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THE EXISTENCE AND UNIQUENESS OF SOLUTIONS TO 3D NAVIER-STOKES EQUATIONS WITH DAMPING AND DELAYS

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Abstract. In this paper we consider the existence and uniqueness of weak solutions to 3D Navier-Stokes equations with damping and delays in $\Omega \subset \mathbb{R}^3$.
Keywords: 3D Navier-Stokes equation, damping, delays, weak solution.

1. Introduction

In this paper we consider the following 3D Navier-Stokes equations with damping and delays in $\Omega \subset \mathbb{R}^3$,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p + \alpha|u|^{\beta-1}u & = G(u(t-\rho(t))) + h(x), \text{ in } (0,T)\times\Omega, \\ \text{div} u & = 0 \qquad \text{in } (0,T)\times\Omega, \\ u(x,t) & = 0 \qquad \text{in } (0,T)\times\Gamma, \\ u(0,x) & = u_0(x), \qquad x\in\Omega, \\ u(t,x) & = \phi(t,x), \qquad t\in(-r,0), x\in\Omega. \end{cases} \tag{1.1}$$

where $\nu > 0$ is the kinematic viscocity, $\beta \geq 1$ and $\alpha > 0$ are two constants, u is the velocity field of the fluid and $u = u(x,t) = (u_1, u_2, u_3)$, p is the pressure, h is a nondelayed external force field, G is another external force term and contains some memory effects during a fixed interval of time of length $r > 0$, ρ is an adequate given delay function, u_0 is the initial velocity and ϕ the initial datum on the interval.

Note that the case $\alpha = 0$ and $G = 0$ corresponds to the classical Navier-Stokes problem for which the existence of smooth solutions is an open problem. The case of $G = 0$ is studied in [1], which is 3D Navier-Stokes equations with damping. The damping is from the resistance to the motion of the flow. It describes various physical situations such as porous media flow, drag or friction effects, and some dissipative mechanisms (see [2-5] and references therein).

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The 2D Navier-Stokes equations with delays has studied by T. Caraballo et al. [6]. In this paper, we continue to study 3D Navier-Stokes equations of forcing term with bounded variable delay. Let $G : \mathbb{R}^3 \to \mathbb{R}^3$ be a measurable function satisfing $G(0) = 0$, and assume that there exists $L_G > 0$ such that

$$|G(u) - G(v)|_{\mathbb{R}^3} \leq L_G|u - v|_{\mathbb{R}^3}, \forall u, v \in \mathbb{R}^3. \tag{1.2}$$

Consider a function $\rho(t) \in C^1([0, T])$, $\rho(t) \geq 0$ for all $t \in [0, T]$, $r = \max_{t\in[0,T]} \rho(t) > 0$ and $\rho_* = \max_{t\in[0,T]} \rho'(t) < 1$. We will use the Galerkin method to study the existence of weak solutions to (1.1) (see e.g., [6, 7]). To show the priori estimates for $u(t - \rho(t))$, we use the technique of changing variable in [8].

In this paper, we will prove the existence of weak solutions to (1.1) in the case of $\beta \geq 1$. If $\beta > 3$, we will get at the uniqueness of this solutions to (1.1).

2. Preliminaries

We define the following abstract spaces:

- $\mathcal{V} = \{u \in (C_0^\infty(\Omega))^3 : \text{div} u = 0\}$.

- H is the closure of $\mathcal{V}$ in $(L^2(\Omega))^3$ with the norm $|.|$, and inner product $(., .)$ defined by
$$(u, v) = \sum_{j=1}^{3} \int_\Omega u_j(x)v_j(x)dx \text{ for } u, v \in (L^2(\Omega))^3.$$

- V is the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^3$ with norm $\|.\|$, and associated scalar product $((.))$ defined by
$$((u, v)) = \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_j}{\partial x_j}\frac{\partial v_j}{\partial x_i}dx \text{ for } u, v \in (H_0^1(\Omega))^3.$$

We denote $a(u, v) = ((u, v))$, and define the trilinear form b on $V \times V \times V$ by

$$b(u, v, w) = \sum_{i,j=1}^{3} \int_\Omega u_i\frac{\partial v_j}{\partial x_i}w_j dx, \forall u, v, w \in V.$$

