HNUE JOURNAL OF SCIENCE DOI: 10.18173/2354-1059.2019-0070
Natural Science, 2019, Volume 64, Issue 10, pp. 17-35
This paper is available online at http://stdb.hnue.edu.vn

THE GENERALIZED CONVOLUTION FOR h-LAPLACE TRANSFORM ON TIME SCALE

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Abstract. In this paper we study generalized convolution for h-Laplace transform on time scale $\mathbb{T}_h^+$ and obtain some of its properties as well as applications in solving some linear equations of convolution type.
Keywords: Fourier cosine transform, time scales, convolution, Laplace transform, h-Laplace transform.

1. Introduction

The Laplace transform theory has been studied from the 17^{th} century. The Fourier transform has been studied from the 19^{th} century together with the Fourier cosine, Fourier sine transforms and convolution of two functions for Fourier transform. The Laplace transform, the Fourier transform, Fourier cosine and Fourier sine transforms play important roles in mathematics and have many applications in science and engineering.

There are many interesting results related to Laplace transform (see [1-4]), Fourier, Fourier cosine and Fourier sine transforms [4-8].

A time scale is an arbitrary nonempty closed subset of real numbers. Time scale analysis unifies and extends continuous and discrete analyses; see [9].

The subject of transforms on time scale for the continuous case has been studied long ago and there are many results for continuous dynamic systems. However the subject of transforms on time scale for the discrete case has only been studied recently and there are not many works about transforms on discrete time scales.

Let h be a positive real number. An important time scale is the following:

Received October 11, 2019. Revised October 24, 2019. Accepted October 29, 2019.
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Definition 1.1. *[9] Time scale $\mathbb{T}_h$ is determined by*

$$\mathbb{T}_h = \begin{cases} 0 & \textit{if } h = \infty \\ h\mathbb{Z} & \textit{if } h > 0 \\ \mathbb{R} & \textit{if } h = 0 \end{cases}$$

We denote

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\} \text{ is the set of all natural numbers}, \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$
$$\mathbb{T}_h^+ = \{hk : \ k \in \mathbb{N}_0\}$$

Note that $\mathbb{T}_h^+$ is also a time scale obtained from time scale $\mathbb{T}_h$ where we only take non-negative points.

The first one who works on the subject of integral transformation on time scales is Stefan Hilger in 1988 in his PhD dissertation. His work aimed to do away with the discrepancies between continuous and discrete dynamic systems.

The Laplace transform on time scales was introduced by Hilger in [10] in a form that tries to unify the (continuous) Laplace transform and the (discrete) Z-transform. The Laplace transform on time scales was further investigated by Martin Bohner, Allan Peterson and Gusein Sh. Guseinov in [9, 11, 12].

In this paper we study generalized convolution for h-Laplace transform on time scale $\mathbb{T}_h^+$ and obtain some of its properties as well as applications in solving some linear equations of convolution type. This paper is organized as follows. In Section 2, we review some properties of h-Laplace and Fourier cosine transforms on time scale $\mathbb{T}_h^+$. In Section 3 we introduce and study generalized convolution for h-Laplace transform. In Section 4 we give some applications of this generalized convolution in solving some linear equations of convolution type.

2. h-Laplace and Fourier transforms on time scale $\mathbb{T}_h^+$

In this paper we use the following spaces:

Definition 2.1. *Let $\alpha > 0$ be a fixed positive number. We define*

$$L_1(\mathbb{T}_h^+) = \{x : \mathbb{T}_h^+ \to \mathbb{R} \big| |x(0)| + 2\sum_{n=1}^{\infty} |x(nh)| < \infty\}$$

$$\|x\|_1 = h\big(|x(0)| + 2\sum_{n=1}^{\infty} |x(nh)|\big) \text{ is called the norm of } x \text{ in } L_1(\mathbb{T}_h^+).$$

$$L_1(\mathbb{T}_h^+, e^{\alpha nh}) := \Big\{x : \mathbb{T}_h^+ \to \mathbb{R} \big| 2h\sum_{n=1}^{\infty} e^{\alpha nh}|x(nh)| < \infty\Big\}.$$

$$B(\mathbb{T}_h^+, e^{-\alpha nh}) := \Big\{x : \mathbb{T}_h^+ \to \mathbb{R} \Big| \exists C > 0 \textit{ such that } |x(nh)| \leq Ce^{-\alpha nh},\ \forall n \in \mathbb{N}_0\}.$$

For the case $h = 1$ the space $L_1(\mathbb{T}_h^+)$ and the norm $\frac{1}{2}\|x\|_1$ were used in [13].

Proposition 2.1. *For all $\alpha > 0$ we have*

$$B(\mathbb{T}_h^+, e^{-2\alpha nh}) \subset L_1(\mathbb{T}_h^+, e^{\alpha nh}) \subset L_1(\mathbb{T}_h^+).$$

Proof. (i) If $x \in L_1(\mathbb{T}_h^+, e^{\alpha nh})$ then $\sum\limits_{n=1}^{\infty} e^{\alpha nh}|x(nh)| < \infty$. Since $e^{\alpha nh} > 1$ we get $|x(0)| + 2\sum\limits_{n=1}^{\infty} |x(nh)| < \infty$ and then $f \in L_1(\mathbb{T}_h^+)$. Therefore $L_1(\mathbb{T}_h^+, e^{\alpha nh}) \subset L_1(\mathbb{T}_h^+)$.

(ii) If $x \in B(\mathbb{T}_h^+, e^{-2\alpha nh})$ then there exists $C > 0$ such that

$$|x(nh)| \leq Ce^{-2\alpha nh},\ \forall n \in \mathbb{N}_0.$$

From this inequality we get

$$\sum_{n=1}^{\infty} e^{\alpha nh}|x(nh)| \leq C\sum_{n=1}^{\infty} e^{-\alpha nh} < \infty$$

so $x \in L_1(\mathbb{T}_h^+, e^{\alpha nh})$. Therefore $B(\mathbb{T}_h^+, e^{-2\alpha nh}) \subset L_1(\mathbb{T}_h^+, e^{\alpha nh})$.

□

For $z \in \mathbb{C}$ we denote $\Re z$ the real part of z and $\Im z$ the imaginary part of z.
In [12] Martin Bohner and Gusein Sh. Guseinov gave the concept of h-Laplace transform on time scale $\mathbb{T}_h^+$

Definition 2.2. *[12] If $x : \mathbb{T}_h^+ \to \mathbb{C}$ is a function, then its h-Laplace transform is defined by*

$$\mathcal{L}\{x\}(z) = \frac{h}{1+hz}\sum_{k=0}^{\infty}\frac{x(kh)}{(1+hz)^k} \tag{2.1}$$

for those values of $z \neq -\frac{1}{h}$ for which the series converges.

