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EXTENDED DEGREE OF RATIONAL MAPS

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Abstract. In the article [1], we see that the birational maps of degree d of the projective space $\mathbb{P}^n_{\Bbbk}$ form a locally closed subvariety of the projective space $\mathbb{P}(S_d^{n+1})$, denoted $\mathrm{Cr}_d(n)$. In this paper, we will construct the notion of extended degree of rational maps, then we obtain the functor $\mathrm{cr}_d(n)$ that represents the variety $\mathrm{Cr}_d(n)$.
Keywords: Rational map, birational map, extended degree, representable functor, remona group.

1. Introduction

Let $\mathrm{Cr}(n) = \mathrm{Bir}(\mathbb{P}^n_{\Bbbk})$ denote the set of all birational maps of the projective space $\mathbb{P}^n_{\Bbbk}$, on the field $\Bbbk$. It is clear that $\mathrm{Cr}(n)$ is a group under composition of dominant rational maps called the Cremona group of order n. This group is naturally identified with the Galois group of $\Bbbk$-automorphisms of the field $\Bbbk(x_1, \ldots, x_n)$ of rational fractions in n-variables $x_1, \ldots, x_n$. It was studied in the first time by Luigi Cremona (1830 - 1903), an Italian mathematician. Although it has been studied since the 19th century by many famous mathematicians, it is still not well understood. For example, we still don't know if it has the structure of an algebraic group of infinite dimension.

In the article [2], we constructed the Cremona group $\Bbbk$-functor

$$\begin{array}{rcrcl} \mathrm{cr}(n) & : & \mathfrak{Alg}(\Bbbk) & \longrightarrow & \mathfrak{Gr} \\ & & R & \longmapsto & \mathrm{Bir}_R(\mathbb{P}^n_R). \end{array}$$

from the category $\mathfrak{Alg}(\Bbbk)$ of $\Bbbk$-algebras to the category $\mathfrak{Gr}$ of groups and calculated its Lie algebra. The $\Bbbk$-value points of the Cremona group $\Bbbk$-functor $\mathrm{cr}(n)$ are exactly the elements of the Cremona group $\mathrm{Cr}_{\Bbbk}(n) = \mathrm{Bir}_{\Bbbk}(\mathbb{P}^n_{\Bbbk})$.

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Denote by $S_d = \mathbb{k}[x_0, \dots, x_n]_d$ the $\mathbb{k}$-vector space of homogeneous polynomials of degree d in $(n+1)$-variables $x_0, \dots, x_n$ and zero polynomial over an algebraically closed field $\mathbb{k}$ of characteristic 0. In the article [1], we also knew that the birational maps of degree d of the projective space $\mathbb{P}^n_{\mathbb{k}}$ form a locally closed subvariety of the projective space $\mathbb{P}(S_d^{n+1})$ associated with the $\mathbb{k}$-vector space S_d^{n+1}, denoted $\mathrm{Cr}_d(n)$. In this paper, we will construct the sub-functor $\mathrm{cr}_d(n)$ of the Cremona group $\mathbb{k}$-functor $\mathrm{cr}(n)$ and we will show that this sub-functor $\mathrm{cr}_d(n)$ represents the variety $\mathrm{Cr}_d(n)$. For that, we need extend the notion of degree of rational maps $\varphi : \mathbb{P}^n_R \dashrightarrow \mathbb{P}^n_R$ where R is not necessarily a field but any $\mathbb{k}$-algebra.

2. Content

2.1. Degrees of rational maps

In classic algebraic geometry, we know that a rational map of the projective space $\mathbb{P}^n_{\mathbb{k}}$ is of the form:

$$\mathbb{P}^n_{\mathbb{k}} \ni [x_0 : \dots : x_n] = x \dashrightarrow \varphi(x) = \big[P_0(x) : \dots : P_n(x)\big] \in \mathbb{P}^n_{\mathbb{k}},$$

where $P_0, \dots, P_n$ are homogeneous polynomials of same degree in $(n+1)$-variables $x_0, \dots, x_n$ and are mutually prime. The common degree of P_i is called the degree of φ; denoted by $\deg \varphi$. In the language of linear systems; giving a rational map such as φ is equivalent to giving a linear system without fixed components of $\mathbb{P}^n_{\mathbb{k}}$

$$\varphi^\star |\mathcal{O}_{\mathbb{P}^n}(1)| = \left\{ \sum_{i=0}^{n} \lambda_i P_i | \lambda_i \in \mathbb{k} \right\}.$$

Clearly, the degree of φ is also the degree of a generic element of $\varphi^\star|\mathcal{O}_{\mathbb{P}^n}(1)|$ and the undefined points of φ are exactly the base points of $\varphi^\star|\mathcal{O}_{\mathbb{P}^n}(1)|$.

