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LIOUVILLE TYPE THEOREM FOR STABLE SOLUTIONS TO ELLIPTIC EQUATIONS INVOLVING THE GRUSHIN OPERATOR

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Abstract. We study a Liouville type theorem for stable solutions of the following semilinear equation involving Grushin operators $-(\Delta_x u + a^2|x|^{2\alpha}\Delta_y u) = |u|^{p-1}u, (x,y) \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, where $p > 1, \alpha > 0$ and $a \neq 0$. Basing on the technique of Farina [1], we establish the nonexistence of nontrivial stable solutions under the range $p < p_c(N_\alpha)$ where $N_\alpha = N_1 + (1+\alpha)N_2$, and $p_c(N_\alpha)$ is a certain (explicitly given) positive constant depending on N_α.
Keywords: Liouville type theorem, stable solution, degenerate elliptic equations, Grushin operators.

1. Introduction

In this paper, we study the semilinear degenerate partial differential equation of the form

$$-G_\alpha u = |u|^{p-1}u \tag{1.1}$$

where $G_\alpha = \Delta_x + a^2|x|^{2\alpha}\Delta_y$ is the Grushin operator, Δ_x and Δ_y are Laplace operators with respect to $x \in \mathbb{R}^{N_1}$ and $y \in \mathbb{R}^{N_2}$. Here we always assume that $a \neq 0$, $\alpha > 0$, $p > 1$ and $N_1, N_2 \geq 1$. Recall that G_α is elliptic for $|x| \neq 0$ and degenerates on the manifold $\{0\} \times \mathbb{R}^{N_2}$. This operator belongs to the wide class of subelliptic operators studied by Franchi *et al*. in [2]. In the special case $\alpha = 1$, problem (1.1) is close related to the Heisenberg Laplacian equation $\Delta_H u = f(u)$ in $H^n = \mathbb{C}^n \times \mathbb{R}$, where Δ_H is the Heisenberg Laplacian (see e.g., [3, 4]).

Problem (1.1) has recently attracted much attention in variety of mathematics directions. The most interesting questions are about the existence and non-existence [5]; the multiplicity of solutions [6]; the symmetry properties [7]; the asymptotic behaviour [8]; the regularity estimates [9-11].

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Let us recall that the Liouville-type theorem is the nonexistence of nontrivial solution of problem (1.1) in the whole space $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. In recent years, the Liouville property has emerged as one of the most powerful tools in the study of qualitative properties for nonlinear PDEs. It turns out that one can obtain from Liouville type theorems a variety of results such as universal, pointwise, a priori estimate; universal and singularity estimates; decay estimates, etc., see [12] and references therein. In addition, Liouville type results combined with degree type arguments are useful to obtain the existence of solutions of semilinear boundary value problems in bounded domains (see [13]). In what follows, we make a short review on the recent developments of Liouville-type property for the problem (1.1).

In the class of nonnegative solutions, it has been recently proved by Monticelli [14] for nonnegative classical solutions, and by Yu [15] for nonnegative weak solutions. The optimal condition on the range of the exponent is $p < \frac{N_\alpha+2}{N_\alpha-2}$, where $N_\alpha := N_1 + (1+\alpha)N_2$ is called the homogeneous dimension. The main tool in [14, 15] is the Kelvin transform combined with technique of moving planes. Before that, Dolcetta and Cutrì [16] established the Liouville-type theorem for nonnegative super-solutions under the condition $p \le \frac{N_\alpha}{N_\alpha-2}$ (see also [17]). Further results on Liouville type theorem on manifold was established in [18].

In the the class of sign-changing solutions, the Liouville-type theorem is still open, even in the special case of Laplace operator with $\alpha = 0$. However, in a special class of solutions – the so-called stable solutions, the Liouville type result for $\alpha = 0$ was completely established by Farina [1] (for the properties of stable solutions, we refer to the monograph of Dupaigne [19]).

Theorem 1.1 (Farina [1])**.** *Let $u \in C^2(\mathbb{R}^N)$ be a stable solution of* (1.1) *with*

$$\begin{cases} 1 < p < +\infty & \text{if } N \le 10 \\ 1 < p < p_c(N) = \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N \ge 11. \end{cases}$$

Then $u \equiv 0$.