Definition 2.3. *[12] For given functions $x, y : \mathbb{T}_h^+ \to \mathbb{C}$ their Laplace convolution $x \underset{\mathcal{L}}{*} y$ is defined by*

$$\begin{aligned}&(x \underset{\mathcal{L}}{*} y)(kh) = h\sum_{m=0}^{k-1} x(kh - mh - h)y(mh) \quad \textit{for } k \in \mathbb{N}^*,\\ &(x \underset{\mathcal{L}}{*} y)(0h) = 0.\end{aligned} \tag{2.2}$$

Remark 2.1. *Let $x \in L_1(\mathbb{T}_h^+)$. For $z \in \mathbb{C}$, $\Re z \geq 0$ we have $\left|\dfrac{x(kh)}{(1+hz)^k}\right| \leq |x(kh)|$. Since $x \in L_1(\mathbb{T}_h^+)$ the series $\sum_{k=0}^{\infty} |x(kh)|$ converges. By comparison test $\sum_{k=0}^{\infty} \dfrac{x(kh)}{(1+hz)^k}$ converges. Hence for $z \in \mathbb{C}$, $\Re z \geq 0$ the series $\mathcal{L}\{x\}(z)$ converges.*

Setting $h_* = -\frac{1}{h}$, we can rewrite the formula (2.1) in the form (see [12])

$$\mathcal{L}\{x\}(z) = \frac{1}{z - h_*} \sum_{k=0}^{\infty} \frac{x(kh)}{h^k (z-h_*)^k} \tag{2.3}$$

Remark 2.2. *[12] The domain of existence for the h-Laplace transform (2.1) of function x is investigated as below:*
Set

$$R = \limsup_{k \to \infty} \sqrt[k]{|x(kh)|}.$$

(i) If $0 < R < \infty$ the series (2.3) converges in the region $|z - h_*| > \frac{R}{h}$ and diverges for $|z - h_*| < \frac{R}{h}$.

(ii) If $R = 0$ then the series (2.3) converges everywhere with the exception of $z = h_*$.

(iii) If $R = \infty$ then the series (2.3) diverges everywhere.

Proposition 2.2. *[12] If $\mathcal{L}\{x\}(z)$ exists for $|z - h_*| > A$ and $\mathcal{L}\{y\}(z)$ exists for $|z - h_*| > B$ then the Laplace convolution defined in (2.2) satisfies*

$$\mathcal{L}\{x \underset{\mathcal{L}}{*} y\}(z) = \mathcal{L}\{x\}(z)\mathcal{L}\{y\}(z) \quad \text{for } |z - h_*| > \max\{A, B\}.$$

Lemma 2.1. *If $x \in L_1(\mathbb{T}_h^+)$ then its h-Laplace transform $\mathcal{L}\{x\}(z)$ is analytic in the region $\Re z > 0$.*

Proof. Let us denote

$$\mathcal{L}_n\{x\}(z) = h \sum_{k=0}^{n} \frac{x(kh)}{(1+hz)^{k+1}}.$$

We can see that each function $\mathcal{L}_n\{x\}(z)$ is analytic in the region $\Re z > 0$. For $\Re z > 0$ then $|1 + hz| \geq \Re(1 + hz) \geq 1$ so we have the following estimate:

$$|\mathcal{L}\{x\}(z) - \mathcal{L}_n\{x\}(z)| \leq h \sum_{k=n+1}^{\infty} \frac{|x(kh)|}{|1+hz|^{k+1}} \leq h \sum_{k=n+1}^{\infty} |x(kh)|. \tag{2.4}$$

Since $x \in L_1(\mathbb{T}_h^+)$ from (2.4) the sequence $\mathcal{L}_n\{x\}(z)$ converges uniformly to $\mathcal{L}\{x\}(z)$ with respect to z in the region $\Re z > 0$ therefore $\mathcal{L}\{x\}(z)$ is analytic in the region $\Re z > 0$. $\square$

The Fourier cosine transform on time scale $\mathbb{T}_h^+$ is defined as the following:

Definition 2.4. *[14] For a real valued function $x \in L_1(\mathbb{T}_h^+)$ its Fourier cosine transform is defined by*

$$\mathcal{F}_c\{x\}(u) = hx(0) + 2h\sum_{n=1}^{\infty} x(nh)\cos(unh), \quad u \in \mathbb{R}. \tag{2.5}$$

For the case $h = 1$, (2.5) becomes two times the discrete time Fourier cosine transform studied in [13].

Definition 2.5.

$$\mathcal{A}_c = \{\mathcal{F}_c\{x\}(u), u \in [0, \frac{\pi}{h}] \big| x \in L_1(\mathbb{T}_h^+)\} \tag{2.6}$$

We call $\mathcal{A}_c$ the image space of $L_1(\mathbb{T}_h^+)$ through the Fourier cosine transform $\mathcal{F}_c$.

For $\mathcal{F}_c\{x\} \in \mathcal{A}_c$ the inverse Fourier cosine transform is given by

$$x(nh) = \frac{1}{\pi}\int_0^{\frac{\pi}{h}} \mathcal{F}_c\{x\}(u)\cos(unh)du, \quad n \in \mathbb{N}_0. \tag{2.7}$$

Definition 2.6. *[14] The Fourier cosine convolution on time scale of two functions $x, y \in L_1(\mathbb{T}_h^+)$ is defined as*

$$(x \underset{\mathcal{F}_c}{*} y)(t) = h\Big\{\sum_{n=1}^{\infty} x(nh)\Big[y(|t-nh|) + y(t+nh)\Big] + x(0)y(t)\Big\}, \quad t \in \mathbb{T}_h^+. \tag{2.8}$$

Proposition 2.3. *[14] Let $x, y \in L_1(\mathbb{T}_h^+)$ then $x \underset{\mathcal{F}_c}{*} y \in L_1(\mathbb{T}_h^+)$,*

$$\|x \underset{\mathcal{F}_c}{*} y\|_1 \leq \|x\|_1\|y\|_1$$

and we have the factorization equality

$$\mathcal{F}_c\{x \underset{\mathcal{F}_c}{*} y\}(u) = \mathcal{F}_c\{x\}(u)\mathcal{F}_c\{y\}(u), \quad u \in [0, \frac{\pi}{h}]. \tag{2.9}$$

Lemma 2.2. *[8](Wiener-Levy type Theorem for Fourier cosine series) Let $x \in L_1(\mathbb{T}_h^+)$ and $\Phi(z)$ be an analytic function whose domain contains the range of $\mathcal{F}_c\{x\}(u)$ and satisfies $\Phi(0) = 0$. Then $\Phi(\mathcal{F}_c\{x\}(u))$ is a Fourier cosine transform of a function in $L_1(\mathbb{T}_h^+)$.*