Note that a rational map $\varphi : \mathbb{P}^n_{\mathbb{k}} \dashrightarrow \mathbb{P}^n_{\mathbb{k}}$ is not in general a map from the set $\mathbb{P}^n_{\mathbb{k}}$ to $\mathbb{P}^n_{\mathbb{k}}$; it is only the map defined on its domain of definition $\mathrm{Dom}(\varphi) = \mathbb{P}^n_{\mathbb{k}} \setminus V(P_0, \dots, P_n)$. We say that φ is *dominant* if its image $\varphi(\mathrm{Dom}(\varphi))$ is dense in $\mathbb{P}^n_{\mathbb{k}}$. By the Chevalley theorem, the image $\varphi(\mathrm{Dom}(\varphi))$ is always a constructible subset of $\mathbb{P}^n_{\mathbb{k}}$, hence, it is dense in $\mathbb{P}^n_{\mathbb{k}}$ if and only if it contains a non-empty Zariski open subset of $\mathbb{P}^n_{\mathbb{k}}$ (see the page 94, in [3]). In general, we can not compose two rational maps. However, the composition $\psi \circ \varphi$ is always defined if φ is dominant so that the set of all dominant rational maps $\varphi : \mathbb{P}^n_{\mathbb{k}} \dashrightarrow \mathbb{P}^n_{\mathbb{k}}$ is identified with the set of injective field homomorphisms $\varphi^\star$ of the field $\mathbb{k}(x_1, \dots, x_n)$ of all rational fractions in n-variables $x_1, \dots, x_n$. We say that a rational map $\varphi : \mathbb{P}^n_{\mathbb{k}} \dashrightarrow \mathbb{P}^n_{\mathbb{k}}$ is *birational* (a *birational automorphism*) if there exists a rational map $\psi : \mathbb{P}^n_{\mathbb{k}} \dashrightarrow \mathbb{P}^n_{\mathbb{k}}$ such that $\psi \circ \varphi = \mathrm{id}_{\mathbb{P}^n} = \varphi \circ \psi$ as rational maps. Clearly, if such a ψ exists, then it is unique and is called the *inverse* of φ. Moreover, φ and ψ are both dominant. If we denote by $\mathrm{Cr}(n) = \mathrm{Bir}(\mathbb{P}^n_{\Bbbk})$ the set of all birational maps of the projective space $\mathbb{P}^n_{\Bbbk}$, then $\mathrm{Cr}(n)$ is a group under composition of dominant rational maps an is called the Cremona group of order n. This group is naturally identified with the Galois group of $\Bbbk$-automorphisms of the field $\Bbbk(x_1, \ldots, x_n)$ of rational fractions in n-variables $x_1, \ldots, x_n$.

Now, suppose that R is any $\Bbbk$-algebra and $\varphi : \mathbb{P}^n_R \dashrightarrow \mathbb{P}^n_R$ is a rational map (see the general definition of rational maps in [4]). It gives us a family of rational $\kappa(s)$-maps $\left(\varphi_s : \mathbb{P}^n_{\kappa(s)} \dashrightarrow \mathbb{P}^n_{\kappa(s)}\right)_{s\in\mathrm{Spec}(R)}$ where $\kappa(s)$ is the residue field at s. Roughly, we say that the degree of φ is equal to d if φ_s has the degree d for all $s \in \mathrm{Spec}(R)$. More precisely, we define as follows:

Definition 2.1. *Let $X = \mathit{Spec}(R)$ be an affine $\Bbbk$-scheme and $\varphi : \mathbb{P}^n_R \dashrightarrow \mathbb{P}^n_R$ a rational R-map with the domain of definition $U = \mathit{dom}(\varphi) \subset \mathbb{P}^n_R$*

$$\begin{tikzcd} \mathbb{P}^n_R \supset U \arrow[rr, "\varphi"] \arrow[dr, "p=\pi|_U"'] & & \mathbb{P}^n_R \arrow[dl, "\pi"] \\ & \mathrm{Spec}(R) & \end{tikzcd}$$

satisfying the following condition:

$$\mathit{codim}\left(\mathbb{P}^n_{\kappa(s)} - U_s, \mathbb{P}^n_{\kappa(s)}\right) \geqslant 2, \quad \forall s \in \mathit{Spec}(R) \tag{2.1}$$

where $U_s := U \times_{\mathit{Spec}(R)} \mathit{Spec}\,\kappa(s)$ is the fiber of U at s.
Suppose there exists an invertible sheaf $\mathscr{L}$ on $\mathit{Spec}(R)$ and a positive integer d such that

$$\varphi^{\star}\mathcal{O}_{\mathbb{P}^n_R}(1) \simeq \mathcal{O}_U(d) \otimes_{\mathcal{O}_U} p^{\star}\mathscr{L} \tag{2.2}$$

where $\mathcal{O}_U(d)$ is the restriction to U of $\mathcal{O}_{\mathbb{P}^n_R}(d)$ and p is the restriction of the structural morphism π to U, and here $\varphi : U \to \mathbb{P}^n_R$ becomes a morphism, in such a way that, the notation $\varphi^{\star}\mathcal{O}_{\mathbb{P}^n_R}(1)$ defines an invertible sheaf on U. It is clear that such a couple $(\mathscr{L}, d)$ is uniquely determined if it exists. This positive integer d is called the degree of φ, still denoted by $\deg(\varphi)$.

Remark 2.1. *When $R = \Bbbk$ is any field, $X = \mathit{Spec}(\Bbbk)$ and $\pi : \mathbb{P}^n_{\Bbbk} \to \mathit{Spec}(\Bbbk)$ is the structural morphism, we show that such a couple $(\mathscr{L}, d)$ always exists and it is uniquely determined, so we find again the notion of usual degree in classical algebraic geometry. Indeed, the invertible sheaf $\mathscr{L}$ is quasi-coherent on $X = \mathit{Spec}(\Bbbk)$, therefore, of form $\mathscr{L} = \widetilde{M}$ where M is a locally free $\Bbbk$-module of rank 1 (projective of rank 1), hence, $M \simeq \Bbbk$, that is, $\mathscr{L} = \widetilde{\Bbbk} = \mathcal{O}_{\mathit{Spec}(\Bbbk)}$. Therefore, $p^{\star}\mathscr{L} \simeq p^{\star}\mathcal{O}_{\mathit{Spec}(\Bbbk)} \simeq \mathcal{O}_U$, hence, the isomorphism of sheaves (2.2) becomes: $\varphi^{\star}\mathcal{O}_{\mathbb{P}^n_{\Bbbk}}(1) \simeq \mathcal{O}_U(d)$. Moreover, in this case, (2.1) gives us $\mathit{Pic}(U) = \mathit{Pic}(\mathbb{P}^n_{\Bbbk})$, hence, $|\mathcal{O}_{\mathbb{P}^n_{\Bbbk}}(d)| \equiv |\mathcal{O}_U(d)| \supset \varphi^{\star}|\mathcal{O}_{\mathbb{P}^n_{\Bbbk}}(1)|$, therefore, the* *generic element (generic hypersurface) of the linear system defining φ has for the degree d, ie, the degree of φ is d, as the usual notion of degree.*

Remark 2.2. *The condition (2.2) implicates that the degree of φ_s is equal to d for all $s \in$ Spec(R). Indeed, by taking the fiber at $s \in$ Spec(R), the isomorphism of sheaves (2.2) becomes*

$$\varphi_s^\star \mathcal{O}_{\mathbb{P}^n_{\kappa(s)}}(1) \simeq \mathcal{O}_{U_s}(d) = \mathcal{O}_{\mathbb{P}^n_{\kappa(s)}}(d)|_{U_s}.$$

Therefore, the linear system $\mathcal{C}_{\varphi_s} := \varphi_s^\star |\mathcal{O}_{\mathbb{P}^n_{\kappa(s)}}(1)| \subset |\mathcal{O}_{U_s}(d)| \equiv |\mathcal{O}_{\mathbb{P}^n_{\kappa(s)}}(d)|$ defining φ_s is a linear system of hypersurfaces of degree d in $\mathbb{P}^n_{\kappa(s)}$, hence, $\deg(\varphi_s) = d$.