On the other hand, for $N \ge 11$ and $p \ge p_c(N)$, the equation (1.1) *admits a smooth, positive, bounded, stable and radial solution.*

The exponent $p_c(N)$ stands for the Joseph-Lundgren exponent (see [20]). The main tools in [1] are the nonlinear integral estimates combined with the property of stable solutions. In addition, this technique was employed to obtain the optimal Liouville type theorem for finite Morse index classical solutions. Some applications of Liouville type results on qualitative properties of solutions, such as the universal a priori estimate and the behaviour of solution near an isolated singularity, were also studied in [1].

In this paper, we will extend the result of Farina [1] to the general case $\alpha > 0$ and look for the effect of the degeneracy on the range of the exponent p on Liouville-type theorem. It seems that the presence of the weight term $|x|^\alpha$ makes the problem more challenging. The main difficulty is that the Grushin operator is nonautonomous. This requires suitable scaled test functions in the integral estimate. On the other hand, we make use of the properties of the Grushin divergent and the associated distance to derive the nonlinear integral estimates. To our best knowledge, Liouville type theorems for stable solutions to (1.1) have not been established so far.

2. Formulation of main result

In this section, we state our main result concerning the nonexistence of nontrivial stable solutions of (1.1).

Firstly, without loss of generality, we always assume that the constant a in (1.1) is equal to 1. We then recall the definition of stable solutions, see [19].

Definition 2.1. *Let $u \in C^2(\mathbb{R}^N)$ be a classical solution of* (1.1). *The solution u is said to be stable if*

$$Q_u(\psi) := \int_{\mathbb{R}^N} \left(|\nabla_x \psi|^2 + |x|^{2\alpha}|\nabla_y \psi|^2 - p|u|^{p-1}\psi^2\right) dxdy \geq 0, \text{ for all } \psi \in C_c^1(\mathbb{R}^N). \tag{2.1}$$

Recall that $N_\alpha := N_1 + (\alpha+1)N_2$ is the homogeneous dimension of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. We define the critical exponent

$$p_c(N_\alpha) = \begin{cases} +\infty & \text{if } N_\alpha \leq 10 \\ \frac{(N_\alpha-2)^2 - 4N_\alpha + 8\sqrt{N_\alpha - 1}}{(N_\alpha-2)(N_\alpha-10)} & \text{if } N_\alpha > 10 \end{cases}. \tag{2.2}$$

The main result of this paper is the following:

Theorem 2.1. *Let $u \in C^2(\mathbb{R}^N)$ is a stable solution of* (1.1) *with $1 < p < p_c(N_\alpha)$. Then, u is the trivial solution.*

Remark 2.1.

• In (2.2), we can see the impact of the exponent α to the critical exponent $p_c(N_\alpha)$.

• The result of A.Farina-Theorem 1.1 is a consequence of our main result with $\alpha = 0$.

• Note that Theorem 1.1 is optimal in the sense that, for $\alpha = 0$, $N \geq 11$ and $p \geq p_c(N)$, there exist stable radial solutions to (1.1) in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ (see [1]). In the case $\alpha > 0$, it seems still difficult to prove the existence of stable solutions to (1.1) on $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ under the condition $p \geq p_c(N_\alpha)$ with $N_\alpha > 10$. However, if p is sufficiently large, for example $p \geq p_c(N_1)$, there exist stable solutions which do not depend on the y-variable.

A brief outline of the proof

Here we give the outline of the proof inspired by ideas of A. Farina [1]. Suppose that u is a classical stable solution of (1.1). By using the stability condition (2.1) with the test function $|u|^{\frac{\gamma-1}{2}}u\phi$, $\phi \in C_c^\infty(\mathbb{R}^N)$, we show that for some $\gamma \geq 1$

$$\begin{aligned}\left(p-\frac{(\gamma+1)^2}{4\gamma}\right)&\int_{\mathbb{R}^N}|u|^{p+\gamma}\phi^2dxdy\\&\leq \int_{\mathbb{R}^N}|u|^{\gamma+1}|\nabla_G\phi|^2dxdy+\left(\frac{\gamma+1}{4\gamma}-\frac{1}{2}\right)\int_{\mathbb{R}^N}|u|^{\gamma+1}G_\alpha(\phi^2)dxdy, \quad (2.3)\end{aligned}$$

where $\nabla_G = (\nabla_x, |x|^\alpha\nabla_y)$ is known as Grushin gradient. Let $\phi = \psi^m, \psi \in C_c^2(\mathbb{R}^N;[-1;1])$ and making use of some integral estimates to arrive at