3. Generalized convolution for Fourier cosine and h-Laplace transform on time scale

Notation 1. *For $m, n \in \mathbb{N}_0$ we define*

$$I(n,m) = \int_0^\pi \frac{\cos(nu)}{(1+u)^{m+1}} du. \tag{3.1}$$

Definition 3.1. *The generalized convolution of two functions $x, y \in L_1(\mathbb{T}_h^+)$ with respect to the Fourier cosine and h-Laplace transform on time scale $\mathbb{T}_h^+$ is defined as*

$$(x * y)(kh) = \frac{h}{2\pi} x(0) \sum_{m=0}^{\infty} y(mh)\theta(k,0,m) + \frac{h}{\pi} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} x(nh)y(mh)\theta(k,n,m), \quad k \in \mathbb{N}_0 \tag{3.2}$$

in here

$$\theta(k,n,m) = I(n+k,m) + I(|n-k|,m). \tag{3.3}$$

Notation 2. *For each function x we denote x_1 a function on $\mathbb{T}_h^+$ defined by*

$$x_1(0) = \frac{1}{2} x(0), \quad x_1(nh) = x(nh), \ \textit{for } n \in \mathbb{N}. \tag{3.4}$$

The formula (3.2) can be written in the form

$$(x * y)(kh) = \frac{h}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x_1(nh)y(mh)\theta(k,n,m), \quad k \in \mathbb{N}_0. \tag{3.5}$$

Lemma 3.1. *The following properties for $I(n,m)$ can be obtained straightforward.*

(i) $I(0,0) = \ln(1+\pi)$

(ii) $I(0,m) = \frac{1}{m}\Big[1 - \frac{1}{(1+\pi)^m}\Big], \ m \in \mathbb{N}$

(iii) $|I(n,m)| \le I(0,m) \le \ln(1+\pi), \ m, n \in \mathbb{N}_0.$

(iv) $I(n,m) = \frac{1}{m}\Big[1 + \frac{(-1)^{n+1}}{(1+\pi)^m}\Big] - \frac{n^2}{m(m-1)} I(n,m-2), \ m \ge 2, n \in \mathbb{N}_0.$

We use Lemma 3.1 (iii) and (3.3) to obtain $|\theta(k,n,m)| \le 2\ln(1+\pi)$ and consequently for $x, y \in L_1(\mathbb{T}_h^+)$ the expression (3.2) is well defined.

Lemma 3.2. *For $n \in \mathbb{N}, m \in \mathbb{N}_0$ the following equality holds.*

$$I(n,m) = \frac{1}{m!} \int_0^\infty \frac{t^{m+1} e^{-t}}{n^2 + t^2} \Big[1 - (-1)^n e^{-\pi t}\Big] dt. \tag{3.6}$$

Proof. By changing of variable

$$\int_0^\infty t^m e^{-t(1+u)}dt = \frac{1}{(1+u)^{m+1}}\int_0^\infty z^m e^{-z}dz = \frac{1}{(1+u)^{m+1}}\Gamma(m+1)$$
$$= \frac{m!}{(1+u)^{m+1}} \tag{3.7}$$

Substituting (3.7) into (3.1) to get

$$\begin{aligned} I(n,m) &= \frac{1}{m!}\int_0^\pi \cos(nu)du \int_0^\infty t^m e^{-t(1+u)}dt = \frac{1}{m!}\int_0^\infty t^m e^{-t}dt \int_0^\pi \cos(nu)e^{-tu}du \\ &= \frac{1}{m!}\int_0^\infty t^m e^{-t}\frac{t}{n^2+t^2}\Big[1-(-1)^n e^{-\pi t}\Big]dt \\ &= \frac{1}{m!}\int_0^\infty \frac{t^{m+1}e^{-t}}{n^2+t^2}\Big[1-(-1)^n e^{-\pi t}\Big]dt. \end{aligned}$$

□

Notation 3. *In [15], page 386 we know the following functions:*

$$ci(u) = \int_u^\infty \frac{\cos t}{t}dt, \quad si(u) = -\int_u^\infty \frac{\sin t}{t}dt, \quad u > 0. \tag{3.8}$$

Lemma 3.3. *For $n \in \mathbb{N}$ we have*

(i) $I(n,0) = \cos(n)\big[ci(n) - ci(n+n\pi)\big] + \sin(n)\big[si(n+n\pi) - si(n)\big],$

(ii) *For* $m \in \mathbb{N}$

$$\begin{aligned} I(n,2m) =& \frac{(-1)^m n^{2m}}{(2m)!}\Big\{\cos(n)[ci(n) - ci(n+n\pi)] + \sin(n)[si(n+n\pi) - si(n)]\Big\} + \\ & \frac{1}{(2m)!}\sum_{k=0}^{m-1}(-1)^{m-1-k}n^{2m-2-2k}(2k+1)!\Big[1 - \frac{(-1)^n}{(1+\pi)^{2k+2}}\Big], \end{aligned} \tag{3.9}$$

(iii) *For* $m \in \mathbb{N}_0$

$$\begin{aligned} I(n,2m+1) =& \frac{(-1)^{m+1}n^{2m+2}}{n(2m+1)!}\Big\{\sin(n)[ci(n+n\pi) - ci(n)] + \\ & \cos(n)[si(n+n\pi) - si(n)]\Big\} + \frac{1}{(2m+1)!}\sum_{k=0}^{m}(-1)^{m-k}n^{2m-2k}(2k)!\Big[1 - \frac{(-1)^n}{(1+\pi)^{2k+1}}\Big]. \end{aligned} \tag{3.10}$$

Proof. (i) From formula (3.1)

$$
\begin{aligned}
I(n,0) &= \int_0^{\pi} \frac{\cos(nu)}{1+u}du = \int_1^{1+\pi} \frac{\cos(nv-n)}{v}dv \\
&= \cos(n)\int_1^{1+\pi} \frac{\cos(nv)}{v}dv + \sin(n)\int_1^{1+\pi} \frac{\sin(nv)}{v}dv \\
&= \cos(n)\int_n^{n+n\pi} \frac{\cos(s)}{s}ds + \sin(n)\int_n^{n+n\pi} \frac{\sin(s)}{s}ds \\
&= \cos(n)\big[ci(n) - ci(n+n\pi)\big] + \sin(n)\big[si(n+n\pi) - si(n)\big].
\end{aligned}
$$

