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Quantum Probability based Decision Making in Finance: from Individual Preferences to Market Outcomes

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Abstract

This paper surveys the main directions of the applications in finance of a generalized probability calculus that is derived from the axiomatics of quantum physics, see monographs by [33], [12], [22]. Recently, subjective expected utility with QP (quantum probability) measures of agents' uncertainty and contextuality in preferences was formalized in [3]. The projective measurement scheme that is at the core of QP relaxes some of the core axioms of classical probability, namely the *commutativity* and *distributivity* of events. Hence, QP captures well real decision making scenarios, where agents can have ambiguous and state dependent beliefs. In [8] agents' make comparison between lotteries and *interference effects* between prospects are present that denote risk perceptions from the ambiguity about prospect realisation in the selection process. The notion of *non-commuting* lottery observables has the substantial to explain paradoxical behaviour of individual investors, characterised my myopia in asset return evaluation, as well as inter-asset valuation. Moreover, the interference term of agents' comparison state can provide a quantitative description of the disposition effect from agents' contextual utility perception. Some of the implications of non-classicality in beliefs for the composite market outcomes can be also modelled with the aid of QP. For instance, the emergence of speculative bubbles from investors' sentiment in asset pricing is elaborated in [36, 37]

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1 INTRODUCTION

"Theories which purported to describe the uncertainty [of events] in terms of probabilities would be quite inapplicable...unless quite different operation for measuring probability were devised." (Ellsberg, [13], p. 646)

An array of deviations from classical probability based information processing in economic agents' judgement and decision making has been detected in experiments as well as in real market settings. Broadly speaking, the main causes of contextual or state dependently behaviour where attributed to cognitive and psychological influences coupled with environmental conditions elaborated in the works by, [26], [29], [54] and [58].

Irrationality of preferences that are at variance with EUT ([61]) under risk and SEUT ([49]) under uncertainty is hinged by the state dependence of economic agents' valuation of payoffs with far reaching implications for their trading on the finance market and deviations from rational equilibrium prices.[a] The core question plaguing decision theory could be formulated as following: "Should one rely on the axiomatic of classical probability when describing human beliefs and their dynamics?" .

There is a vast amount of contributions that aimed to address nonclassicality of human beliefs and the impact of ambiguity upon human way of thinking and making decisions. We can mention here the foundational contributions by [17] and [51] that aimed to generalize the classical probability functions to overcome non-additivity of probability. Future studies built upon exiting findings on human beliefs about likelihood of payoffs in risky and uncertain settings to devise a more accurate representation of beliefs, via a probability weighting function that takes into account the outcomes and their cumulative probability distribution, [60], [64] and [46].

Other contributions also focused on the state dependence and unstable nature of individual utility and hence, changing risk preferences, [27], [60], [30]. The above mentioned works aimed to provide a generalization of classical probability scheme in belief formation through a formulation of a more rich framework of human risk and ambiguity preferences. Modifications of EUT and SEUT (together with some assumptions, such as 'coding rules' and 'reference point' in Prospect Theory) give a good fit with empirical data and account for revealed biases and sate dependent preferences. Here we can mention important cognitive features such as e.g., loss aversion and disposition effect, ambiguity dependent beliefs, order effects in information processing and preference formation, as well as intertemporal dynamics of preferences and beliefs.

In the search for a different (more general, yet complete) theory of probability that could be applied to measurement of human beliefs, but also provide a probabilistic description of decisions, researchers from interdisci-

[a] Abbreviation EUT stands foe Expected utility theory and SEUT stands for Subjective expected utility theory.

plinary fields in psychology, economics as well as mathematics and physics adapted quantum probability based calculus that was an original part of the theory of measurement applied to microscopic objects, such as photons and electrons. We can mention here early works by [1], [31], [21] in which the authors conceived that cognitive systems and the flow of information can be modelled by the same calculus that is used to depict the behaviour of microscopic systems and their contextuality.

