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HYERS-ULAM STABILITY FOR NONLOCAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we present a result on Hyers-Ulam stability for a class of nonlocal differential equations in Hilbert spaces by using the theory of integral equations with completely positive kernels together with a new Gronwall inequality type. The paper develops some recent results on fractional differential equations to the ones involving general nonlocal derivatives. Instead of Mittag-Leffler functions, we must utilize the characterization of relaxation function.
Keywords: nonlocal differential equation, mild solution, Hyers-Ulam stability.

1. Introduction

Let H be a separable Hilbert space. Consider the following equation

$$(k * \partial_t u)(t) + Au(t) = f(t, u(t)),\, t \in J := [0, T]. \tag{1.1}$$

where the unknown function u takes values in H, the kernel $k \in L^1_{loc}(\mathbb{R}^+)$, A is an inbounded linear operator, and $f\colon J \times H \to H$ is a given function. Here the notation $*$ denotes the Laplace convolution, i.e., $(k * v)(t) = \int_0^t k(t-s)v(s)ds$.
In [1], authors introduced a result on the existence, regularity and stability for mild solutions to (1.1) where f depends only on u and the initial condition is given by

$$u(0) = u_0. \tag{1.2}$$

Our goal in this paper is to consider the Hyers-Ulam stability for (1.1).

The Hyers-Ulam stability for functional equations was founded in 1940 by S.M Ulam in a talk at Wisconsin University (see [2]) and by D. H Hyers' answer a year later for additive functions defined on Banach spaces (see [3]). However, the first result on the Hyers-Ulam stability of a differential equation was addressed by C.Alsina and R. Ger in 1998 (see [4]). In this remarkable work, they proved that if a differentiable function

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$y : I \to \mathbb{R}$ satisfies $|y'(t) - y(t)| \leq \epsilon$ for all $t \in I$, where $\epsilon > 0$ is a given number and I is an open interval of $\mathbb{R}$, then there exists a differentiable function $g : I \to \mathbb{R}$ satisfying both $g'(t) = g(t)$ and $|y(t) - g(t)| \leq 3\epsilon$ for all $t \in I$. It then has attracted attention of mathematicians for decades (see [5-13]) to study this type of stability for differential equations systematically.

In order to deal with (1.1), we use the following standing hypotheses:

(A) *The operator $A : D(A) \to H$ is densely defined, self-adjoint, and positively definite.*

(K) *The kernel function $k \in L^1_{loc}(\mathbb{R}^+)$ is nonnegative and nonincreasing, and there exists a function $l \in L^1_{loc}(\mathbb{R}^+)$ such that $k * l = 1$ on $(0, \infty)$.*

(F) *The continuous function $f\colon J \times H \to H$ is Lipschitzian , i.e , there is $L_f > 0$ such that*

$$\|f(t, v_1) - f(t, v_2)\| \leq L_f \|v_1 - v_2\|, \ \forall t \in J, \forall v_1, v_2 \in H.$$

2. Preliminaries

2.1. The resolvent families and the Gronwall type inequality

Consider the following scalar integral equations

$$s(t) + \mu(l * s)(t) = 1, \ t \geq 0, \tag{2.1}$$

$$r(t) + \mu(l * r)(t) = l(t), \ t > 0. \tag{2.2}$$

The existence and uniqueness of s and r were analyzed in [8]. Recall that the function l is called a completely positive kernel iff $s(\cdot)$ and $r(\cdot)$ take nonnegative values for every $\mu > 0$. The complete positivity of l is equivalent to that (see [14]), there exist $\alpha \geq 0$ and $k \in L^1_{loc}(\mathbb{R}^+)$ nonnegative and nonincreasing which satisfy $\alpha l + l * k = 1$. So the hypothesis (K) implies that l is completely positive.

Denote by $s(\cdot, \mu)$ and $r(\cdot, \mu)$ the solutions of (2.1) and (2.1), respectively. As mentioned in [15], the functions $s(\cdot, \mu)$ and $r(\cdot, \mu)$ take nonnegative values even in the case $\mu \leq 0$. We collect some additional properties of these functions.

Proposition 2.1. *[1, 15] Let the hypothesis (K) hold. Then for every $\mu > 0$, $s(\cdot, \mu), r(\cdot, \mu) \in L^1_{loc}(\mathbb{R}^+)$. In addition, we have*

1. *The function $s(\cdot, \mu)$ is nonnegative and nonincreasing. Moreover,*

$$s(t, \mu) \left[1 + \mu \int_0^t l(\tau) d\tau\right] \leq 1, \ \forall t \geq 0, \tag{2.3}$$

hence if $l \notin L^1(\mathbb{R}^+)$ then $\lim_{t \to \infty} s(t, \mu) = 0$ for every $\mu > 0$.

2. *The function $r(\cdot, \mu)$ is nonnegative and one has*

$$s(t, \mu) = 1 - \mu \int_0^t r(\tau, \mu) d\tau = k * r(\cdot, \mu)(t), \ t \geq 0, \tag{2.4}$$

so $\int_0^t r(\tau,\mu)d\tau \le \mu^{-1},\ \forall t>0$. *If* $l \notin L^1(\mathbb{R}^+)$ *then* $\int_0^\infty r(\tau,\mu)d\tau = \mu^{-1}$ *for every* $\mu > 0$.