$$\begin{aligned}&\int_{\mathbb{R}^N}|u|^{p+\gamma}\psi^{2m}dxdy\\&\qquad\leq C\int_{\mathbb{R}^N}\left(|\nabla_x\psi|^2+|x|^{2\alpha}|\nabla_y\psi|^2+|\psi|(|\Delta_x\psi|+|x|^{2\alpha}|\Delta_y\psi|)\right)^{\frac{p+\gamma}{p-1}}dxdy, \quad (2.4)\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}|u|^{\gamma+1}G_\alpha(\psi^{2m})dxdy\\&\qquad\leq C\int_{\mathbb{R}^N}\left(|\nabla_x\psi|^2+|x|^{2\alpha}|\nabla_y\psi|^2+|\psi|(|\Delta_x\psi|+|x|^{2\alpha}|\Delta_y\psi|)\right)^{\frac{p+\gamma}{p-1}}dxdy. \quad (2.5)\end{aligned}$$

The following key estimate is then deduced from (2.4) and (2.5):

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(\left|\nabla_G\left(|u|^{\frac{\gamma-1}{2}}.u\right)\right|^2+|u|^{p+\gamma}\right)\psi^{2m}dxdy\\&\leq C_{p,m,\gamma}\int_{\mathbb{R}^N}\left(|\nabla_x\psi|^2+|x|^{2\alpha}|\nabla_y\psi|^2+|\psi|\left(|\Delta_x\psi|+|x|^{2\alpha}|\Delta_y\psi|\right)\right)^{\frac{p+\gamma}{p-1}}dxdy.\end{aligned} \quad (2.6)$$

Finally, by choosing suitable scaled test functions depending on a large parameter R, the right hand side of (2.6) is bounded by $CR^{N_\alpha-2\frac{p+\gamma}{p-1}}$. The constant γ is thus taken such that $N_\alpha - 2\frac{p+\gamma}{p-1} < 0$ and the proof is finished if we let $R \to +\infty$.

3. Proof of Theorem 2.1

We first establish a key tool to prove the main result. The following proposition is an extension of Proposition 4 in [1].

Proposition 3.1. *Let $p > 1$ and $u \in C^2(\mathbb{R}^N)$ be a stable solution of* (1.1). *Fix a real number $\gamma \in \left[1, 2p+2\sqrt{p(p-1)}-1\right)$ and an integer $m \geq \frac{p+\gamma}{p-1}$. Then there is a constant*

$C_{p,m,\gamma} > 0$ *depending only on* p, m *and* γ*, such that*

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\left| \nabla_x \left(|u|^{\frac{\gamma-1}{2}}.u \right) \right|^2 + |x|^{2\alpha} \left| \nabla_y \left(|u|^{\frac{\gamma-1}{2}}.u \right) \right|^2 + |u|^{p+\gamma} \right) \psi^{2m} dxdy \\
&\leq C_{p,m,\gamma} \int_{\mathbb{R}^N} \left(|\nabla_x \psi|^2 + |x|^{2\alpha} |\nabla_y \psi|^2 + |\psi| \left(|\Delta_x \psi| + |x|^{2\alpha} |\Delta_y \psi| \right) \right)^{\frac{p+\gamma}{p-1}} dxdy,
\end{aligned} \tag{3.1}$$

for all $\psi \in C_c^2(\mathbb{R}^N; [-1; 1])$.

Proof. Denote the Grushin gradient by $\nabla_G = (\nabla_x, |x|^\alpha \nabla_y)$ and let $\phi \in C_c^2(\mathbb{R}^N)$. Multiplying (1.1) by $|u|^{\gamma-1} u \phi^2$ and integrating over $\mathbb{R}^N$, we obtain

$$\int_{\mathbb{R}^N} -G_\alpha u.|u|^{\gamma-1} u \phi^2 dxdy = \int_{\mathbb{R}^N} |u|^{p+\gamma} \phi^2 dxdy. \tag{3.2}$$

