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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO A CLASS OF QUASILINEAR DEGENERATE PARABOLIC EQUATIONS

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Abstract. In this paper we prove the existence and uniqueness of weak solutions to a class of quasilinear degenerate parabolic equations involving weighted p-Laplacian operators by combining compactness and monotonicity methods.
Keywords: Quasilinear degenerate parabolic equation, weighted p-Laplacian operator, weak solution, compactness method, monotonicity method.

1. Introduction

In this paper we consider the following parabolic problem:

$$\begin{cases} u_t - \text{div}(a(x)|\nabla u|^{p-2}\nabla u) + f(u) = g(x), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$, $2 \leq p \leq N$, $u_0 \in L^2(\Omega)$ given, the coefficient $a(\cdot)$, the nonlinearity f and the external force g satisfy the following conditions:

(H1) The function $a : \Omega \to \mathbb{R}$ satisfies the following assumptions: $a \in L^1_{\text{loc}}(\Omega)$ and $a(x) = 0$ for $x \in \Sigma$, and $a(x) > 0$ for $x \in \overline{\Omega} \setminus \Sigma$, where Σ is a closed subset of $\overline{\Omega}$ with meas$(\Sigma) = 0$. Furthermore, we assume that

$$\int_\Omega \frac{1}{[a(x)]^{\frac{N}{\alpha}}} dx < \infty \text{ for some } \alpha \in (0, p); \tag{1.2}$$

(H2) $f : \mathbb{R} \to \mathbb{R}$ is a C^1-function satisfying

$$C_1|u|^q - C_0 \leq f(u)u \leq C_2|u|^q + C_0, \quad \text{for some } q \geq 2, \tag{1.3}$$

$$f'(u) \geq -\ell, \tag{1.4}$$

where C_0, C_1, C_2, ℓ are positive constants;

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(H3) $g \in L^s(\Omega)$, where $s \geq \min\left(\dfrac{q}{q-1}, \dfrac{pN}{(N+1)p-N+\alpha}\right)$.

The degeneracy of problem (1.1) is considered in the sense that the measurable, nonnegative diffusion coefficient $a(x)$ is allowed to vanish somewhere. The physical motivation of the assumption $(H1)$ is related to the modeling of reaction diffusion processes in composite materials, occupying a bounded domain Ω, in which at some points they behave as *perfect insulator*. Following [1, p. 79], when at some points the medium is perfectly insulating, it is natural to assume that $a(x)$ vanishes at these points. As mentioned in [2], the assumption $(H1)$ implies that the degenerate set may consist of an infinite many number of points, which is different from the weight of Caldiroli-Musina type in [3, 4] that is only allowed to have at most a finite number of zeroes. A typical example of the weight a is dist$(x, \partial\Omega)$.

Problem (1.1) contains some important classes of parabolic equations, such as the semilinear heat equation (when $a = 1$, $p = 2$), semilinear degenerate parabolic equations (when $p = 2$), the p-Laplacian equations (when $a = 1$, $p \neq 2$), etc. It is noticed that the existence and long-time behavior of solutions to (1.1) when $p = 2$, the semilinear case, have been studied recently by Li *et al.* in [2]. We also refer the interested reader to [4-11] for related results on degenerate parabolic equations.

2. Preliminary results

To study problem (1.1), we introduce the weighted Sobolev space $W_0^{1,p}(\Omega, a)$, defined as the closure of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{W_0^{1,p}(\Omega,a)} := \left(\int_\Omega a(x)|\nabla u|^p dx\right)^{\frac{1}{p}},$$

and denote by $W^{-1,p'}(\Omega, a)$ its dual space.

We now prove some embedding results, which are generalizations of the corresponding results in the case $p = 2$ of Li *et al.* [2].

Proposition 2.1. *Assume that Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, and $a(\cdot)$ satisfies $(H1)$. Then the following embeddings hold:*

(i) $W_0^{1,p}(\Omega, a) \hookrightarrow W_0^{1,\beta}(\Omega)$ continuously if $1 \leq \beta \leq \frac{pN}{N+\alpha}$;

(ii) $W_0^{1,p}(\Omega, a) \hookrightarrow L^r(\Omega)$ continuously if $1 \leq r \leq p_\alpha^*$, where $p_\alpha^* = \frac{pN}{N-p+\alpha}$.