The field of application of QP (quantum probability) to social science has grown rapidly, with a diversity of contributions to decision making in games, voting behaviour and information processing in various contexts. Finance applications of quantum mechanical calculus are also wide ranging, and utilize both classical (Copenhagen) interpretation of quantum probability and pilot-wave models of deterministic nature that are inspired by Bohemian quantum mechanics. For an in depth introduction and references the reader is invited to consult the monographs by [33], [22], [12] and surveys by [32], [45]. The focus of this survey is on applications of QP as a basis to decision theoretic models in economics and finance, to mention few, we refer to works by [11], [44], [65], [34] and [59].[b]

While quantum probability showed to provide a good descriptive account for, i) ambiguity perception; ii) state dependence of beliefs and preferences combined with instances of non-Bayesian update, the ultimate goal was to develop a theoretical framework of decision making based on QP and decision contextuality. The latest contributions in economics and finance addressed well the Ellsberg and Machina type ambiguity, see works by [23], [3], [8]. Also, collected experimental evidence on disjunctive investment preferences under risk was successfully modelled with aid of QP in [24].

State dependence has been extensively explored in questionnaires and opinion polls. QP model for order effects that accounts for specific QP regularity in preference frequency from non-commutativity is devised [59] and [62] and further explored in terms of predictions in the work by [34]. The roots of state dependence are identified and testable quantitative predictions for modelling the endowment effect are established in the recent contribution by [3]. Non-commutativity of projectors as a source of state dependence in belief formation serves as a good explanation for the heterogeneity in agents' information processing that yields the 'agree to disagree' paradox among agents, see QP mode in [35]. Other implications of non-Bayesian update with a sub-additive treatment of complimentary beliefs are experimentally explored in the setting of 'zero prior' paradox in [7]. Financial implications such as deviations from rational expectations equilibrium resulting from incomplete information and ambiguous beliefs of agents are theorized in [37].

The remainder of this survey is structured as follows: in the next sec-

[b] There are also many applications of quantum probability and the dynamics of complex probability amplitudes to game theory, economics and asset pricing, e.g., [43], [55] and [4].

tion, section, 2 we sketch an overview of the behavioural paradoxes in economics and finance and approaches to modelling them via non EUT theories. In section 3 we present a non technical introduction to the latest advances in QP based decision theory that was developed in works by [3] and [8]. This framework provides the core mathematical rules, pertaining to lottery selection from an agent's (indefinite) comparison state.

The main causes of non-rational behaviour in finance, pertaining among other to inflationary and deflationary asset prices that deviate from a fundamental valuation of assets. In section, 3 we summarize assumptions of the proposed QP based model of subjective expected utility and define the core mathematical rules pertaining to lottery selection from an agent's (indefinite) comparison state. In section 4 we discuss the implications of the model for the disparity of WTA (Willingness to accept a certain payment for a lot) and WPA (Willingness to pay for the same lot) and the emergence of endowment effect that also gives raise to disposition effect in the context of asset trading. In section 5, we focus on complementarity of beliefs about an asset' returns returns of complimentary assets in the setting of portfolio holding. In the section 6, we outline a QP rule of belief formation, that serves as a contribution to theoretical models of composite market outcomes, characterized by speculative bubbles and volatility.

Finally, in section, 7 we conclude to consider some possible future venues of research in the domain of application of QP based decision making in asset pricing and behavioural finance.

2 BEHAVIOURAL FINANCE AND PARADOXES

Starting with the seminal paradoxes revealed in thought experiments by [2] and [13] the classical neo-economic theory was preoccupied with modelling of the impact of ambiguity and risk upon agent's probabilistic belief and preference formation. In classical decision theories due to [61] and [49] there are two core components of a decision making process: i) agents' form beliefs about subjective and objective risks via classical probability measures. They update their beliefs via a Bayesian scheme; ii) preference formation is derived from optimization via an attachment of a utility value to each (monetary) outcome. These two building blocks of rational decision making serve as the core pillars behind asset trading frameworks in finance, starting with Modern Portfolio theory that is based on mean-variance optimization and Capital Asset Pricing model that presumes a representative agents' asset valuation.[c] The core premise of the frameworks is that beliefs about the returns suppose a similar historical pattern in the absence of new information, and are homogeneous across economic agents. The predictions of asset allocation and asset trading are grounded in the assumption of all agents being

[c] For a comprehensive introduction to asset pricing frameworks and references we refer the interested reader to core texts in finance, e.g. [10].