It is easy to check that if $u, v, w \in V$, then

$$b(u, v, w) = -b(u, w, v),$$

and

$$b(u, v, v) = 0, \forall u, v \in V. \tag{2.1}$$

In what follows, we will frequently use the following inequalities:

Young's inequality

$$ab \leq \frac{\varepsilon}{p}a^p + \frac{1}{q\varepsilon^{1/(p-1)}}b^q, \text{ for all } a, b, \varepsilon > 0, \text{ with } q = \frac{p}{p-1}, 1 < p < +\infty.$$

Ladyzhenskaya's inequality (when $n = 3$) (see, e.g., [10])

$$\|u\|_{L^3} \leq c|u|^{1/2}\|u\|^{1/2}, \forall u \in V,$$
$$\|u\|_{L^4} \leq c|u|^{1/4}\|u\|^{3/4}, \forall u \in V.$$

From [9, 10], we will use some results in the following lemma.

Lemma 2.1. *If $n = 3$ then*

$$|b(u, v, w)| \leq \begin{cases} C|u|^{1/2}\|u\|^{1/2}\|v\|\|w\|^{1/2}\|w\| \\ C\|u\|\|v\||w|^{1/2}\|w\|^{1/2} & \forall u, v, w \in V, \\ C\lambda_1^{1/4}\|u\|\|v\|\|w\|^{1/2} \end{cases} \tag{2.2}$$

and

$$|b(u, v, u)| \leq \sqrt{2}|u|\|u\|\|v\| \text{ for all } u, v \in V. \tag{2.3}$$

3. Existence and uniqueness of weak solutions

We first give the definition of weak solution.

Definition 3.1. *A function*

$$u \in C([-r, T]; H) \cap L^2_{loc}([-r, T]; V) \cap L^{\beta+1}_{loc}([-r, T]; L^{\beta+1}(\Omega))$$

is said to be a weak solution of (1.1) *if for all $T > 0$,*

$$u \in L^2(-r, T; V) \cap L^\infty(0, T; H)$$

such that, for all $v \in V$,

$$\begin{aligned} \frac{d}{dt}(u(t), v) + \nu((u(t), v)) + b(u(t), u(t), v) + \alpha(|u|^{\beta-1}u, v) \\ = (G(u(t - \rho(t))), v) + (h, v), \\ u(0) = u_0, \qquad u(t) = \phi(t), \quad t \in [-r, 0]. \end{aligned} \tag{3.1}$$

We now prove the following theorem.

Theorem 3.1. *Let* (1.2) *hold, $u_0, h \in H, \phi \in L^2(-r, 0; H)$. Then there exists a unique solution to* (1.1) *if $\nu^2 > \dfrac{L_G}{\lambda_1^2(1 - \rho_*)}$.*

Proof. Existence.

Let us consider $\{w_j\} \subset V \cap (H^2(\Omega))^3$, the orthonormal basis of H of all the eigenfunctions of the Stokes problem in Ω with homogeneous Dirichlet conditions. The subspace of V spanned by $w_1, w_2, ..., w_m$ will be denoted V_m. Consider the projector $P_m : H \to V_m$ given by $P_m u = \sum_{j=1}^m (u, w_j)w_j$, and define $u_m(t) = \sum_{j=1}^m \gamma_{mj}(t)w_j$, where $u_m \in L^2(-r, T; V_m) \cap C^0([0, T]; V_m)$ satisfies

$$\begin{cases} \dfrac{d}{dt}(u_m(t), w_j) + \nu((u_m(t), w_j)) + b(u_m(t), u_m(t), w_j) + \alpha(|u_m|^{\beta-1}u_m(t), w_j) \\ \qquad\qquad = (G(u_m(t-\rho(t))), w_j) + (h(x), w_j) \text{ in } D'(0,T), 1 \le j \le m, \\ u_m(0) = P_m u_0, \quad u_m(t) = P_m\phi(t), \quad t \in (-r, 0). \end{cases} \tag{3.2}$$