3. *For each* $t>0$*, the functions* $\mu \mapsto s(t,\mu)$ *and* $\mu \mapsto r(t,\mu)$ *are nonincreasing.*

4. *Equation (2.1) is equivalent to the problem*

$$\frac{d}{dt}[k*(s-1)] + \mu s = 0,\ s(0)=1.$$

5. *Let* $v(t) = s(t,\mu)v_0 + (r(\cdot,\mu)*g)(t)$*, here* $g \in L^\infty_{loc}(\mathbb{R}^+)$*. Then* v *solves the problem*

$$\frac{d}{dt}[k*(v-v_0)](t) + \mu v(t) = g(t),\ v(0)=v_0.$$

Let us mention that, the hypothesis (A) ensures the existence of an orthonormal basis of H consisting of eigenfunctions $\{e_n\}_{n=1}^\infty$ of the operator A and we have

$$Av = \sum_{n=1}^\infty \lambda_n v_n e_n,$$

where $\lambda_n > 0$ is the eigenvalue corresponding to the eigenfunction e_n of A,

$$D(A) = \{v = \sum_{n=1}^\infty v_n e_n : \sum_{n=1}^\infty \lambda_n^2 v_n^2 < \infty\}.$$

We can assume that $0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_n \to \infty$ as $n \to \infty$.

For $\gamma \in \mathbb{R}$, one can define the fractional power of A as follows:

$$D(A^\gamma) = \left\{ v = \sum_{n=1}^\infty v_n e_n : \sum_{n=1}^\infty \lambda_n^{2\gamma} v_n^2 < \infty \right\},$$

$$A^\gamma v = \sum_{n=1}^\infty \lambda_n^\gamma v_n e_n.$$

Let $V_\gamma = D(A^\gamma)$. Then V_γ is a Banach space endowed with the norm

$$\|v\|_\gamma = \|A^\gamma v\| = \left(\sum_{n=1}^\infty \lambda_n^{2\gamma} v_n^2\right)^{\frac{1}{2}}.$$

Furthermore, for $\gamma > 0$, we can identify the dual space V_γ^* of V_γ with $V_{-\gamma}$.

We now define the following operators:

$$S(t)v = \sum_{n=1}^\infty s(t,\lambda_n)v_n e_n,\ t \ge 0, v \in H, \tag{2.5}$$

$$R(t)v = \sum_{n=1}^\infty r(t,\lambda_n)v_n e_n,\ t > 0, v \in H. \tag{2.6}$$

It is easily seen that $S(t)$ and $R(t)$ are linear. We collect some basic properties of these operators in the following lemma.

Lemma 2.1. *[1] Let $\{S(t)\}_{t\geq 0}$ and $\{R(t)\}_{t>0}$, be the families of linear operators defined by (2.5) and (2.6), respectively. Then*

1. *For each $v \in H$ and $T > 0$, $S(\cdot)v \in C([0,T];H)$ and $AS(\cdot)v \in C((0,T];H)$. Moreover,*

$$\|S(t)v\| \leq s(t,\lambda_1)\|v\|,\ t \in [0,T], \tag{2.7}$$

$$\|AS(t)v\| \leq \frac{\|v\|}{(1*l)(t)},\ t \in (0,T]. \tag{2.8}$$

2. *Let $v \in H, T > 0$ and $g \in C([0,T];H)$. Then $R(\cdot)v \in C((0,T];H)$, $R*g \in C([0,T];H)$ and $A(R*g) \in C([0,T];V_{-\frac{1}{2}})$. Furthermore,*

$$\|R(t)v\| \leq r(t,\lambda_1)\|v\|,\ t \in (0,T], \tag{2.9}$$

$$\|(R*g)(t)\| \leq \int_0^t r(t-\tau,\lambda_1)\|g(\tau)\|d\tau,\ t \in [0,T], \tag{2.10}$$

$$\|A(R*g)(t)\|_{-\frac{1}{2}} \leq \left(\int_0^t r(t-\tau,\lambda_1)\|g(\tau)\|^2 d\tau\right)^{\frac{1}{2}},\ t \in [0,T]. \tag{2.11}$$

The following proposition shows a Gronwall type inequality.