(ii) Using (3.6) and the equality

$$
\frac{t^{2m+1} - t(-n^2)^m}{n^2+t^2} = \sum_{k=0}^{m-1} t^{2k+1}(-n^2)^{m-1-k}
$$

we get

$$
\begin{aligned}
I(n,2m) &= \frac{1}{(2m)!}\int_0^{\infty} \frac{t^{2m+1}e^{-t}}{n^2+t^2}\Big[1-(-1)^n e^{-\pi t}\Big]dt. \\
&= \frac{(-1)^m n^{2m}}{(2m)!}\int_0^{\infty} \frac{te^{-t}}{n^2+t^2}\Big[1-(-1)^n e^{-\pi t}\Big]dt + \\
&\quad \frac{1}{(2m)!}\sum_{k=0}^{m-1}(-1)^{m-1-k}n^{2m-2-2k}\int_0^{\infty} t^{2k+1}e^{-t}\Big[1-(-1)^n e^{-\pi t}\Big]dt.
\end{aligned}
\tag{3.11}
$$

We compute the integrals inside (3.11)

$$
\int_0^{\infty} t^{2k+1}e^{-t}dt = \Gamma(2k+2) = (2k+1)! \tag{3.12}
$$

$$
\begin{aligned}
\int_0^{\infty} t^{2k+1}e^{-t}e^{-\pi t}dt &= \int_0^{\infty} t^{2k+1}e^{-(1+\pi)t}dt \\
&= \frac{1}{(1+\pi)^{2k+2}}\int_0^{\infty} s^{2k+1}e^{-s}ds = \frac{\Gamma(2k+2)}{(1+\pi)^{2k+2}} = \frac{(2k+1)!}{(1+\pi)^{2k+2}}.
\end{aligned}
\tag{3.13}
$$

Using the formula for Laplace transform in [15], page 135 for $\alpha = n,\ p = 1,\ A = 1,\ B = 0$ we have

$$
\int_0^{\infty} \frac{te^{-t}}{n^2+t^2}dt = \cos(n)ci(n) - \sin(n)si(n). \tag{3.14}
$$

Using the formula for Laplace transform in [15], page 135 for $\alpha = n,\ p = 1 + \pi,\ A = 1,\ B = 0$ we have

$$\int_0^\infty \frac{te^{-t}e^{-\pi t}}{n^2+t^2}dt = \cos(n+n\pi)ci(n+n\pi) - \sin(n+n\pi)si(n+n\pi)$$
$$= (-1)^n\Big[\cos(n)ci(n+n\pi) - \sin(n)si(n+n\pi)\Big]. \quad (3.15)$$

Plugging (3.12), (3.13), (3.14) and (3.15) into (3.11) we get (3.9).

(iii) Using (3.6) and the equality

$$\frac{t^{2m+2}-(-n^2)^{m+1}}{n^2+t^2} = \sum_{k=0}^{m} t^{2k}(-n^2)^{m-k}$$

we get

$$I(n,2m+1) = \frac{1}{(2m+1)!}\int_0^\infty \frac{t^{2m+2}e^{-t}}{n^2+t^2}\Big[1-(-1)^n e^{-\pi t}\Big]dt$$
$$= \frac{(-1)^{m+1}n^{2m+2}}{(2m+1)!}\int_0^\infty \frac{e^{-t}}{n^2+t^2}\Big[1-(-1)^n e^{-\pi t}\Big]dt +$$
$$\frac{1}{(2m+1)!}\sum_{k=0}^{m}(-1)^{m-k}n^{2m-2k}\int_0^\infty t^{2k}e^{-t}\Big[1-(-1)^n e^{-\pi t}\Big]dt. \quad (3.16)$$

We compute the integrals inside (3.16)

$$\int_0^\infty t^{2k}e^{-t}dt = \Gamma(2k+1) = (2k)! \quad (3.17)$$

$$\int_0^\infty t^{2k}e^{-t}e^{-\pi t}dt = \int_0^\infty t^{2k}e^{-(1+\pi)t}dt$$
$$= \frac{1}{(1+\pi)^{2k+1}}\int_0^\infty s^{2k}e^{-s}ds = \frac{\Gamma(2k+1)}{(1+\pi)^{2k+1}} = \frac{(2k)!}{(1+\pi)^{2k+1}}. \quad (3.18)$$

Using the formula for Laplace transform in [15], page 135 for $\alpha = n,\ p = 1,\ A = 0,\ B = \frac{1}{n}$ we have

$$\int_0^\infty \frac{e^{-t}}{n^2+t^2}dt = -\frac{1}{n}\sin(n)ci(n) - \frac{1}{n}\cos(n)si(n). \quad (3.19)$$

Using the formula for Laplace transform in [15], page 135 for $\alpha = n,\ p = 1 + \pi,\ A = 0,\ B = \frac{1}{n}$ we have

$$\int_0^\infty \frac{e^{-t}e^{-\pi t}}{n^2+t^2}dt = -\frac{1}{n}\sin(n+n\pi)ci(n+n\pi) - \frac{1}{n}\cos(n+n\pi)si(n+n\pi)$$
$$= \frac{(-1)^{n+1}}{n}\Big[\sin(n)ci(n+n\pi) + \frac{1}{n}\cos(n)si(n+n\pi)\Big]. \quad (3.20)$$

Plugging (3.17), (3.18), (3.19) and (3.20) into (3.16) we get (3.10).

□

Lemma 3.4. *(i) For $m, n \in \mathbb{N}_0$ we have*

$$I(n, m) > 0, \tag{3.21}$$

(ii) For $m \in \mathbb{N}_0$ we have

$$\sum_{n=1}^{\infty} I(n, m) < \pi. \tag{3.22}$$

Proof. (i) For $n = 0$ from the result in Lemma 3.1 (i) and (ii) we have $I(0, m) > 0\ \forall m \in \mathbb{N}_0$.

For $n > 0$ from (3.6)

$$I(n, m) = \frac{1}{m!} \int_0^{\infty} \frac{t^{m+1}e^{-t}}{n^2 + t^2} \Big[1 - (-1)^n e^{-\pi t}\Big] dt.$$

For $t > 0$ we have $0 < 1 - (-1)^n e^{-\pi t}$. Hence $I(n, m) > 0$

(ii) For $t > 0$ we have $1 - (-1)^n e^{-\pi t} < 2$. Then

$$I(n, m) < \frac{2}{m!} \int_0^{\infty} \frac{t^{m+1}e^{-t}}{n^2 + t^2} dt. \tag{3.23}$$

Moreover

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + t^2} \leq \sum_{n=1}^{\infty} \int_{n-1}^{n} \frac{dx}{x^2 + t^2} = \int_0^{\infty} \frac{dx}{x^2 + t^2} = \frac{1}{t} \Big[\arctan \frac{x}{t}\Big]_{x=0}^{\infty} = \frac{\pi}{2t}. \tag{3.24}$$

Combining (3.23) with (3.24) the following inequality holds

$$\sum_{n=1}^{\infty} I(n, m) < \frac{\pi}{m!} \int_0^{\infty} t^m e^{-t} dt = \frac{\pi}{m!} \Gamma(m + 1) = \pi.$$