Observe that (3.2) is a system of ordinary functional differential equations in the unknown $\gamma^m(t) = (\gamma_{m1}(t), ..., \gamma_{mm}(t))$. Now we can ensure that problem (3.2) has a unique solution defined in an interval $[0, t^*]$ with $0 < t^* \le T$. However, as can be deduced by the a priori estimates below, we can set $t^* = T$.
In fact, multiplying in (3.2) by $\gamma_{mj}(t)$ and summing up, by using (2.1) we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \nu\|u_m(t)\|^2 + \alpha\int_\Omega |u_m(t)|^{\beta+1}dx \\ &\qquad = \int_\Omega G(u_m(t-\rho(t)))u_m(t)dx + \int_\Omega h(x)u_m(t)dx. \end{aligned}$$

And we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \nu\|u_m(t)\|^2 + \alpha\int_\Omega |u_m|^{\beta+1}dx \\ &\qquad \le |G(u_m(t-\rho(t)))| \cdot |u_m(t)| + |h| \cdot |u_m(t)|. \end{aligned} \tag{3.3}$$

Assumption (1.2) implies that

$$|G(\xi)| \le L_G|\xi|. \tag{3.4}$$

Then we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \nu\|u_m(t)\|^2 + \alpha\int_\Omega |u_m|^{\beta+1}dx \\ &\qquad \le L_G|u_m(t-\rho(t))| \cdot |u_m(t)| + |h| \cdot |u_m(t)|. \end{aligned} \tag{3.5}$$

By the Cauchy inequality, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \nu\|u_m(t)\|^2 + \alpha\int_\Omega |u_m|^{\beta+1}dx \\ &\qquad \le \frac{L_G}{\lambda_1\nu}|u_m(t-\rho(t))|^2 + \frac{\lambda_1\nu}{4}|u_m(t)|^2 + \frac{1}{\lambda_1\nu}|h|^2 + \frac{\lambda_1\nu}{4}|u_m(t)|^2, \end{aligned}$$

where λ_1 is first eigenvalue of $-\Delta$. Intergrating from 0 to t and using (3.4) we obtain

$$\begin{aligned}|u_m(t)|^2 + 2\nu\int_0^t \|u_m(s)\|^2 ds + 2\alpha\int_0^t \|u_m(s)\|_{L^{\beta+1}}^{\beta+1} ds \le |u_0|^2 \\ + \frac{2L_G}{\lambda_1\nu}\int_0^t |u_m(s-\rho(s))|^2 ds + \frac{2}{\lambda_1\nu}\int_0^t |h|^2 ds + \lambda_1\nu\int_0^t |u_m(s)|^2 ds.\end{aligned} \tag{3.6}$$

From $\|u_m\|^2 \ge \lambda_1 |u_m|^2$, we obtain

$$\begin{aligned}|u_m(t)|^2 + \nu\int_0^t \|u_m(s)\|^2 ds + 2\alpha\int_0^t \|u_m(s)\|_{L^{\beta+1}}^{\beta+1} ds \\ \le |u_0|^2 + \frac{2L_G}{\lambda_1\nu}\int_0^t |u_m(s-\rho(s))|^2 ds + \frac{2}{\lambda_1\nu}\int_0^t |h|^2 ds.\end{aligned} \tag{3.7}$$

Let $\tau = s-\rho(s)$, in view of $\rho(s)\in[0,r]$ and $\frac{1}{1-\rho'} \le \frac{1}{1-\rho_*}$, we obtain

$$\begin{aligned}\int_0^t |u(s-\rho(s))|^2 ds = \frac{1}{1-\rho'}\int_{-r}^t |u(\tau)|^2 d\tau \le \frac{1}{1-\rho_*}\int_{-r}^t |u(\tau)|^2 d\tau \\ = \frac{1}{1-\rho_*}\int_{-r}^0 |u(\tau)|^2 d\tau + \frac{1}{1-\rho_*}\int_0^t |u(\tau)|^2 d\tau\end{aligned} \tag{3.8}$$