Bayesian rational in their wealth maximization.

The main assumption that allows these elegant frameworks to provide as benchmark for fair prices of risky assets is context independence of beliefs and preferences. Agents ought to form joint probability distribution of all asset class returns in regard to the whole investment period in order to assess the mean returns and standard deviations. The agents also dislike idiosyncratic risk and hence prefer only to hold the market portfolio (in combination with a risk free asset depending on their risk aversion profile).

After extensive empirical evidence documented an existence of market inefficiencies, such as deviations from equilibrium asset prices, characterised by bubbles or abrupt market corrections the School of Behavioural Finance endeavoured to explain the observed anomalies in human behaviour. We can mention to streams of research, with contributions focused on individual agent's beliefs and preferences, as well as investigation of the implications for the composite finance market behaviour characterized by excess trading and excess volatility, asymmetric and incomplete information and agents' reaction, etc., see some fundamental works in this direction by [26], [53], [52], [42], [57]. Bubbles and high return rates as a result of agents' heterogeneous beliefs were firstly addressed in the works by [20], [50] as well as in works based changing risk preferences in [54], and [9]. A disposition effect characterising 'sticky behaviour' in respect to negative return stocks was explained via loss aversion and desire to break-even as postulated in the prominent 'Prospect theory' ([27], [60]). Prospect theory contains a generalization of classical utility function from [61]. Two value functions of a different curvature exits, with the one in the loss domain being 2.5 times more curved than the one in the gain domain, to depict the extra 'pain' associated with foregoing a monetary amount or an object in one's possession, see extensive experimental evidence and analysis in [28]. The idea that a loss can have such a strong effect upon agents' preferences, attracted a vast attention in asset pricing studies. Loss aversion was attributed to trigger the notable disposition effect, manifest in an unwillingness of the investor to e.g., sell shares that depreciated in value, yielding in high returns for the wining stocks and vice versa, with a general effect of creating and in other periods attenuating the price trends, [53]. Another peculiarity in investors' behaviour was triggered by their non-classical belief formation that deviates from Kolmogorovian probability theory, [38]. These deviations where often coined as 'noisy' with an assumption that on average, the effects of the positive and the negative noise in agents' beliefs cancels out there are minimal influences on the composite capital markets.

Non-linearity in beliefs, as well as their dependence on the negative, or positive changes in wealth was well captured via an inflected probability weighting function, devised in the works by [27], [60], and advanced in [46], [19], [64]. This type of probability weighing function provides a viable explanation for common ratio effect [2] and ambiguity aversion in [13].

Non-additivity in beliefs is not confined to 'laboratory experiments' only and has been detected among professional traders as well, [16]. Moreover, it was found that economic agents can exhibit other information processing fallacy, coined 'myopia'. Myopia corresponds to narrow framing, or more formally an inability to form a joint sample space for an asset's returns over a set of investment periods. Agents can also show state dependence, as they employ different 'evaluation rules' in respect to the assessment of previous losses and gains, see experimental findings in [40] and [57].[d] When the economic agents tend to display a joint myopia and loss aversion bias (MLA), the implications for the composite finance markets can be far-reaching, as the risky assets become under-prised and agents demand higher risk premium. This is the result of their narrow framing in the evaluation of the returns for each investment period in isolation, rather than over the whole planned investment horizon, [9]. Market experiments document as well that agents, who do not receive frequent feedback about their investment, will exhibit lower degree of MLA and as a result the asset prices appreciate, [63] and [18].