Noticing that $\nabla_x |u|^{\gamma+1} = \frac{1}{\gamma+1} \nabla_x u.u|u|^{\gamma-1}$ and $\nabla_x (|u|^{\frac{\gamma-1}{2}} u) = \frac{\gamma+1}{2} \nabla_x u.|u|^{\frac{\gamma-1}{2}}$. Using this and the integration by parts to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} (-\Delta_x u).|u|^{\gamma-1} u \phi^2 dxdy \\
&\qquad = \int_{\mathbb{R}^N} (\nabla_x u) \nabla_x \left(|u|^{\gamma-1} u \right) \phi^2 dxdy + \int_{\mathbb{R}^N} (\nabla_x u) \left(|u|^{\gamma-1} u \right) \nabla_x (\phi^2) dxdy \\
&\qquad = \gamma \int_{\mathbb{R}^N} |\nabla_x u|^2 |u|^{\gamma-1} \phi^2 dxdy + \int_{\mathbb{R}^N} (\nabla_x u) \left(|u|^{\gamma-1} u \right) \nabla_x (\phi^2) dxdy \\
&\qquad = \frac{\gamma}{\left(\frac{\gamma+1}{2}\right)^2} \int_{\mathbb{R}^N} \left| \nabla_x \left(|u|^{\frac{\gamma-1}{2}} u \right) \right|^2 \phi^2 dxdy + \frac{1}{\gamma+1} \int_{\mathbb{R}^N} \nabla_x |u|^{\gamma+1} \nabla_x (\phi^2) dxdy \\
&\qquad = \frac{4\gamma}{(\gamma+1)^2} \int_{\mathbb{R}^N} \left| \nabla_x \left(|u|^{\frac{\gamma-1}{2}} u \right) \right|^2 \phi^2 dxdy - \frac{1}{\gamma+1} \int_{\mathbb{R}^N} |u|^{\gamma+1} \Delta_x (\phi^2) dxdy.
\end{aligned}$$

The same arguments also follows that

$$\begin{aligned}
\int_{\mathbb{R}^N} (-|x|^{2\alpha} \Delta_y u).|u|^{\frac{\gamma-1}{2}} u \phi^2 dxdy = \frac{4\gamma}{(\gamma+1)^2} \int_{\mathbb{R}^N} |x|^{2\alpha} \left| \nabla_y \left(|u|^{\frac{\gamma-1}{2}} u \right) \right|^2 \phi^2 dxdy \\
- \frac{1}{\gamma+1} \int_{\mathbb{R}^N} |x|^{2\alpha} |u|^{\gamma+1} \Delta_y (\phi^2) dxdy.
\end{aligned}$$

Inserting the above computations into (3.2), we have

Lemma 3.1. *There holds*

$$\int_{\mathbb{R}^N} \left| \nabla_G \left(|u|^{\frac{\gamma-1}{2}} u \right) \right|^2 \phi^2 dxdy = \frac{(\gamma+1)^2}{4\gamma} \int_{\mathbb{R}^N} |u|^{p+\gamma} \phi^2 dxdy + \frac{\gamma+1}{4\gamma} \int_{\mathbb{R}^N} |u|^{\gamma+1} G_\alpha (\phi^2) dxdy. \tag{3.3}$$

We next use the fact that u is a stable solution of (1.1). In (2.1), we choose the test function $|u|^{\frac{\gamma-1}{2}}u\phi \in C_c^1(\mathbb{R}^N)$ and get

$$\int_{\mathbb{R}^N} \left(\left| \nabla_G \left(|u|^{\frac{\gamma-1}{2}} u\phi \right) \right|^2 - p|u|^{p+\gamma}\phi^2 \right) dxdy \geq 0. \tag{3.4}$$

A straightforward computation gives

$$\nabla_G \left(|u|^{\frac{\gamma-1}{2}} u\phi \right) = \nabla_G \left(|u|^{\frac{\gamma-1}{2}} u \right) \phi + |u|^{\frac{\gamma-1}{2}} u \nabla_G \phi.$$

and

$$\left| \nabla_G \left(|u|^{\frac{\gamma-1}{2}} u\phi \right) \right|^2 = \left| \nabla_G \left(|u|^{\frac{\gamma-1}{2}} u \right) \right|^2 \phi^2 + |u|^{\gamma+1} |\nabla_G \phi|^2 + \frac{1}{2} \nabla_G \left(|u|^{\gamma+1} \right) \nabla_G \phi^2.$$

Combining this with the integration by parts, the inequality (3.4) becomes

$$\begin{aligned} \int_{\mathbb{R}^N} \left| \nabla_G \left(|u|^{\frac{\gamma-1}{2}} u \right) \right|^2 \phi^2 dxdy + \int_{\mathbb{R}^N} |u|^{\gamma+1} |\nabla_G \phi|^2 dxdy - \frac{1}{2} \int_{\mathbb{R}^N} |u|^{\gamma+1} G_\alpha(\phi^2) dxdy \\ \geq \int_{\mathbb{R}^N} p|u|^{p+\gamma} \phi^2 dxdy. \end{aligned} \tag{3.5}$$