□

Theorem 3.1. *Let x, y be any two functions in $L_1(\mathbb{T}_h^+)$ then their generalized convolution defined in (3.2) satisfies $x * y \in L_1(\mathbb{T}_h^+)$ and we have the estimate*

$$\|x * y\|_1 \leq \Big[2 + \frac{\ln(1 + \pi)}{\pi}\Big] \|x\|_1 \|y\|_1. \tag{3.25}$$

Moreover the following factorization equality holds

$$\mathcal{F}_c\{x * y\}(u) = \mathcal{F}_c\{x\}(u) \mathcal{L}\{y\}(u),\ \forall u \in [0, \frac{\pi}{h}]. \tag{3.26}$$

Proof. Firstly we will prove that $x * y \in L_1(\mathbb{T}_h^+)$.
We define function x_1 as in (3.4). From (3.2) and (3.21)

$$|(x*y)(0)| + 2\sum_{k=1}^{\infty} |(x*y)(kh)| \le \frac{h}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|x_1(nh)||y(mh)|\Big[\theta(0,n,m)+ 2\sum_{k=1}^{\infty}\theta(k,n,m)\Big]. \tag{3.27}$$

The expression inside bracket can be estimated using (3.22)

$$\begin{aligned}\theta(0,n,m) + 2\sum_{k=1}^{\infty}\theta(k,n,m) &= 2I_{n,m} + 2\sum_{k=1}^{\infty}\big[I(n+k,m) + I(|n-k|,m)\big] \\ &= 2\big[I(0,m) + 2\sum_{s=1}^{\infty} I(s,m)\big] < 2\big[\ln(1+\pi) + 2\pi\big].\end{aligned} \tag{3.28}$$

Substituting (3.28) into (3.27) we obtain

$$\begin{aligned}|(x*y)(0)| + 2\sum_{k=1}^{\infty}|(x*y)(kh)| &\le \frac{2h}{\pi}\big[2\pi + \ln(1+\pi)\big]\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|x_1(nh)||y(mh)| \\ &\le 2h\big[2 + \frac{\ln(1+\pi)}{\pi}\big]\frac{\|x\|_1}{2h}\frac{\|g\|_1}{h}.\end{aligned} \tag{3.29}$$

Multiplying (3.29) by h we have

$$\|x*y\|_1 \le \Big[2 + \frac{\ln(1+\pi)}{\pi}\Big]\|x\|_1\|y\|_1.$$

For $k \in \mathbb{N}_0$ it follows from (3.5)

$$\begin{aligned}(x*y)(kh) &= \frac{h}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_1(nh)y(mh)\theta(k,n,m) \\ &= \frac{h}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_1(nh)y(mh)\Big[I(n+k,m) + I(|n-k|,m)\Big] \\ &= \frac{h}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_1(nh)y(mh)\int_0^{\pi}\frac{\cos(n+k)u + \cos(n-k)u}{(1+u)^{m+1}}du \\ &= \frac{h^2}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_1(nh)y(mh)\int_0^{\frac{\pi}{h}}\frac{\cos(n+k)uh + \cos(n-k)uh}{(1+hu)^{m+1}}du \\ &= \frac{2h^2}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_1(nh)y(mh)\int_0^{\frac{\pi}{h}}\frac{\cos(nuh)\cos(kuh)}{(1+hu)^{m+1}}du \\ &= \frac{1}{\pi}\int_0^{\frac{\pi}{h}} 2h^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\cos(unh)}{(1+hu)^{m+1}}x_1(nh)y(mh)\cos(kuh)du.\end{aligned} \tag{3.30}$$

We compute the product of Fourier cosine and h-Laplace transform of two functions x, y using formulas (2.1) and (2.5)

$$\begin{aligned}\mathcal{F}_c\{x\}(u)\mathcal{L}\{y\}(u) &= 2h^2\sum_{n=0}^{\infty}x_1(nh)\cos(unh)\sum_{m=0}^{\infty}\frac{y(mh)}{(1+hu)^{m+1}}\\ &= 2h^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\cos(unh)}{(1+hu)^{m+1}}x_1(nh)y(mh).\end{aligned}\tag{3.31}$$

Substituting (3.31) into (3.30) we get

$$(x*y)(kh) = \frac{1}{\pi}\int_0^{\frac{\pi}{h}}\mathcal{F}_c\{x\}(u)\mathcal{L}\{y\}(u)\cos(kuh)du,\ \forall k\in\mathbb{N}_0.\tag{3.32}$$

Moreover from inverse Fourier cosine transform (2.7) we have

$$(x*y)(kh) = \frac{1}{\pi}\int_0^{\frac{\pi}{h}}\mathcal{F}_c\{x*y\}(u)\cos(kuh)du,\ \forall k\in\mathbb{N}_0.\tag{3.33}$$

By (3.32) and (3.33) we then get the factorization equality (3.26). □

Theorem 3.2. ***(Titchmarsh's type Theorem)*** : *Let $x\in L_1(\mathbb{T}_h^+, e^{\alpha nh})$ and $y\in L_1(\mathbb{T}_h^+)$. If $x*y\equiv 0$ then $x\equiv 0$ or $y\equiv 0$.*

Proof. Since $x*y\equiv 0$ we have

$$\mathcal{F}_c\{x*y\}(u) = 0,\ \ \text{for all } u\in[0,\frac{\pi}{h}].\tag{3.34}$$

Using (3.26) and (3.34)

$$\mathcal{F}_c\{x\}(u)\mathcal{L}\{y\}(u)\equiv 0,\ \ \text{for all } u\in[0,\frac{\pi}{h}].\tag{3.35}$$

Applying Lemma 2.1 then $\mathcal{L}\{y\}(u)$ is an analytic function in the region $\Re u>0$. We have

$$\mathcal{F}_c\{x\}(u) = hx(0) + 2h\sum_{n=1}^{\infty}x(nh)\cos(unh).\tag{3.36}$$

For $k\in\mathbb{N}$ by calculation

$$\begin{aligned}\Big|\frac{d^k}{du^k}\big[x(nh)\cos(unh)\big]\Big| &= \Big|x(nh)(nh)^k\cos(unh+k\frac{\pi}{2})\Big| \le |x(nh)|(nh)^k\\ &\le e^{\alpha nh}|x(nh)|\frac{(\alpha nh)^k e^{-\alpha nh}}{\alpha^k}.\end{aligned}\tag{3.37}$$

We see that

$$0 \leq (\alpha nh)^k e^{-\alpha nh} = e^{-\alpha nh}\frac{(\alpha nh)^k}{k!}k! \leq k!. \tag{3.38}$$