(iii) $W_0^{1,p}(\Omega, a) \hookrightarrow L^r(\Omega)$ compactly if $1 \leq r < p_\alpha^*$.

Proof. Applying the Hölder inequality, we have

$$\begin{aligned}\int_\Omega |\nabla u|^{\frac{pN}{N+\alpha}} dx &= \int_\Omega \frac{1}{[a(x)]^{\frac{N}{N+\alpha}}} [a(x)]^{\frac{N}{N+\alpha}} |\nabla u|^{\frac{pN}{N+\alpha}} dx \\ &\leq \left(\int_\Omega \frac{1}{[a(x)]^{\frac{N}{\alpha}}} dx\right)^{\frac{\alpha}{N+\alpha}} \left(\int_\Omega a(x)|\nabla u|^p dx\right)^{\frac{N}{\alpha}}.\end{aligned}$$

Using the assumption $(H1)$, we complete the proof of (i).

The conclusions (ii) and (iii) follow from (i) and the well-known embedding results for the classical Sobolev spaces. □

Putting

$$L_{p,a}u = -\text{div}(a(x)|\nabla u|^{p-2}\nabla u), \quad u \in W_0^{1,p}(\Omega, a).$$

The following proposition, its proof is straightforward, gives some important properties of the operator $L_{p,a}$.

Proposition 2.2. *The operator $L_{p,a}$ maps $W_0^{1,p}(\Omega, a)$ into its dual $W^{-1,p'}(\Omega, a)$. Moreover,*

(i) $L_{p,a}$ is hemicontinuous, i.e., for all $u, v, w \in W_0^{1,p}(\Omega, a)$, the map $\lambda \mapsto \langle L_{p,a}(u+\lambda v), w\rangle$ is continuous from $\mathbb{R}$ to $\mathbb{R}$;

(ii) $L_{p,a}$ is strongly monotone when $p \geq 2$, i.e.,

$$\langle L_{p,a}u - L_{p,a}v, u - v\rangle \geq \delta \|u - v\|^p_{W_0^{1,p}(\Omega,a)} \text{ for all } u, v \in W_0^{1,p}(\Omega, a).$$

3. Existence and uniqueness of global weak solutions

Denote

$$\begin{aligned} \Omega_T &= \Omega \times (0, T), \\ V &= L^p(0, T; W_0^{1,p}(\Omega, a)) \cap L^q(0, T; L^q(\Omega)), \\ V^* &= L^{p'}(0, T; W^{-1,p'}(\Omega, a)) + L^{q'}(0, T; L^{q'}(\Omega)). \end{aligned}$$

Definition 3.1. *A function u is called a weak solution of problem* (1.1) *on the interval $(0, T)$ if*

$$u \in V, \quad \frac{du}{dt} \in V^*,$$
$$u|_{t=0} = u_0 \text{ a.e. in } \Omega,$$

and

$$\int_{\Omega_T} \left(\frac{\partial u}{\partial t}\eta + a(x)|\nabla u|^{p-2}\nabla u \nabla \eta + f(u)\eta - g\eta\right) dxdt = 0, \tag{3.1}$$

for all test functions $\eta \in V$.

It is known (see e.g. [4]) that if $u \in V$ and $\dfrac{du}{dt} \in V^*$, then $u \in C([0, T]; L^2(\Omega))$. This makes the initial condition in problem (1.1) meaningful.

Lemma 3.1. *Let $\{u_n\}$ be a bounded sequence in $L^p(0, T; W_0^{1,p}(\Omega, a))$ such that $\{u_n'\}$ is bounded in V^*. If $(H1)$ and $(H3)$ hold, then $\{u_n\}$ converges almost everywhere in Ω_T up to a subsequence.*