By (3.7) and (3.8), and using $u(t)=\phi(t)$, $t\in(-r,0)$, we have

$$\begin{aligned}|u_m(t)|^2 + \nu\int_0^t \|u_m\|^2 ds + 2\alpha\int_0^t \|u_m(s)\|_{L^{\beta+1}}^{\beta+1} ds \le |u_0|^2 \\ + \frac{2L_G}{\lambda_1\nu(1-\rho_*)}\int_{-r}^0 |\phi(\tau)|^2 d\tau + \frac{2L_G}{\lambda_1\nu(1-\rho_*)}\int_0^t |u(\tau)|^2 d\tau + \frac{2}{\lambda_1\nu}\int_0^t |h|^2 ds.\end{aligned} \tag{3.9}$$

Using $\|u_m\|^2 \ge \lambda_1|u_m|^2$ again, we obtain

$$\begin{aligned}|u_m(t)|^2 + \Big(\nu - \frac{2L_G}{\lambda_1^2\nu(1-\rho_*)}\Big)\int_0^t \|u_m\|^2 ds + 2\alpha\int_0^t \|u_m(s)\|_{L^{\beta+1}}^{\beta+1} ds \\ \le |u_0|^2 + \frac{2L_G}{\lambda_1\nu(1-\rho_*)}\int_{-r}^0 |\phi(\tau)|^2 d\tau + \frac{2}{\lambda_1\nu}\int_0^t |h|^2 ds.\end{aligned} \tag{3.10}$$

For $\nu^2 > \dfrac{2L_G}{\lambda_1^2(1-\rho_*)}$ and from $\phi\in L^2(-r,0;V)$, we obtain K_1, K_2 and K_3 (depending only on ϕ, h, G, T, but not on m or t^*) such that

$$\sup_{t\in[0,t^*]} |u_m(t)|^2 \le K_1, \quad \int_0^t \|u_m(s)\|^2 ds \le K_2, \quad \int_0^t \|u_m(s)\|_{L^{\beta+1}}^{\beta+1} \le K_3.$$

Thus we can take $t_* = T$ to obtain that

$$\{u_m\} \text{ is bounded in } L^2(0,T;V)\cap L^\infty(0,T;H)\cap L^{\beta+1}(0,T;L^{\beta+1}(\Omega)).$$

Moreover, observe that $u_m = P_m\phi$ in $(-r, 0)$ and, by the choice of the basis $\{w_j\}$, the sequence $\{u_m\}$ converges to ϕ in $L^2(-r, 0; V)$.

Note that $G(u_m)$ is bounded in $L^2(0, T; H)$ and it is a straight forward to bound the nonlinear term $\{b(u_m, u_m, .)\}$. By using the same argument as in Constantin and Foias [10], we can obtain that $\{\frac{du_m}{dt}\}$ is bounded in $L^{4/3}(0, T; V')$. Using the compactness of the injection of the space $W = \{u \in L^2(0, T; V); \frac{du}{dt} \in L^{4/3}(0, T; V')\}$ into $L^2(0, T; H)$, from the preceding analysis and the assumptions on G, we can deduce that there exists a subsequence (denote again $\{u_m\}$ and $u \in L^2(-r, T; V)$) such that

$$\begin{aligned}
&u_m \to u \text{ weakly in } L^2(-r, T; V)\\
&u_m \to u \text{ weakly star in } L^\infty(0, T; H)\\
&u_m \to u \text{ in } L^2(-r, T; H),\\
&G(u_m) \to G(u) \text{ weakly in } L^2(0, T; V').
\end{aligned}$$

Arguing now as in the non-delay case, we can take the limits in (3.2) after intergrating over the interval $(0, t)$ (for $t \in (0, T)$), and obtain that u is a solution to our problem (1.1).

Uniqueness.

Let u and v be two solutions and let $w = u - v$. Then, this function solves

$$\begin{cases}
\dfrac{dw}{dt} - \nu\Delta w + (w, \nabla)u + (v, \nabla)w + \nabla(p_u - p_v) + \alpha(|u|^{\beta-1}u - |v|^{\beta-1}v)\\
\qquad\qquad\qquad\qquad\qquad = [G(u(t-\rho(t))) - G(v(t-\rho(t)))],\\
\text{div} w = 0.
\end{cases}$$