From (3.36), (3.37) and (3.38) and Definition 2.1

$$\Big|\frac{d^k\big(\mathcal{F}_c\{x\}(u)\big)}{du^k}\Big| \leq \frac{k!}{\alpha^k}\Big(2h\sum_{n=1}^{\infty} e^{\alpha nh}|x(nh)|\Big) \leq C\frac{k!}{\alpha^k}, \quad \text{for all } u \in [0, \frac{\pi}{h}].$$

The Taylor expansion of $\mathcal{F}_c\{x\}(u)$ is

$$\mathcal{F}_c\{x\}(u) = \mathcal{F}_c\{x\}(u_0) + \sum_{n=1}^{\infty}\frac{1}{n!}\frac{d^n\big(\mathcal{F}_c\{x\}(u)\big)}{du^n}\Big|_{u=u_0}(u-u_0)^n, \ u_0 \in (0, \frac{\pi}{h}). \tag{3.39}$$

We estimate the general component of the series as the following:

$$\Big|\frac{1}{n!}\frac{d^n\big(\mathcal{F}_c\{x\}(u)\big)}{du^n}\Big|_{u=u_0}(u-u_0)^n\Big| \leq \frac{1}{n!}C\frac{n!}{\alpha^n}|u-u_0|^n = C\big(\frac{|u-u_0|}{\alpha}\big)^n.$$

Therefore, the series (3.39) converges if $|u-u_0| < \alpha$, it means that $\mathcal{F}_c\{x\}(u)$ is analytic for all $u \in (0, \frac{\pi}{h})$. Moreover we know that $\mathcal{L}\{y\}(u)$ is an analytic function in the region $\Re u > 0$. Hence from (3.35) we get $\mathcal{F}_c\{x\}(u) \equiv 0$ or $\mathcal{L}\{y\}(u) \equiv 0$ for all $u \in [0, \frac{\pi}{h}]$. Therefore $x(nh) = 0, \forall n$ or $y(mh) = 0, \forall m$. This completes the Theorem.

□

4. Some applications

4.1. Two linear equations of convolution type

In this subsection we will study two linear equations

$$\frac{hx(0)}{2\pi}\sum_{m=0}^{\infty} y(mh)\theta(k,0,m) + \frac{h}{\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} x(nh)y(mh)\theta(k,n,m) = z(kh), \quad \forall k \in \mathbb{N}_0 \tag{4.1}$$

$$x(kh) + \frac{hx(0)}{2\pi}\sum_{m=0}^{\infty} y(mh)\theta(k,0,m) + \frac{h}{\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} x(nh)y(mh)\theta(k,n,m) = z(kh), \quad \forall k \in \mathbb{N}_0 \tag{4.2}$$

Here $y, z \in L_1(\mathbb{T}_h^+)$ are given functions and $x \in L_1(\mathbb{T}_h^+)$ is an unknown function.

Theorem 4.1. *Let $y, z \in L_1(\mathbb{T}_h^+)$ and $\mathcal{L}\{y\}(u) \neq 0$ on $[0, \frac{\pi}{h}]$. Then the necessary and sufficient condition for the equation (4.1) to have a solution in $L_1(\mathbb{T}_h^+)$ is $\dfrac{\mathcal{F}_c\{z\}(u)}{\mathcal{L}\{y\}(u)} \in \mathcal{A}_c$*

where $\mathcal{A}_c$ is defined in (2.6).

Moreover the solution is of the form

$$x(nh) = \frac{1}{\pi}\int_0^{\frac{\pi}{h}} \frac{\mathcal{F}_c\{z\}(u)}{\mathcal{L}\{y\}(u)}\cos(unh)du,\ n \in \mathbb{N}_0. \tag{4.3}$$

Proof. Using Definition 3.1, the equation (4.1) can be written in the form

$$(x * y)(kh) = z(kh),\ \forall k \in \mathbb{N}_0. \tag{4.4}$$

- The necessary condition. Applying the Fourier cosine transform to both sides of (4.4) and using the factorization equality (3.26) we get

  $$\mathcal{F}_c\{x\}(u)\mathcal{L}\{y\}(u) = \mathcal{F}_c\{z\}(u),\ u \in [0, \frac{\pi}{h}].$$

  Hence $\mathcal{F}_c\{x\}(u) = \dfrac{\mathcal{F}_c\{z\}(u)}{\mathcal{L}\{y\}(u)} \in \mathcal{A}_c$ and the solution is given by (4.3).

- The sufficient condition. If $\dfrac{\mathcal{F}_c\{z\}(u)}{\mathcal{L}\{y\}(u)} \in \mathcal{A}_c$ then there exists $x \in L_1(\mathbb{T}_h^+)$ such that $\mathcal{F}_c\{x\}(u) = \dfrac{\mathcal{F}_c\{z\}(u)}{\mathcal{L}\{y\}(u)},\ u \in [0, \frac{\pi}{h}]$. Therefore

  $$\mathcal{F}_c\{x * y\}(u) = \mathcal{F}_c\{x\}(u)\mathcal{L}\{y\}(u) = \mathcal{F}_c\{z\}(u),\ u \in [0, \frac{\pi}{h}]. \tag{4.5}$$

  Taking the inverse Fourier cosine transform of (4.5) we have $(x * y)(kh) = z(kh),\ \forall k \in \mathbb{N}_0$.

□

Lemma 4.1. *Let $f \in L_1(\mathbb{T}_h^+)$ then there exists $g \in L_1(\mathbb{T}_h^+)$ such that*

$$\mathcal{F}_c\{g\}(u) = \mathcal{L}\{f\}(u),\ \forall u \in [0, \frac{\pi}{h}], \tag{4.6}$$

$$\|g\|_1 \leq \Big(\frac{\ln(1+\pi)}{\pi} + 2\Big)\|f\|_1. \tag{4.7}$$

Proof. We choose a function g defined on $\mathbb{T}_h^+$ by

$$g(nh) = \frac{1}{\pi}\int_0^{\frac{\pi}{h}} \mathcal{L}\{f\}(u)\cos(unh)du,\ n \in \mathbb{N}_0. \tag{4.8}$$

We will prove that $g \in L_1(\mathbb{T}_h^+)$. Using the definition of h-Laplace transform in (2.1) and substituting to (4.8)

$$\begin{aligned} g(nh) &= \frac{h}{\pi}\sum_{k=0}^{\infty} f(kh)\int_0^{\frac{\pi}{h}} \frac{\cos(unh)}{(1+hu)^{k+1}}du \\ &= \frac{1}{\pi}\sum_{k=0}^{\infty} f(kh)\int_0^{\pi} \frac{\cos(vn)}{(1+v)^{k+1}}dv = \frac{1}{\pi}\sum_{k=0}^{\infty} f(kh)I(n,k). \end{aligned} \tag{4.9}$$