Proof. By Proposition 2.1, one can take a number $r \in [2, p_\alpha^*)$ such that

$$W_0^{1,p}(\Omega, a) \hookrightarrow\hookrightarrow L^r(\Omega). \tag{3.2}$$

Since $r' \leq 2$, we have

$$L^p(\Omega) \cap L^q(\Omega) \hookrightarrow L^{r'}(\Omega),$$

and therefore,

$$L^r(\Omega) \hookrightarrow L^{p'}(\Omega) + L^{q'}(\Omega). \tag{3.3}$$

Using Proposition 2.1 once again and noticing that $p \leq p_\alpha^*$ since $\alpha \in (0, p)$, we see that

$$W_0^{1,p}(\Omega, a) \hookrightarrow L^p(\Omega).$$

This and (3.3) follow that

$$L^r(\Omega) \hookrightarrow W^{-1,p'}(\Omega, a) + L^{q'}(\Omega).$$

Now with (3.2), we have an evolution triple

$$W_0^{1,p}(\Omega, a) \hookrightarrow\hookrightarrow L^r(\Omega) \hookrightarrow W^{-1,p'}(\Omega, a) + L^{q'}(\Omega).$$

The assumption of $\{u_n'\}$ in V^* implies that

$$\{u_n'\} \text{ is also bounded in } L^s(0, T; W^{-1,p'}(\Omega, a) + L^{q'}(\Omega)), \text{ where } s = \min\{p', q'\}.$$

Thanks to the well-known Aubin-Lions compactness lemma (see [12, p. 58]), $\{u_n\}$ is precompact in $L^p(0, T; L^r(\Omega))$ and therefore in $L^t(0, T; L^t(\Omega)),\ t = \min(p, r)$, so it has an a.e. convergent subsequence. □

The following lemma is a direct consequence of Young's inequality and the embedding $W_0^{1,p}(\Omega, a) \hookrightarrow L^{p_\alpha^*}(\Omega)$, where $p_\alpha^* = \frac{pN}{N-p+\alpha}$, which is frequently used later.

Lemma 3.2. *Let condition $(H3)$ hold and $u \in W_0^{1,p}(\Omega, a) \cap L^q(\Omega)$. Then for any $\varepsilon > 0$, we have*

$$\left| \int_\Omega gudx \right| \leq \begin{cases} \varepsilon \|u\|^p_{W_0^{1,p}(\Omega,a)} + C(\varepsilon)\|g\|^s_{L^s(\Omega)} & \text{if } s \geq \frac{pN}{(N+1)p-N+\alpha}, \\ \varepsilon \|u\|^q_{L^q(\Omega)} + C(\varepsilon)\|g\|^s_{L^s(\Omega)} & \text{if } s \geq \frac{q}{q-1}. \end{cases}$$

The following theorem is the main result of the paper.

Theorem 3.1. *Under assumptions $(H1)-(H3)$, for each $u_0 \in L^2(\Omega)$ and $T > 0$ given, problem* (1.1) *has a unique weak solution on $(0, T)$. Moreover, the mapping $u_0 \mapsto u(t)$ is continuous on $L^2(\Omega)$.*

Proof. *(i) Existence.* Consider the approximating solution $u_n(t)$ in the form

$$u_n(t) = \sum_{k=1}^{n} u_{nk}(t) e_k,$$

where $\{e_j\}_{j=1}^{\infty}$ is a basis of $W_0^{1,p}(\Omega, a) \cap L^q(\Omega)$, which is orthogonal in $L^2(\Omega)$. We get u_n from solving the problem

$$\begin{cases} \langle \dfrac{du_n}{dt}, e_k\rangle + \langle L_{p,a} u_n, e_k\rangle + \langle f(u_n), e_k\rangle = \langle g, e_k\rangle, \\ (u_n(0), e_k) = (u_0, e_k), \ k = 1, \dots, n. \end{cases}$$

By the Peano theorem, we obtain the local existence of u_n.

We now establish some *a priori* estimates for u_n. Since

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2_{L^2(\Omega)} + \int_\Omega a(x)|\nabla u_n|^p dx + \int_\Omega f(u_n) u_n dx = \int_\Omega g u_n dx.$$

Using (1.3) and Lemma 3.2, we have

$$\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + C\left(\int_\Omega a(x)|\nabla u_n|^p dx + \int_\Omega |u_n|^q dx\right) \le C(\|g\|_{L^s(\Omega)}, |\Omega|).$$

Integrating from 0 to t, $0 \le t \le T$ and using the fact that $\|u_n(0)\|_{L^2(\Omega)} \le \|u_0\|_{L^2(\Omega)}$, we obtain

$$\|u_n(t)\|^2_{L^2(\Omega)} + C\int_0^t\int_\Omega a(x)|\nabla u_n|^p dxdt + C\int_0^t\int_\Omega |u_n|^q dxdt$$
$$\le \|u_0\|^2_{L^2(\Omega)} + TC(\|g\|_{L^s(\Omega)}, |\Omega|).$$

It follows that

- $\{u_n\}$ is bounded in $L^\infty(0, T; L^2(\Omega))$;
- $\{u_n\}$ is bounded in $L^p(0, T; W_0^{1,p}(\Omega, a))$;
- $\{u_n\}$ is bounded in $L^q(0, T; L^q(\Omega))$.