It is well known (see, e.g. [9]) that there exists a nonnegative constant $\kappa = \kappa(\alpha, \beta)$ such that

$$\alpha\int_\Omega (|u|^{\beta-1}u - |v|^{\beta-1}v)(u-v)dx \geq \kappa\int_\Omega (|u|^{\beta-1} + |v|^{\beta-1})|u-v|^2 dx \geq 0 \tag{3.11}$$

Multiplying this equations by w, intergrating by parts and using (3.11), we have

$$\begin{aligned}
\frac{d}{dt}|w|^2 + 2\nu\|w\|^2 + \kappa\int_\Omega (|u|^{\beta-1} + |v|^{\beta-1})|u-v|^2 dx &\leq 2\int_\Omega |((w\cdot\nabla)u)\cdot w|dx\\
&+ 2\int_\Omega |G(u(t-\rho(t))) - G(v(t-\rho(t)))|\cdot|w(t)|dx
\end{aligned} \tag{3.12}$$

Using Lemma 2.1 and assume that $\beta > 3$, we have

$$\begin{aligned}
2\int_\Omega |((w\cdot\nabla)u)\cdot w|dx &\leq 2\int_\Omega |u||w||\nabla w|dx \leq \nu|\nabla w|^2 + C(|u|^2\cdot|w|^2)\\
&\leq \nu|\nabla w|^2 + \kappa(|u|^{\beta-1} + |v|^{\beta-1})|w|^2 + C|w|^2,
\end{aligned} \tag{3.13}$$

where $C = C(\nu)$. Thus, (3.12) implies that

$$\begin{aligned}\frac{d}{dt}|w|^2 + 2\nu\|w\|^2 \leq& \nu|\nabla w|^2 + C|w|^2\\ &+ 2\int_\Omega |G(u(t-\rho(t))) - G(v(t-\rho(t)))| \cdot |w(t)|dx.\end{aligned}$$

Using (1.2), we get

$$\frac{d}{dt}|w|^2 + \nu\|w\|^2 \leq C|w|^2 + 2\int_\Omega L_G|u(t-\rho(t)) - v(t-\rho(t))| \cdot |w(t)|dx.$$

By the Cauchy inequality, we obtain

$$\frac{d}{dt}|w|^2 + \nu\|w\|^2 \leq C|w|^2 + \frac{2L_G}{\lambda_1\nu}|w(t-\rho(t))|^2 + \frac{\lambda_1\nu}{2}|w|^2.$$

Using $\|w\|^2 \geq \lambda_1|w|^2$, we have

$$\frac{d}{dt}|w|^2 + \nu\|w\|^2 \leq C|w|^2 + \frac{2L_G}{\lambda_1\nu}|w(t-\rho(t))|^2.$$

Intergrating from 0 to t, we obtain

$$|w|^2 + \nu\int_0^t \|w\|^2 ds \leq |w(0)|^2 + C\int_0^t |w|^2 ds + \frac{2L_G}{\lambda_1\nu}\int_0^t |w(s-\rho(s))|^2 ds.$$

Using (3.8) again, we have

$$|w|^2 + \nu\int_0^t \|w\|^2 ds \leq |w(0)|^2 + C\int_0^t |w|^2 ds + \frac{2L_G}{\lambda_1\nu(1-\rho_*)}\int_{-r}^t |w(\tau)|^2 d\tau.$$

Note that $w(\tau) = 0$ for $\tau \in (-r, 0)$ and $\|w\|^2 \geq \lambda_1|w|^2$, we obtain

$$\begin{aligned}|w|^2 + \nu\int_0^t \|w\|^2 ds \leq& |w(0)|^2 + C\int_0^t |w(s)|^2 ds\\ &+ \frac{2L_G}{\lambda_1^2\nu(1-\rho_*)}\int_0^t \|w(\tau))\|^2 d\tau.\end{aligned}$$

Thus,

$$|w|^2 + \left(\nu - \frac{2L_G}{\lambda_1^2\nu(1-\rho_*)}\right)\int_0^t \|w\|^2 ds \leq |w(0)|^2 + C\int_0^t |w(s)|^2 ds.$$

Note that $\nu^2 > \dfrac{2L_G}{\lambda_1^2(1-\rho_*)}$, we have the uniqueness of solutions by the Gronwall lemma.

$\square$

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