Remark 2.3. *We consider the following example $\varphi : \mathbb{P}^2_{\mathbb{A}^1_{\Bbbk}} \dashrightarrow \mathbb{P}^2_{\mathbb{A}^1_{\Bbbk}}, \varphi = [xz : y(z+sx) : z^2]$ where the parameter $s \in \mathbb{A}^1_{\Bbbk} = X =$ Spec$\Bbbk[t]$, we have*

$$\textit{codim}\left(\mathbb{P}^2_{\kappa(s)} - U_s, \mathbb{P}^2_{\kappa(s)}\right) = 2, \quad \deg(\varphi_s) = 2, \quad \forall s \neq 0$$

$$\textit{codim}\left(\mathbb{P}^2_{\kappa(0)} - U_0, \mathbb{P}^2_{\kappa(0)}\right) = 1, \quad \deg(\varphi_0) = 1 \neq 2.$$

Then, this rational map φ has not degree. This example also shows us that the condition (2.1), that gives us: Pic$(U_s) =$ Pic$(\mathbb{P}^n_{\kappa(s)}), \ \forall s \in$ Spec(R), is necessary.

2.2. Main results

We consider the Cremona group $\Bbbk$-functor

$$\begin{array}{rcccl} \mathrm{cr}(n) & : & \mathfrak{Alg}(\Bbbk) & \longrightarrow & \mathfrak{Gr} \\ & & R & \longmapsto & \mathrm{Bir}_R(\mathbb{P}^n_R) \end{array}$$

from the category $\mathfrak{Alg}(\Bbbk)$ of $\Bbbk$-algebras to the category $\mathfrak{Gr}$ of groups. Here, we denote by $\mathrm{Bir}_R(\mathbb{P}^n_R)$ the group of birational R-maps of the projective space $\mathbb{P}^n_R$ on R.

According to Definition 2.1, we can denote by $\mathrm{Bir}_{d,R}\big(\mathbb{P}^n_R\big)$ the set of birational R-maps of degree d of the projective space $\mathbb{P}^n_R$ on the $\Bbbk$-algebra R. Therefore, we obtain the sub-functor following of the Cremona group $\Bbbk$-functor $\mathrm{cr}(n)$

$$\begin{array}{rcccl} \mathrm{cr}_d(n) & : & \mathfrak{Alg}(\Bbbk) & \longrightarrow & \mathfrak{Set} \\ & & R & \longmapsto & \mathrm{cr}_d(n)(R) := \mathrm{Bir}_{d,R}\big(\mathbb{P}^n_R\big). \end{array}$$

Here, we denote by $\mathfrak{Set}$ the category of sets. We can also regard $\mathrm{cr}_d(n)$ as a $\Bbbk$-functor defined on the category $\mathfrak{Sch}(\Bbbk)$ of $\Bbbk$-schemes. Now, we obtain the main result following:

Theorem 2.1. *The restriction of the $\Bbbk$-functor cr$_d(n)$ to the category of noetherian $\Bbbk$-schemes is a $\Bbbk$-functor representable by the scheme Cr$_d(n)$.*

Proof. It suffices to establish an isomorphism of $\Bbbk$-functors $\mathrm{cr}_d(n) \simeq h_{\mathrm{Cr}_d(n)}$ with the representable functor $h_{\mathrm{Cr}_d(n)}$, that is, for all noetherian $\Bbbk$-algebra R, a bijection

$$\mathrm{cr}_d(n)(R) = \mathrm{Bir}_{d,R}\big(\mathbb{P}^n_R\big) \xrightarrow{\sim} \mathrm{Mor}_{\mathfrak{Sch}(\Bbbk)}\left(\mathrm{Spec}(R), \mathrm{Cr}_d(n)\right) = h_{\mathrm{Cr}_d(n)}\big(\mathrm{Spec}(R)\big)$$

and for all homomorphism of noetherian $\Bbbk$-algebras $T \to R$, a commutative square

$$\begin{array}{ccc} \mathrm{Bir}_{d,R}\big(\mathbb{P}^n_R\big) & \xrightarrow{\sim} & \mathrm{Mor}_{\mathfrak{Sch}(\Bbbk)}\left(\mathrm{Spec}R, \mathrm{Cr}_d(n)\right) \\ \uparrow & & \uparrow \\ \mathrm{Bir}_{d,T}\big(\mathbb{P}^n_T\big) & \xrightarrow{\sim} & \mathrm{Mor}_{\mathfrak{Sch}(\Bbbk)}\left(\mathrm{Spec}T, \mathrm{Cr}_d(n)\right). \end{array}$$