From Lemma 3.1 and (3.5), we have

Lemma 3.2. *There holds*

$$\begin{aligned} &\left(p - \frac{(\gamma+1)^2}{4\gamma} \right) \int_{\mathbb{R}^N} |u|^{p+\gamma} \phi^2 dxdy \\ &\qquad \leq \int_{\mathbb{R}^N} |u|^{\gamma+1} |\nabla_G \phi|^2 dxdy + \left(\frac{\gamma+1}{4\gamma} - \frac{1}{2} \right) \int_{\mathbb{R}^N} |u|^{\gamma+1} G_\alpha(\phi^2) dxdy. \end{aligned} \tag{3.6}$$

Here, the assumptions on p, γ imply that $p - \frac{(\gamma+1)^2}{4\gamma} > 0$ and $\frac{\gamma+1}{4\gamma} - \frac{1}{2} < 0$.

Let $m \geq \frac{p+\gamma}{p-1}$ be a fixed integer. For $\psi \in C_c^2(\mathbb{R}^N; [-1; 1])$, put $\phi = \psi^m$. Hence,

$$|\nabla_x \psi^m|^2 = m^2 |\nabla_x \psi|^2 \psi^{2m-2}; |x|^{2\alpha} |\nabla_y \psi^m|^2 = m^2 |x|^{2\alpha} |\nabla_y \psi|^2 \psi^{2m-2}$$

and

$$\Delta_x \psi^{2m} = 2m\psi^{2m-2} \left((2m-1)|\nabla_x \psi|^2 + \psi \Delta_x \psi \right),$$
$$|x|^{2\alpha} \Delta_y \psi^{2m} = 2m\psi^{2m-2} \left((2m-1)|x|^{2\alpha} |\nabla_y \psi|^2 + \psi |x|^{2\alpha} \Delta_y \psi \right).$$

The right hand side of (3.6) is smaller than or equal to

$$C_{m,\gamma}\int_{\mathbb{R}^N}|u|^{\gamma+1}\psi^{2m-2}\left(|\nabla_x\psi|^2+|x|^{2\alpha}|\nabla_y\psi|^2+|\psi|(|\Delta_x\psi|+|x|^{2\alpha}|\Delta_y\psi|)\right)dxdy.$$

Consequently,

$$\begin{aligned}&\int_{\mathbb{R}^N}|u|^{p+\gamma}\psi^{2m}dxdy\\ &\le C_{m,\gamma,p}\int_{\mathbb{R}^N}|u|^{\gamma+1}\psi^{2m-2}\left(|\nabla_x\psi|^2+|x|^{2\alpha}|\nabla_y\psi|^2+|\psi|(|\Delta_x\psi|+|x|^{2\alpha}|\Delta_y\psi|)\right)dxdy.\end{aligned} \tag{3.7}$$

Applying Hölder's inequality to the right hand side of (3.7), we get

$$\begin{aligned}\int_{\mathbb{R}^N}|u|^{p+\gamma}\psi^{2m}dxdy &\le C_{m,\gamma,p}\left[\int_{\mathbb{R}^N}\left(|u|^{\gamma+1}\psi^{2m-2}\right)^{\frac{p+\gamma}{\gamma+1}}dxdy\right]^{\frac{1+\gamma}{p+\gamma}}\\ &\times\left[\int_{\mathbb{R}^N}\left(|\nabla_x\psi|^2+|x|^{2\alpha}|\nabla_y\psi|^2+|\psi|(|\Delta_x\psi|+|x|^{2\alpha}|\Delta_y\psi|)\right)^{\frac{p+\gamma}{p-1}}dxdy\right]^{\frac{p-1}{p+\gamma}}.\end{aligned} \tag{3.8}$$

Moreover, since

$$\begin{aligned}(2m-2)\frac{p+\gamma}{\gamma+1}-2m &= 2m\left(\frac{p+\gamma}{\gamma+1}-1\right)-2\frac{p+\gamma}{\gamma+1}\\ &= 2m\frac{p-1}{\gamma+1}-2\frac{p+\gamma}{\gamma+1}\ge 2\frac{p+\gamma}{\gamma+1}-2\frac{p+\gamma}{\gamma+1}=0\end{aligned} \tag{3.9}$$

and $|\psi|\le 1$, it implies that

$$\left(|u|^{\gamma+1}\psi^{2m-2}\right)^{\frac{p+\gamma}{\gamma+1}}=|u|^{p+\gamma}\psi^{(2m-2)\frac{p+\gamma}{\gamma+1}}\le|u|^{p+\gamma}\psi^{2m}. \tag{3.10}$$

