lemma 3.1, we conclude that

$$\begin{aligned}\frac{1}{K} < \|u_0\| \le \lim_{t\to\infty} e^{-\lambda_0 t} + L \sup_{[0,T_0]} \|U(\tau, u_0)\| \limsup_{t\to\infty} \int_{t-T_0}^t \varkappa(s) ds + \\ + \limsup_{t\to\infty} \frac{1 - u_{-L}(t)}{K} \le \frac{1}{K},\end{aligned}$$

which is a contradiction. Hence, there does not exist any nontrivial solution which converges to the origin with an exponential rate. This completes the proof. □

We now can verify a stronger result on the decaying rate of solutions to (1.1).

Theorem 3.2. *Let* $U(\cdot, u_0)$ *denote an arbitrary solution of Equation* (1.1) *with the initial condition* $U(0, u_0) = u_0 \neq 0$ *and* $\beta > 1 + \nu$ *be an arbitrary positive number. Then*

$$\limsup_{t\to+\infty} (\log t)^\beta \|U(t, u_0)\| = +\infty. \tag{3.11}$$

Proof. Assume the contrary that there exists a $\beta > 1 + \nu$ such that

$$\limsup_{t\to+\infty} (\log t)^\beta \|U(t, u_0)\| = M < +\infty. \tag{3.12}$$

Therefore, there exists $T_3 > 0$ such that for all $t \ge T_3$, it holds

$$\|U(t, u_0)\| < \frac{M+1}{(\log t)^\beta}. \tag{3.13}$$

Again, according to (2.4) and condition $\beta > 1 + \nu$, there exists a positive constant T_4 such that

$$\frac{M+1}{(\log t)^{\beta}} < \frac{u_{-L}(t)}{K}, \text{ for all } t \geq T_4. \tag{3.14}$$

The estimates (3.13) and (3.14) play the same roles as (3.8) and (3.9), respectively. We proceed as in the proof of Lemma 3.2 and obtain that for all $\widehat{T}_0 = \max\{T_3, T_4\}$

$$\frac{1}{K} < \|u_0\| \leq \lim_{t\to\infty} \frac{M+1}{(\log t)^{\beta}} + L \sup_{[0,\widehat{T}_0]} \|U(\tau, u_0)\| \limsup_{t\to\infty} \int_{t-\widehat{T}_0}^{t} \varkappa(s) ds + \\ + \limsup_{t\to\infty} \frac{1-u_{-L}(t)}{K} \leq \frac{1}{K},$$

which is a contradiction.

Hence, $\limsup\limits_{t\to+\infty} (\log t)^{\beta} \|U(t, u_0)\| = +\infty$. □

4. Conclusions

In this paper, we establish the decaying rate of the solutions to the equilibrium $u \equiv 0$ for distributed order equations in $\mathbb{R}^d$. Constrast to the fractional differential case, where the typical decaying rate is power, the new logarithmic decaying rate is proved to be optimal for distributed derivative. These results can be extended to more general equations with completely positive kernels or equations with more complicated structures, for instance, equations involving delays or impulses.

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