$\square$

Let $\varphi \in \mathrm{Bir}_{d,R}\big(\mathbb{P}^n_R\big)$ be a birational R-map of degree d of the projective space $\mathbb{P}^n_R$ on some noetherian $\Bbbk$-algebra R.

$$\begin{tikzcd} \mathbb{P}^n_R \arrow[rr, dashed, "\varphi"] \arrow[dr, "\pi"'] & & \mathbb{P}^n_R \arrow[dl, "\pi"] \\ & \mathrm{Spec}R. & \end{tikzcd}$$

We try to construct a morphism of $\Bbbk$-schemes $\varphi' : \mathrm{Spec}R \to \mathrm{Cr}_d(n) \subset \mathbb{P}\big(S^{n+1}_d\big)$, that is, a morphism into the projective space $\varphi' : \mathrm{Spec}R \to \mathbb{P}\big(S^{n+1}_d\big)$, whose image is contained in $\mathrm{Cr}_d(n)$. Such a morphism will be defined by the data containing an invertible R-module $H^0\big(\mathrm{Spec}R, \mathscr{L}\big)$ and an epimorphism of R-modules $S^{n+1}_d \otimes_{\Bbbk} R \twoheadrightarrow H^0\big(\mathrm{Spec}R, \mathscr{L}\big)$. According to the definition of degree (Definition 2.1), there exists always such an invertible sheaf $\mathscr{L}$. In order to verify that $\mathscr{L}$ is suitable, we need prove some complementary results following:

Lemma 2.1. *Let $A \to B$ be a local homomorphism of local noetherian rings such that B is a flat A-module. Then, if we denote by $\kappa(A)$ the residue field of A, we have the equality following of depth:*

$$\textit{depth}(B) = \textit{depth}(A) + \textit{depth}\big(\kappa(A) \otimes_A B\big).$$

Proof of Lemma 2.1 is also the corollary of Proposition 11, page AC X-13, in [5].

Lemma 2.2. *Suppose that X is a locally noetherian scheme, Y is a closed subscheme of X and $\mathscr{F}$ is a coherent $\mathcal{O}_X$-module. Then, the following conditions are equivalent:*

(i) *For all $y \in Y$, depth$\big(\mathscr{F}_y\big) \geq 2$.*

(ii) *For all open subset V of X, the natural homomorphism following is bijective*

$$H^0\big(V, \mathscr{F}\big) \longrightarrow H^0\big(V \cap (X - Y), \mathscr{F}\big).$$

Proof of Lemma 2.2 is also Corollary III-1, page 11, in [6].

Lemma 2.3. *Suppose that R is a noetherian $\Bbbk$-algebra, $\varphi : \mathbb{P}^n_R \overset{\sim}{\dashrightarrow} \mathbb{P}^n_R$ is a birational R-map of degree d with $U = dom(\varphi) \subset \mathbb{P}^n_R$ the domain of definition of φ. Then, the canonical homomorphism of R-modules*

$$\begin{array}{rcrcl} \alpha & : & H^0\big(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(d)\big) & \longrightarrow & H^0\big(U, \mathcal{O}_U(d)\big) \\ & & x & \longmapsto & x|_U \end{array}$$

is an isomorphism.