It then follows from (3.8) and (3.10) that

$$\begin{aligned}\int_{\mathbb{R}^N}|u|^{p+\gamma}\psi^{2m}dxdy &\le C_{m,\gamma,p}\left[\int_{\mathbb{R}^N}|u|^{p+\gamma}\psi^{2m}dxdy\right]^{\frac{1+\gamma}{p+\gamma}}\\ &\times\left[\int_{\mathbb{R}^N}\left(|\nabla_x\psi|^2+|x|^{2\alpha}|\nabla_y\psi|^2+|\psi|(|\Delta_x\psi|+|x|^{2\alpha}|\Delta_y\psi|)\right)^{\frac{p+\gamma}{p-1}}dxdy\right]^{\frac{p-1}{p+\gamma}},\end{aligned} \tag{3.11}$$

or equivalently,

$$\begin{aligned}&\int_{\mathbb{R}^N}|u|^{p+\gamma}\psi^{2m}dxdy\\ &\le C_{m,\gamma,p}^{\frac{p+\gamma}{p-1}}\int_{\mathbb{R}^N}\left(|\nabla_x\psi|^2+|x|^{2\alpha}|\nabla_y\psi|^2+|\psi|(|\Delta_x\psi|+|x|^{2\alpha}|\Delta_y\psi|)\right)^{\frac{p+\gamma}{p-1}}dxdy.\end{aligned} \tag{3.12}$$

Similarly, if we choose $\phi = \psi^m$, then the second term in the right hand side of (3.3) can be estimated as follows

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u|^{\gamma+1} G_\alpha(\psi^{2m}) dxdy \\
&\leq C_m \int_{\mathbb{R}^N} |u|^{\gamma+1} \psi^{2m-2} \left(|\nabla_x \psi|^2 + |x|^{2\alpha} |\nabla_y \psi|^2 + |\psi|(|\Delta_x \psi| + |x|^{2\alpha} |\Delta_y \psi|)\right) dxdy \\
&\leq C_m \left[\int_{\mathbb{R}^N} |u|^{p+\gamma} \psi^{2m} dxdy\right]^{\frac{1+\gamma}{p+\gamma}} \\
&\times \left[\int_{\mathbb{R}^N} \left(|\nabla_x \psi|^2 + |x|^{2\alpha} |\nabla_y \psi|^2 + |\psi|(|\Delta_x \psi| + |x|^{2\alpha} |\Delta_y \psi|)\right)^{\frac{p+\gamma}{p-1}} dxdy\right]^{\frac{p-1}{p+\gamma}} \\
&\leq C_m C_{m,\gamma,p}^{\frac{1+\gamma}{p-1}} \int_{\mathbb{R}^N} \left(|\nabla_x \psi|^2 + |x|^{2\alpha} |\nabla_y \psi|^2 + |\psi|(|\Delta_x \psi| + |x|^{2\alpha} |\Delta_y \psi|)\right)^{\frac{p+\gamma}{p-1}} dxdy,
\end{aligned} \tag{3.13}$$

where $C_m = 2m(2m-1)$ and in the last inequality we have used (3.12).

The estimations (3.3) and (3.13) yield

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left|\nabla_G \left(|u|^{\frac{\gamma-1}{2}} u\right)\right|^2 \psi^{2m} dxdy \\
&\qquad \leq C \int_{\mathbb{R}^N} \left(|\nabla_x \psi|^2 + |x|^{2\alpha} |\nabla_y \psi|^2 + |\psi|(|\Delta_x \psi| + |x|^{2\alpha} |\Delta_y \psi|)\right)^{\frac{p+\gamma}{p-1}} dxdy,
\end{aligned} \tag{3.14}$$

where C depends only on m, p, γ.

Finally, Proposition 3.1 results from (3.12) and (3.14). □

Proof of Theorem 2.1.

Let $\chi \in C_c^\infty(\mathbb{R}; [0,1])$ have the properties

$$\chi(t) = 1 \text{ for } |t| \leq 1; \chi(t) = 0 \text{ for } |t| \geq 2.$$

For R large enough, we choose $\psi_R(x,y) = \chi\left(\frac{|x|}{R}\right) \chi\left(\frac{|y|}{R^{\alpha+1}}\right) \in C_c^\infty(\mathbb{R}^N; [0,1])$. Then,

$$|\nabla_x \psi_R(x,y)| = \frac{1}{R} \left|\chi'(\frac{|x|}{R}) \chi(\frac{|y|}{R^{\alpha+1}})\right|; |\nabla_y \psi_R(x,y)| = \frac{1}{R^{1+\alpha}} \left|\chi(\frac{|x|}{R}) \chi'(\frac{|y|}{R^{\alpha+1}})\right|,$$