From (4.9) and (3.22)

$$\sum_{n=1}^{\infty}|g(nh)| \leq \frac{1}{\pi}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}|f(kh)|I(n,k) = \frac{1}{\pi}\sum_{k=0}^{\infty}|f(kh)|\sum_{n=1}^{\infty}I(n,k) \leq \sum_{k=0}^{\infty}|f(kh)|, \tag{4.10}$$

$$|g(0)| \leq \frac{1}{\pi}\sum_{k=0}^{\infty}|f(kh)|I(0,k) \leq \frac{\ln(1+\pi)}{\pi}\sum_{k=0}^{\infty}|f(kh)|. \tag{4.11}$$

From the estimates (4.10) and (4.11)

$$|g(0)| + 2\sum_{n=1}^{\infty}|g(nh)| \leq \Big(\frac{\ln(1+\pi)}{\pi}+2\Big)\sum_{k=0}^{\infty}|f(kh)| \leq \Big(\frac{\ln(1+\pi)}{\pi}+2\Big)\frac{\|f\|_1}{h} < \infty.$$

Therefore $g \in L_1(\mathbb{T}_h^+)$ and inequality (4.7) holds.
From (4.8) and the inverse Fourier cosine transform formula (2.7) and we get g is the inverse Fourier cosine transform of $\mathcal{L}\{f\}$ so equality (4.6) holds. $\square$

Theorem 4.2. *The necessary and sufficient condition for the equation (4.2) to have a unique solution in $L_1(\mathbb{T}_h^+)$, for all right hand side $z \in L_1(\mathbb{T}_h^+)$, is*

$$1 + \mathcal{L}\{y\}(u) \neq 0,\ \forall u \in [0, \frac{\pi}{h}]. \tag{4.12}$$

The solution of (4.2) has the form

$$x(nh) = z(nh) - (z \underset{\mathcal{F}_c}{*} \ell)(nh),\ \forall n \in \mathbb{N}_0,$$

where $\ell \in L_1(\mathbb{T}_h^+)$ is defined by

$$\mathcal{F}_c\{\ell\}(u) = \frac{\mathcal{L}\{y\}(u)}{1+\mathcal{L}\{y\}(u)},\ u \in [0, \frac{\pi}{h}].$$

Proof. Using Definition 3.1, we can rewrite the equation (4.2) as

$$x(kh) + (x * y)(kh) = z(kh),\ \forall k \in \mathbb{N}_0.$$

Applying the Fourier cosine transform on both sides of the previous equation and using equality (3.26) we get

$$\mathcal{F}_c\{x\}(u) + \mathcal{F}_c\{x\}(u)\mathcal{L}\{y\}(u) = \mathcal{F}_c\{z\}(u),\ u \in [0, \frac{\pi}{h}].$$

It means that $\mathcal{F}_c\{x\}(u)\big[1 + \mathcal{L}\{y\}(u)\big] = \mathcal{F}_c\{z\}(u),\ \forall u \in [0, \frac{\pi}{h}]$.
Therefore (4.12) is the necessary condition for the equation (4.2) to have a unique solution in $L_1(\mathbb{T}_h^+)$, for all right hand side $z \in L_1(\mathbb{T}_h^+)$.
Moreover if the condition (4.12) holds then

$$\mathcal{F}_c\{x\}(u) = \frac{\mathcal{F}_c\{z\}(u)}{1 + \mathcal{L}\{y\}(u)}, \forall u \in [0, \frac{\pi}{h}]. \tag{4.13}$$

Since $y \in L_1(\mathbb{T}_h^+)$, by Lemma 4.1 there exists a function $g \in L_1(\mathbb{T}_h^+)$ such that $\mathcal{F}_c\{g\}(u) = \mathcal{L}\{y\}(u), \forall u \in [0, \frac{\pi}{h}]$. Now (4.13) is equivalent to

$$\mathcal{F}_c\{x\}(u) = \mathcal{F}_c\{z\}(u) - \mathcal{F}_c\{z\}(u)\frac{\mathcal{F}_c\{g\}(u)}{1 + \mathcal{F}_c\{g\}(u)},\ \forall u \in [0, \frac{\pi}{h}]. \tag{4.14}$$

By Wiener-Levy type Theorem, there exists a function $\ell \in L_1(\mathbb{T}_h^+)$ such that

$$\mathcal{F}_c\{\ell\}(u) = \frac{\mathcal{F}_c\{g\}(u)}{1 + \mathcal{F}_c\{g\}(u)} = \frac{\mathcal{L}\{y\}(u)}{1 + \mathcal{L}\{y\}(u)},\ u \in [0, \frac{\pi}{h}].$$

Hence (4.14) is equivalent to

$$\mathcal{F}_c\{x\}(u) = \mathcal{F}_c\{z\}(u) - \mathcal{F}_c\{z\}(u)\mathcal{F}_c\{\ell\}(u),\ \forall u \in [0, \frac{\pi}{h}].$$

Using the factorization equality (2.9) we obtain

$$\mathcal{F}_c\{x\}(u) = \mathcal{F}_c\{z\}(u) - \mathcal{F}_c\{z \underset{\mathcal{F}_c}{*} \ell\}(u).$$

This leads to

$$x(nh) = z(nh) - (z \underset{\mathcal{F}_c}{*} \ell)(nh),\ \forall n \in \mathbb{N}_0.$$

$\square$

4.2. System of two linear equations of convolution type

In this subsection we will study system of two linear equations of convolution type. Consider the system of two linear equations in the following form:

$$\begin{cases} x(kh) + h\Big\{ \sum\limits_{n=1}^{\infty} y(nh)\big[u(|kh-nh|) + u(kh+nh)\big] + y(0)u(kh)\Big\} = z(kh), \\ \dfrac{hx(0)}{2\pi} \sum\limits_{m=0}^{\infty} v(mh)\theta(k,0,m) + \dfrac{h}{\pi} \sum\limits_{n=1}^{\infty}\sum\limits_{m=0}^{\infty} x(nh)v(mh)\theta(k,n,m) + y(kh) = w(kh) \end{cases} \tag{4.15}$$

for all $k \in \mathbb{N}_0$ where $u, v, z, w \in L_1(\mathbb{T}_h^+)$ are given functions and $x, y \in L_1(\mathbb{T}_h^+)$ are unknown functions.