Proof. This is a direct consequence of Lemma 2.1 and Lemma 2.2. Indeed, we need only apply Lemma 2.2 with $X = \mathbb{P}^n_R, Y = \mathbb{P}^n_R - U$ and $\mathscr{F} = \mathcal{O}_{\mathbb{P}^n_R}(d)$. So the only problem remains verifying that for all $y \in Y$,

$$\text{depth}\left(\mathcal{O}_{y,\mathbb{P}^n_R}(d)\right) \geq 2.$$

However, $\mathcal{O}_{\mathbb{P}^n_R}(d)$ is an invertible sheaf, $\mathcal{O}_{y,\mathbb{P}^n_R}(d) \simeq \mathcal{O}_{y,\mathbb{P}^n_R}$. Now, by applying Lemma 2.1 to the canonical homomorphism $A = \mathcal{O}_{s,\text{Spec}R} \to B = \mathcal{O}_{y,\mathbb{P}^n_R}$ where $\pi(y) = s$. We have

$$\begin{array}{rcl} \text{depth}\left(\mathcal{O}_{y,\mathbb{P}^n_R}\right) & = & \text{depth}\left(\mathcal{O}_{s,\text{Spec}R}\right) + \text{depth}\left(\kappa(s) \otimes_{\mathcal{O}_{s,\text{Spec}R}} \mathcal{O}_{y,\mathbb{P}^n_R}\right) \\ & = & \text{depth}\left(\mathcal{O}_{s,\text{Spec}R}\right) + \text{depth}\left(\mathcal{O}_{y_s,\mathbb{P}^n_{\kappa(s)}}\right) \\ & \geq & \text{depth}\left(\mathcal{O}_{y_s,\mathbb{P}^n_{\kappa(s)}}\right) = \dim_{\text{Krull}}\left(\mathcal{O}_{y_s,\mathbb{P}^n_{\kappa(s)}}\right) \\ & \geq & \text{codim}\left(\mathbb{P}^n_{\kappa(s)} - U_s, \mathbb{P}^n_{\kappa(s)}\right) \geqslant 2. \end{array}$$

Here, we have the equality: $\text{depth}\left(\mathcal{O}_{y_s,\mathbb{P}^n_{\kappa(s)}}\right) = \dim_{\text{Krull}}\left(\mathcal{O}_{y_s,\mathbb{P}^n_{\kappa(s)}}\right)$ because $\mathbb{P}^n_{\kappa(s)}$ is smooth. □

Lemma 2.4. *(Projection Formula) (see in [7]). Suppose that $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, $\mathscr{F}$ is an $\mathcal{O}_X$-module and $\mathscr{E}$ is an $\mathcal{O}_Y$-module locally free of finite rank. Then*

$$f_\star\big(\mathscr{F} \otimes_{\mathcal{O}_X} f^\star\mathscr{E}\big) \simeq f_\star\left(\mathscr{F}\right) \otimes_{\mathcal{O}_Y} \mathscr{E}.$$

In fact, we have also the isomorphism

$$R^i f_\star\big(\mathscr{F} \otimes_{\mathcal{O}_X} f^\star\mathscr{E}\big) \simeq R^i f_\star\left(\mathscr{F}\right) \otimes_{\mathcal{O}_Y} \mathscr{E}, \quad \forall i.$$

Lemma 2.5. *Let R be a noetherian $\Bbbk$-algebra, we have*

(a) $H^0\big(\textit{Spec}R, \pi_\star\mathcal{O}_{\mathbb{P}^n_R}(d)\big) \simeq S_d \otimes_\Bbbk R$,

(b) $H^0\big(\textit{Spec}R, p_\star\mathcal{O}_U(d)\big) \simeq S_d \otimes_\Bbbk R$.

Proof. (a) Indeed: $H^0\big(\mathrm{Spec}R, \pi_\star\mathcal{O}_{\mathbb{P}^n_R}(d)\big) \simeq H^0\big(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(d)\big) \simeq R\big[x_0,\ldots,x_n\big]_d \simeq S_d\otimes_{\Bbbk} R$.

(b) By applying Lemma 2.3, we have

$$H^0\big(\mathrm{Spec}R, p_\star\mathcal{O}_U(d)\big) \simeq H^0\big(U, \mathcal{O}_U(d)\big) \simeq H^0\big(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(d)\big) \simeq R\big[x_0,\ldots,x_n\big]_d \simeq S_d\otimes_{\Bbbk} R.$$

$\square$

Now, we come back to prove main Theorem 2.1. By applying the three previous lemmas and by taking $p_\star$ of two sides of the isomorphism (2.2), we have

$$\begin{array}{rcl} p_\star\varphi^\star\mathcal{O}_{\mathbb{P}^n_R}(1) & \simeq & p_\star\big(\mathcal{O}_U(d)\otimes_{\mathcal{O}_U} p^\star\mathscr{L}\big) \\ & \simeq & p_\star\big(\mathcal{O}_U(d)\big)\otimes_{\mathcal{O}_{\mathrm{Spec}R}}\mathscr{L} \\ & \simeq & S_d\otimes_{\Bbbk}\mathscr{L}. \end{array}$$