The Hölder inequality yields

$$\begin{aligned} |\int_0^T \langle L_{p,a} u_n, v\rangle dt| &= |\int_0^T\int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n \nabla v dxdt| \\ &\le \int_0^T\int_\Omega \big(a(x)^{\frac{p-1}{p}}|\nabla u_n|^{p-1}\big)\big(a(x)^{\frac{1}{p}}|\nabla v|\big) dxdt \\ &\le \|u_n\|^{\frac{p}{p'}}_{L^p(0,T;W_0^{1,p}(\Omega,a))}\|v\|_{L^p(0,T;W_0^{1,p}(\Omega,a))}, \end{aligned}$$

for any $v \in L^p(0,T;W_0^{1,p}(\Omega,a))$. Using the boundedness of $\{u_n\}$ in $L^p(0,T;W_0^{1,p}(\Omega,a))$, we infer that $\{L_{p,a}u_n\}$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega,a))$. From (1.3), we have

$$|f(u)| \leq C(|u|^{p-1}+1).$$

Hence, since $\{u_n\}$ is bounded in $L^q(0,T;L^q(\Omega))$, one can check that $\{f(u_n)\}$ is bounded in $L^{q'}(0,T;L^{q'}(\Omega))$. Rewriting (1.1) in V^* as

$$u_n' = g - L_{p,a}u_n - f(u_n) \tag{3.4}$$

and using the above estimates, we deduce that $\{u_n'\}$ is bounded in V^*.

From the above estimates, we can assume that

- $u_n' \rightharpoonup u'$ in V^*;
- $L_{p,a}u_n \rightharpoonup \psi$ in $L^{p'}(0,T;W^{-1,p'}(\Omega,a))$;
- $f(u_n) \rightharpoonup \chi$ in $L^{q'}(\Omega_T)$.

By Lemma 3.1, $u_n \to u$ a.e. in Ω_T, so $f(u_n) \to f(u)$ a.e. in Ω_T since $f(\cdot)$ is continuous. Thus, $\chi = f(u)$ thanks to Lemma 1.3 in [12]. Now taking (3.4) into account, we obtain the following equation in V^*,

$$u' = g - \psi - f(u). \tag{3.5}$$

We now show that $\psi = L_{p,a}u$. We have for every $v \in L^p(0,T;W_0^{1,p}(\Omega,a))$,

$$X_n := \int_0^T \langle L_{p,a}u_n - L_{p,a}v, u_n - v\rangle \geq 0.$$

Noticing that

$$\begin{aligned}\int_0^T \langle L_{p,a}u_n, u_n\rangle dt &= \int_0^T\int_\Omega a(x)|\nabla u_n|^p dxdt\\ &= \int_0^T\int_\Omega (gu_n - f(u_n)u_n - u_n'u_n)dxdt\\ &= \int_0^T\int_\Omega (gu_n - f(u_n)u_n)dxdt + \frac{1}{2}\|u_n(0)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_n(T)\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.6}$$

Therefore,

$$\begin{aligned}X_n = &\int_0^T\int_\Omega (gu_n - f(u_n)u_n)dxdt + \frac{1}{2}\|u_n(0)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_n(T)\|^2_{L^2(\Omega)}\\ &- \int_0^T \langle L_{p,a}u_n, v\rangle dt - \int_0^T \langle L_{p,a}v, u_n - v\rangle dt.\end{aligned}$$

It follows from the formulation of $u_n(0)$ that $u_n(0) \to u_0$ in $L^2(\Omega)$. Moreover, by the lower semi-continuity of $\|.\|_{L^2(\Omega)}$ we obtain

$$\|u(T)\|_{L^2(\Omega)} \leq \liminf_{n\to\infty} \|u_n(T)\|_{L^2(\Omega)}. \tag{3.7}$$