and

$$|\Delta_x \psi_R(x,y)| = \left|\frac{n-1}{|x|R} \chi'(\frac{|x|}{R}) \chi(\frac{|y|}{R^{\alpha+1}}) + \frac{1}{R^2} \chi''(\frac{|x|}{R}) \chi(\frac{|y|}{R^{\alpha+1}})\right|,$$

$$|\Delta_y \psi_R(x,y)| = \left|\frac{n-1}{|y|R^{1+\alpha}} \chi(\frac{|x|}{R}) \chi'(\frac{|y|}{R^{\alpha+1}}) + \frac{1}{R^{2(1+\alpha)}} \chi(\frac{|x|}{R}) \chi''(\frac{|y|}{R^{\alpha+1}})\right|.$$

These computations together with the boundedness of χ and all its derivatives deduce that

$$\left(|\nabla_x\psi_R|^2+|x|^{2\alpha}|\nabla_y\psi_R|^2+|\psi_R|\left(|\Delta_x\psi_R|+|x|^{2\alpha}|\Delta_y\psi_R|\right)\right)\leq\frac{C_1}{R^2}, \tag{3.15}$$

where C_1 is independent of R.

Hence, (3.1) and (3.15) give

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(\left|\nabla_x\left(|u|^{\frac{\gamma-1}{2}}.u\right)\right|^2+|x|^{2\alpha}\left|\nabla_y\left(|u|^{\frac{\gamma-1}{2}}.u\right)\right|^2+|u|^{p+\gamma}\right)\psi_R^{2m}dxdy\\&\leq\int_{\{|x|\leq 2R;|y|\leq 2R^{1+\alpha}\}}\left(|\nabla_x\psi_R|^2+|x|^{2\alpha}|\nabla_y\psi_R|^2+|\psi_R|\left(|\Delta_x\psi_R|+|x|^{2\alpha}|\Delta_y\psi_R|\right)\right)dxdy\\&\leq\frac{C}{R^{2(\frac{p+\gamma}{p-1})}}R^{N_1}R^{N_2(1+\alpha)}=CR^{N_\alpha-2\frac{p+\gamma}{p-1}},\end{aligned} \tag{3.16}$$

where C does not depend on R.

Finally, we need to show that, given N_α and p in Theorem 2.1, there is a constant $\gamma\in[1,2p+2\sqrt{p(p-1)}-1)$ such that

$$N_\alpha-2\frac{p+\gamma}{p-1}<0. \tag{3.17}$$

Case $N_\alpha\leq 10$: for any $p>1$, the function $\gamma\mapsto\frac{p+\gamma}{p-1}$ is continuous, increasing and $\frac{p+2p+2\sqrt{p(p-1)}-1}{p-1}>\frac{5p-3}{p-1}>5$. Thus, there is γ verifying (3.17).

Case $N_\alpha>10$*:* consider the inequality

$$N_\alpha-2\frac{p+2p+2\sqrt{p(p-1)}-1}{p-1}<0 \text{ or } N_\alpha(p-1)-2(p+2p+2\sqrt{p(p-1)}-1)<0. \tag{3.18}$$

By changing of variable $t=p-1>0$, it is equivalent to

$$(N_\alpha-6)t-4<4\sqrt{t^2+t}. \tag{3.19}$$

If $t<\frac{4}{N_\alpha-6}$, then (3.19) is true. If $t\geq\frac{4}{N_\alpha-6}$, then (3.19) is equivalent to

$$(N_\alpha^2-12N_\alpha+20)t^2-8(N_\alpha-4)t+16<0$$

and then

$$\frac{4(N_\alpha-4)-8\sqrt{N_\alpha-1}}{N_\alpha^2-12N_\alpha+20}<t<\frac{4(N_\alpha-4)+8\sqrt{N_\alpha-1}}{N_\alpha^2-12N_\alpha+20}.$$

Note that for $N_\alpha>10$, we have

$$\frac{4(N_\alpha-4)-8\sqrt{N_\alpha-1}}{N_\alpha^2-12N_\alpha+20}\leq\frac{4(N_\alpha-4)-24}{N_\alpha^2-12N_\alpha+20}=\frac{4}{N_\alpha-2}<\frac{4}{N_\alpha-6}.$$