According to the property of the adjoint functor, we alway have a morphism of sheaves

$$\mathcal{O}_{\mathbb{P}^n_R}(1) \to \varphi_\star\varphi^\star\mathcal{O}_{\mathbb{P}^n_R}(1)$$

hence, a morphism of sheaves

$$\pi_\star\mathcal{O}_{\mathbb{P}^n_R}(1) \longrightarrow \pi_\star\varphi_\star\varphi^\star\mathcal{O}_{\mathbb{P}^n_R}(1) = p_\star\varphi^\star\mathcal{O}_{\mathbb{P}^n_R}(1) \simeq S_d\otimes_{\Bbbk}\mathscr{L}.$$

By taking the global sections, we have the following homomorphism:

$$S_1\otimes_{\Bbbk} R \longrightarrow S_d\otimes_{\Bbbk} H^0\mathscr{L}.$$

By taking the duality, we obtain

$$\Theta \quad : \quad (S_d)^\star\otimes_{\Bbbk}(S_1\otimes_{\Bbbk} R) \longrightarrow H^0\mathscr{L}.$$

The monomials $x_I = x_0^{i_0}\cdots x_n^{i_n}$ with $\mathrm{Card}(I) = i_0+\cdots+i_n = d$ form a canonical basis of the vector $\Bbbk$-space $\Bbbk[x_0,\ldots,x_n]_d = S_d$, hence, we obtain the dual basis $(x_I)^\star$ of $(S_d)^\star$

$$\begin{array}{rcccl} (x_I)^\star & : & S_d & \longrightarrow & \Bbbk \\ & & x_J & \longmapsto & (x_I)^\star(x_J) := \delta_{I,J} \end{array}$$

where $\delta_{I,J}$ is the Kronecker symbol. In the end, the homomorphism Θ is well defined by its images on the canonical basis $(x_I)^\star\otimes x_i$ as follows:

$$\begin{array}{rcccl} \Theta & : & (S_d)^\star\otimes_{\Bbbk}(S_1\otimes_{\Bbbk} R) & \twoheadrightarrow & H^0\mathscr{L} \\ & & (x_I)^\star\otimes x_i & \longmapsto & \Theta\big((x_I)^\star\otimes x_i\big) = \mathrm{Coeff}_{x_I}(P_i) \end{array}$$

where $\mathrm{Coeff}_{x_I}(P_i)$ is the coefficient of x_I in P_i if $\varphi = \left[P_0 : \ldots : P_n\right]$. This shows that Θ is an epimorphism of R-modules.

In summary, we showed that the birational R-map $\varphi = \left[P_0 : \ldots : P_n\right]$ of degree d determines an epimorphism of R-modules $S_d^{n+1} \otimes_{\mathbb{k}} R \twoheadrightarrow H^0\mathscr{L}$. So, this epimorphism defines a morphism of $\mathbb{k}$-schemes $\varphi' : \mathrm{Spec}R \to \mathbb{P}\left(S_d^{n+1}\right)$. As the set map, the underlying mapping of the corresponding morphism φ' is defined as follows: $\mathrm{Spec}R \ni s \mapsto \varphi'(s) = \varphi_s \in \mathrm{Cr}_d(n)$ where φ_s is a member of the family of the birational $\mathbb{k}$-maps φ. Therefore, the image of φ' is contained in $\mathrm{Cr}_d(n)$.

Conversely, all the morphisms of $\mathbb{k}$-schemes from $\mathrm{Spec}R$ to $\mathrm{Cr}_d(n)$ are deduced from this way. Moreover, the fact that the square being commutative is obvious.

3. Conclusions

In this paper, the author has acquired the two main results. The first is Definition 2.1 that is the notion of extended degree of rational maps. The second is Theorem 3.1, which states that the restriction of the $\mathbb{k}$-functor $\mathrm{cr}_d(n)$ to the category of noetherian $\mathbb{k}$-schemes is a $\mathbb{k}$-functor representable by the scheme $\mathrm{Cr}_d(n)$.

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