Proposition 2.2. *Let v be a nonnegative continuous function satisfying*

$$v(t) \leq C_1 + C_2\int_0^t r(t-\tau,\mu)v(\tau)d\tau,\ t \in J, \tag{2.12}$$

for given nonnegative numbers C_1, C_2 and $\mu > 0$. Then

$$v(t) \leq s(t,-C_2)C_1.$$

Proof. From (2.2) and the positivity of $r(\cdot,\mu)$ and $l(\cdot)$, we get

$$r(t,\mu) \leq l(t), \forall t \in J, \text{ and } \mu > 0.$$

Combining this inequality with (2.12) yield

$$v(t) \leq C_1 + C_2(l*v)(t). \tag{2.13}$$

Consider the following equation

$$\xi(t) = C_1 + C_2(l*\xi)(t), t \in J.$$

Obviously $\xi(0) = C_1$ and the equation is equivalent to

$$\xi(t) - C_1 = C_2(l * \xi)(t).$$

Taking the convolution with the kernel k gives us

$$k * (\xi - C_1) = C_2(1 * \xi)(t).$$

Then ξ is a solution to the following systems

$$\frac{d}{dt}[k * (\xi - C_1)] = C_2\xi(t)$$
$$\xi(0) = C_1.$$

So $\xi(t) = s(t, -C_2)C_1$. Therefore, we arrive at

$$v(t) \le s(t, -C_2)C_1, \ \forall t \in J,$$

thanks to the comparison principle. □

2.2. Existence result to system (1.1) - (1.2)

Definition 2.1. *A function $u \in C((0, T]; H)$ is said to be a mild solution to (1.1)-(1.2) on $[0, T]$ iff*

$$u(t) = S(t)u_0 + \int_0^t R(t - \tau)f(\tau, u(\tau))d\tau,$$

for $t \in [0, T]$.

Theorem 2.1. *Let* (A), (K) *and* (F) *hold. Then the mild solution to (1.1)-(1.2) is unique.*

Proof. To get the result, we use the same arguments as in [1]. □

3. Hyers-Ulam stability on $[0, T]$

We first define of Hyer-Ulam stability for nonlocal differential equation (1.1) and then we show our main result.

We consider (1.1) and the following inequality

$$\| (k * \partial_t v)(t) + Av(t) - f(t, v(t))\| \le \epsilon, \ t \in J, \tag{3.1}$$

where $\epsilon > 0$ is given. We now give the definition of mild solution to the above inequality.

Definition 3.1. *A continuous funtion $v : J \to H$ is said to be a mild solution to (3.1) if there exists a function $g \in L^1_{loc}(J, H)$ such that $\|g(t)\| \le \epsilon$ and*

$$v(t) = S(t)v(0) + \int_0^t R(t - \tau)[f(\tau, v(\tau)) + g(\tau)]d\tau, \ t \in J.$$

Definition 3.2. *Equation (1.1) is called Hyers-Ulam stable, with respect to s defined on J, if there exists a real number $C > 0$ such that for each $\epsilon > 0$ and for every mild solution v of (3.1), there is a mild solution u of (1.1) with*

$$\|v(t) - u(t)\| \leq C\epsilon s(t, \nu), \forall t \in [0, T],$$

for some $\nu \in \mathbb{R}$.

Definition 3.3. *Equation (1.1) is called generalized Hyers-Ulam stable, with respect to $s(t, \nu)$, if there exists $\theta \in C\left(\mathbb{R}^+, \mathbb{R}^+\right)$, $\theta(0) = 0$ such that for each mild solution v of (3.1) there exists a mild solution u of (1.1) with $\|v(t) - u(t)\| \leq \theta(\epsilon) s(t, \nu)$, for all $t \in J$.*

Remark 3.1. *It is clear that if equation (1.1) is Hyers-Ulam stable then it is also generalized Hyers-Ulam stable.*

The following Theorem is the main result in this paper.

Theorem 3.1. *If* (A), (K) *and* (F) *hold, then the equation (1.1) is Hyers-Ulam stable.*

Proof. Let v be a mild solution to (3.1). By Theorem 2.1, the following problem

$$\begin{gathered} \left(k * \partial_t u\right)(t) + Au(t) = f(t, u(t)),\ t \in J, \\ u(0) = v(0), \end{gathered}$$

admits a unique mild solution given by

$$u(t) = S(t)v(0) + \int_0^t R(t - \tau) f(\tau, u(\tau)) d\tau,\ \forall t \in J.$$

Therefore, we have

$$\begin{aligned} \|v(t) - u(t)\| &\leq \left\| \int_0^t R(t - \tau)[f(\tau, v(\tau)) + g(\tau) - f(\tau, u(\tau))] d\tau \right\| \\ &\leq \epsilon \int_0^t r(t - \tau, \lambda_1) d\tau + L_f \int_0^t r(t - \tau, \lambda_1) \|v(\tau) - u(\tau)\| d\tau \\ &\leq \epsilon \frac{1}{\lambda_1} + L_f \int_0^t r(t - \tau, \lambda_1) \|v(\tau) - u(\tau)\| d\tau, \end{aligned}$$

thanks to (F) and Proposition 2.1.

It comes from the Gronwall type inequality stated in Proposition 2.2 that

$$\|v(t) - u(t)\| \leq \frac{\epsilon}{\lambda_1} s(t, -L_f).$$

The proof is complete. $\square$

4. Conclusions

In this paper, the Hyers-Ulam stability has been discussed for a class of nonlocal evolution equations in Hilbert space. The result may be extended to more general models and concepts. It is very interesting to investigate these types of stabilities for nonlocal differential equations in Banach spaces, where the new methods and ideas are needed due to the lack of Hilbertian structrure on phase spaces.

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