Meanwhile, by the Lebesgue dominated theorem, one can check that

$$\int_0^T \int_\Omega (gu - f(u)u)dxdt = \lim_{n\to\infty} \int_0^T \int_\Omega (gu_n - f(u_n)u_n)dxdt.$$

This fact and (3.6), (3.7) imply that

$$\begin{aligned}\limsup_{n\to\infty} X_n \leq \int_0^T \int_\Omega (gu - f(u)u)dxdt + \frac{1}{2}\|u(0)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u(T)\|^2_{L^2(\Omega)} \\ - \int_0^T \langle \psi, v\rangle dt - \int_0^T \langle L_{p,a}v, u - v\rangle dt.\end{aligned} \tag{3.8}$$

In view of (3.5), we have

$$\int_0^T \int_\Omega (gu - f(u)u)dxdt + \frac{1}{2}\|u(0)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u(T)\|^2_{L^2(\Omega)} = \int_0^T \langle \psi, u\rangle dt.$$

This and (3.8) deduce that

$$\int_0^T \langle \psi - L_{p,a}v, u - v\rangle dt \geq 0. \tag{3.9}$$

Putting $v = u - \lambda w$, $w \in L^p(0,T;W_0^{1,p}(\Omega,a))$, $\lambda > 0$. Since (3.9) we have

$$\lambda \int_0^T \langle \psi - L_{p,a}(u - \lambda w), w\rangle dt \geq 0.$$

Then

$$\int_0^T \langle \psi - L_{p,a}(u - \lambda w), w\rangle dt \geq 0.$$

Taking the limit $\lambda \to 0$ and noticing that $L_{p,a}$ is hemicontinuous, we obtain

$$\int_0^T \langle \psi - L_{p,a}u, w\rangle dt \geq 0,$$

for all $w \in L^p(0,T;W_0^{1,p}(\Omega,a))$. Thus, $\psi = L_{p,a}u$.

We now prove $u(0) = u_0$. Choosing some test function $\varphi \in C^1([0,T];W_0^{1,p}(\Omega,a) \cap L^q(\Omega))$ with $\varphi(T) = 0$ and integrating by parts in t in the approximate equations, we have

$$\int_0^T -\langle u_n, \varphi'\rangle dt + \int_0^T \langle L_{p,a}u_n, \varphi\rangle dt + \int_{\Omega_T} (f(u_n)\varphi - g\varphi)dxdt = (u_n(0), \varphi(0)).$$

Taking limits as $n \to \infty$, we obtain

$$\int_0^T -\langle u, \varphi' \rangle dt + \int_0^T \langle L_{p,a}u, \varphi \rangle dt + \int_{\Omega_T} (f(u)\varphi - g\varphi) dxdt = (u_0, \varphi(0)), \qquad (3.10)$$

since $u_n(0) \to u_0$. On the other hand, for the "limiting equation", we have

$$\int_0^T -\langle u, \varphi' \rangle dt + \int_0^T \langle L_{p,a}u, \varphi \rangle dt + \int_{\Omega_T} (f(u)\varphi - g\varphi) dxdt = (u(0), \varphi(0)). \qquad (3.11)$$

Comparing (3.10) and (3.11), we get $u(0) = u_0$.

(ii) Uniqueness and continuous dependence. Let u, v be two weak solutions of problem (1.1) with initial data u_0, v_0 in $L^2(\Omega)$. Then $w := u - v$ satisfies

$$\begin{cases} \dfrac{dw}{dt} + (L_{p,a}u - L_{p,a}v) + (f(u) - f(v)) = 0, \\ w(0) = u_0 - v_0. \end{cases}$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)} + \langle L_{p,a}u - L_{p,a}v, u - v \rangle + \int_\Omega (f(u) - f(v))(u - v) dx = 0.$$

Using (1.4) and the monotonicity of the operator $L_{p,a}$, we have

$$\frac{d}{dt}\|w\|^2_{L^2(\Omega)} \le 2\ell \|w\|^2_{L^2(\Omega)}.$$

Applying the Gronwall inequality, we obtain

$$\|w(t)\|_{L^2(\Omega)} \le \|w(0)\|_{L^2(\Omega)} e^{2\ell t} \text{ for all } t \in [0, T].$$

This completes the proof. □

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