Since $v \in L_1(\mathbb{T}_h^+)$, by Lemma 4.1 there exists a function $v_1 \in L_1(\mathbb{T}_h^+)$ such that

$$\mathcal{F}_c\{v_1\}(\omega) = \mathcal{L}\{v\}(\omega),\ \forall\omega \in [0, \frac{\pi}{h}]. \tag{4.16}$$

Theorem 4.3. *If* $u, v, z, w \in L_1(\mathbb{T}_h^+)$ *and satisfy*

$$\triangle = 1 - \mathcal{F}_c\{u\}(\omega)\mathcal{L}\{v\}(\omega) \neq 0,\ \forall\omega \in [0, \frac{\pi}{h}]$$

then the system of equations (4.15) has unique solution $x, y \in L_1(\mathbb{T}_h^+)$

$$\begin{cases} x(kh) = z(kh) - (u \underset{\mathcal{F}_c}{*} w)(kh) + (z \underset{\mathcal{F}_c}{*} \ell)(kh) - \big[(u \underset{\mathcal{F}_c}{*} w) \underset{\mathcal{F}_c}{*} \ell\big](kh) \\ y(kh) = w(kh) - (z \underset{\mathcal{F}_c}{*} v_1)(kh) + (w \underset{\mathcal{F}_c}{*} \ell)(kh) - \big[(z \underset{\mathcal{F}_c}{*} v_1) \underset{\mathcal{F}_c}{*} \ell\big](kh) \end{cases}$$

for all $k \in \mathbb{N}_0$ *where* v_1 *is defined by (4.16) and* $\ell \in L_1(\mathbb{T}_h^+)$ *is defined by*

$$\mathcal{F}_c\{\ell\}(\omega) = \frac{\mathcal{F}_c\{u\}(\omega)\mathcal{L}\{v\}(\omega)}{1 - \mathcal{F}_c\{u\}(\omega)\mathcal{L}\{v\}(\omega)}\ \forall\omega \in [0, \frac{\pi}{h}].$$

Proof. Let $v_2 = u \underset{\mathcal{F}_c}{*} v_1 \in L_1(\mathbb{T}_h^+)$. From Wiener-Levy Type Theorem for Fourier cosine series there exists a function $\ell \in L_1(\mathbb{T}_h^+)$ such that

$$\begin{aligned} \mathcal{F}_c\{\ell\}(\omega) &= \frac{\mathcal{F}_c\{v_2\}(\omega)}{1 - \mathcal{F}_c\{v_2\}(\omega)} = \frac{\mathcal{F}_c\{u\}(\omega)\mathcal{F}_c\{v_1\}(\omega)}{1 - \mathcal{F}_c\{u\}(\omega)\mathcal{F}_c\{v_1\}(\omega)} \\ &= \frac{\mathcal{F}_c\{u\}(\omega)\mathcal{L}\{v\}(\omega)}{1 - \mathcal{F}_c\{u\}(\omega)\mathcal{L}\{v\}(\omega)},\ \forall\omega \in [0, \frac{\pi}{h}]. \end{aligned}$$

Using (2.8) and (3.2) the system of equations (4.15) can be written in the form

$$\begin{cases} x(kh) + (y \underset{\mathcal{F}_c}{*} u)(kh) = z(kh) \\ (x * v)(kh) + y(kh) = w(kh). \end{cases} \tag{4.17}$$

Applying the Fourier cosine transform to both sides of the first and the second equations of (4.17) we obtain

$$\begin{cases} \mathcal{F}_c\{x\}(\omega) + \mathcal{F}_c\{y\}(\omega)\mathcal{F}_c\{u\}(\omega) = \mathcal{F}_c\{z\}(\omega),\ \forall \omega \in [0, \frac{\pi}{h}] \\ \mathcal{F}_c\{x\}(\omega)\mathcal{L}\{v\}(\omega) + \mathcal{F}_c\{y\}(\omega) = \mathcal{F}_c\{w\}(\omega),\ \forall \omega \in [0, \frac{\pi}{h}]. \end{cases} \tag{4.18}$$

We have

$$\triangle = 1 - \mathcal{F}_c\{u\}(\omega)\mathcal{L}\{v\}(\omega) \neq 0,$$

$$\triangle = 1 - \mathcal{F}_c\{u\}(\omega)\mathcal{F}_c\{v_1\}(\omega) = 1 - \mathcal{F}_c\{v_2\}(\omega),$$

$$\frac{1}{\triangle} = 1 + \frac{\mathcal{F}_c\{v_2\}(\omega)}{1 - \mathcal{F}_c\{v_2\}(\omega)} = 1 + \mathcal{F}_c\{\ell\}(\omega).$$

$$\triangle_1 = \mathcal{F}_c\{z\}(\omega) - \mathcal{F}_c\{u\}(\omega)\mathcal{F}_c\{w\}(\omega) = \mathcal{F}_c\{z - u \underset{\mathcal{F}_c}{*} w\}(\omega),$$

$$\begin{aligned} \triangle_2 &= \mathcal{F}_c\{w\}(\omega) - \mathcal{F}_c\{z\}(\omega)\mathcal{L}\{v\}(\omega) \\ &= \mathcal{F}_c\{w\}(\omega) - \mathcal{F}_c\{z\}(\omega)\mathcal{F}_c\{v_1\}(\omega) = \mathcal{F}_c\{w - z \underset{\mathcal{F}_c}{*} v_1\}(\omega). \end{aligned}$$

The solution of the system (4.18) is

$$\mathcal{F}_c\{x\}(\omega) = \frac{\triangle_1}{\triangle} = \mathcal{F}_c\{z - u \underset{\mathcal{F}_c}{*} w\}(\omega)\big[1 + \mathcal{F}_c\{\ell\}(\omega)\big],$$

$$\mathcal{F}_c\{y\}(\omega) = \frac{\triangle_2}{\triangle} = \mathcal{F}_c\{w - z \underset{\mathcal{F}_c}{*} v_1\}(\omega)\big[1 + \mathcal{F}_c\{\ell\}(\omega)\big].$$

Therefore

$$\begin{cases} x(kh) = (z - u \underset{\mathcal{F}_c}{*} w)(kh) + \big[(z - u \underset{\mathcal{F}_c}{*} w) \underset{\mathcal{F}_c}{*} \ell\big](kh) \\ y(kh) = (w - z \underset{\mathcal{F}_c}{*} v_1)(kh) + \big[(w - z \underset{\mathcal{F}_c}{*} v_1) \underset{\mathcal{F}_c}{*} \ell\big](kh). \end{cases}$$

We obtain

$$\begin{cases} x(kh) = z(kh) - (u \underset{\mathcal{F}_c}{*} w)(kh) + (z \underset{\mathcal{F}_c}{*} \ell)(kh) - \big[(u \underset{\mathcal{F}_c}{*} w) \underset{\mathcal{F}_c}{*} \ell\big](kh) \\ y(kh) = w(kh) - (z \underset{\mathcal{F}_c}{*} v_1)(kh) + (w \underset{\mathcal{F}_c}{*} \ell)(kh) - \big[(z \underset{\mathcal{F}_c}{*} v_1) \underset{\mathcal{F}_c}{*} \ell\big](kh). \end{cases}$$

$\square$

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