Thus, (3.19) is true for all $0 < t < \frac{4(N_\alpha - 4) + 8\sqrt{N_\alpha - 1}}{N_\alpha^2 - 12N_\alpha + 20}$. This shows that (3.18) is true for

$$1 < p < 1 + \frac{4(N_\alpha - 4) + 8\sqrt{N_\alpha - 1}}{N_\alpha^2 - 12N_\alpha + 20} = p_c(N_\alpha).$$

As above, combining the fact that the function $\gamma \mapsto N_\alpha - \frac{p+\gamma}{p-1}$ is continuous, decreasing and (3.18), there exists $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ such that

$$N_\alpha - 2\frac{p+\gamma}{p-1} < 0.$$

Letting $R \to \infty$ in (3.16) and using (3.17), we obtain $u \equiv 0$ on $\mathbb{R}^N$. The proof of Theorem 2.1 is finished. $\square$

REFERENCES

[1] A. Farina., 2007. *On the classification of solutions of the Lane-Emden equation on unbounded domains of* $\mathbb{R}^N$. J. Math. Pures Appl. (9), 87(5):537-561.

[2] B. Franchi, C.E. Gutiérrez, and R.L. Wheeden, 1994. *Weighted Sobolev-Poincaré inequalities for Grushin type operators.* Comm. Partial Differential Equations, 19(3-4):523-604.

[3] I. Birindelli, I. Capuzzo Dolcetta, and A. Cutrì, 1997. *Liouville theorems for semilinear equations on the Heisenberg group.* Ann. Inst. H. Poincaré Anal. Non Linéaire, 14(3):295-308.

[4] I. Birindelli and J. Prajapat, 1999. *Nonlinear Liouville theorems in the Heisenberg group via the moving plane method.* Comm. Partial Differential Equations, 24(9-10):1875-1890.

[5] Nguyen Minh Tri, 1998. *On the Grushin equation.* Mat. Zametki, 63(1):95-105.

[6] C.Wang, Q. Wang, and J. Yang, 2015. *On the Grushin critical problem with a cylindrical symmetry.* Adv. Differential Equations, 20(1-2):77-116.

[7] R. Monti and D. Morbidell, 2006. *Kelvin transform for Grushin operators and critical semilinear equations.* Duke Math. J., 131(1):167-202.

[8] A. Loiudice, 2009. *Asymptotic behaviour of solutions for a class of degenerate elliptic critical problems.* Nonlinear Anal., 70(8):2986-2991.

[9] B.Franchi, 1991. *Weighted Sobolev-Poincaré inequalities and pointwise estimates for a class of degenerate elliptic equations.* Trans. Amer. Math. Soc., 327(1):125-158.

[10] L. Wang, 2003. *Hölder estimates for subelliptic operators.* J. Funct. Anal., 199(1):228-242.

[11] Q. Song, Y. Lu, J. Shen, and L. Wang, 2011. *Regularity of a class of degenerate elliptic equations.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 10(3):645-667.

[12] P.Poláčik, P.Quittner, and P. Souplet, 2007. *Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems.* Duke Math. J., 139(3):555-579.

[13] H. Berestycki, I. Capuzzo-Dolcetta, and L. Nirenberg,1994. *Superlinear indefinite elliptic problems and nonlinear Liouville theorems*. Topol. Methods Nonlinear Anal., 4(1):59-78.

[14] D. Monticelli, 2010. *Maximum principles and the method of moving planes for a class of degenerate elliptic linear operators.* J. Eur. Math. Soc. (JEMS), 12(3):611-654.

[15] X.Yu, 2014. *Liouville type theorem for nonlinear elliptic equation involving Grushin operators.* Communications in Contemporary Mathematics.

[16] A. Cutrì I. CapuzzoDolcetta, 1997. *On the liouville property for sublaplacians.* Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, 25(1-2):239-256.

[17] L. D'Ambrosio and S. Lucente, 2003. *Nonlinear Liouville theorems for Grushin and Tricomi operators.* J. Differential Equations, 193(2):511-541.

[18] D.Morbidelli, 2009. *Liouville theorem, conformally invariant cones and umbilical surfaces for Grushin-type metrics.* Israel J. Math., 173:379-402.

[19] L. Dupaigne, 2011. *Stable solutions of elliptic partial differential equations*, Volume 143 of Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics. Chapman & Hall/CRC, Boca Raton, FL.

[20] D. D. Joseph and T. S. Lundgren, 1973. *Quasilinear Dirichlet problems driven by positive sources.* Arch. Rational Mech. Anal., 49:241-269.