Recently, the notion of belief state dependence, as result of previously experienced gains, or losses was detected in a set market experiments by [39]. The findings of this study showed that individual belief update can deviate from the Bayesian scheme, and moreover, the deviations are interrelated to the sign of the experienced return. Essentially, one can witness that state dependence of beliefs is of a more non-separable character than conceived by the classical utility theories and their generalizations, such as Prospect Theory. There decision theoretic frameworks separate between the representation of beliefs about state-outcomes and the attached utility/value.[e]

The notion of ambiguity that surrounds future events, and its possible implications for agents' beliefs about the future returns of risky assets also attracted fast attention in finance literature. Most of these frameworks are endeavouring to model Ellsberg-type ambiguity aversion that results more pessimistic beliefs and in shunning of complex risks. The celebrated "Max-min expected utility" due to [17] provides a good account for the representation of the pessimistic beliefs that can ex-

[d] Previous gains and losses, i.e. positive or negative returns should not have any effect upon investors' subsequent preferences, besides becoming a part of her existing wealth.

[e] To put it differently, the realized states and corresponding outcomes can affect beliefs and preferences of the agents. Beliefs can also be influenced by the probabilistic set-up of complementary prospects (lotteries) as shown in [3], [8]. One should note that this effect is different from the non-linearity of prior beliefs, as captured in the probability weighting functionals, in [27], [60]. We apply the word 'context' or 'state dependence' as an umbrella for coining these effects.

[f] For instance, agents' can be ambiguous in respect to the prior likelihoods, as well as being affected by ambiguous information that produces deviations of asset prices from the rational equilibrium, [14]. We also refer to works on ambiguity markets for risky assets detected in ex-

plain an additional ‘ambiguity premiums’ on assets with complex and unknown risks.[f]

3 QP LOTTERY SELECTION FROM AN AMBIGUOUS STATE

The main premises of vNM utility theory due to [61] imply: i) separability in evaluation of mutually exclusive lottery outcomes; b) the evaluations of outcomes may be quantified by the cardinal utility function that attaches real utility numbers to consequences $U(c)$; c) utilities may be obtained by firstly computing the expectations of each (monetary) consequence, with respect to the risk encoded in the objective probabilities; and finally d) the utilities of the considered outcomes are aggregated across the decision tree, see [30] for a more technical treatment. The above formalisation suggests only the consequences matter when agents are computing utilities combined with the existence of a joint probabilistic distribution of the consequences of several lotteries that are chosen concurrently, or are part of a compound lottery.

The QP lottery selection theory developed in [3], [8] can be considered as a generalization of Prospect Theory in following respects: i) non-additivity of beliefs and a non-neutral attitude to the lottery outcome risk is modelled via complex probability amplitudes and interference effects can exist between them. Interference term λ quantitatively captures the ‘fear’ to obtain an undesirable outcome before the lottery choice has been made and the outcome realized. One can interpret it as an agent’s *Degree of Evaluation of Risk* (DER); ii) an agent’s comparison state (that is modelled as a ψ vector) in the process of lottery selection considers the lotteries as complimentary in the process of decision making. This assumption relaxes requirement of an existence of a joint probability distribution across lottery outcomes. The utility of each lottery outcome depends on the lottery composition and the comparison of the lotteries is driven by a process of reflections about the possible outcome realization and relative utility that is will generate for the decision maker. This process is operationally given by a comparison operator, D. Hence, the main premise of the QP framework is that the subjects attach a *state dependent utility* to the realization of the lottery outcomes and their subjective beliefs can deviate from the objective probability distribution of lottery outcomes.

3.1 Standard EUT Maximization via Classical Probability Calculus

There are two lots, say $A = (x_i, p_i)$ and $B = (y_i, p_i)$, where (x_i) and (y_i) are outcomes and (p_i) and (q_i) are probabilities of these outcomes. All of the outcomes are different from each other. The agent is confronted with following question when dealing with this simple decision making task: *Which lot do you*

perimental studies by [41], [48] and [47]. The latter study detects a more rare phenomenon of ‘ambiguity seeking’, as a result agents’ shifts in reference points due to experienced gains or losses.

select? What decision rule to use, in order to be able to rank the lots in terms of desirability? An agent, can simulate her experience that she draws the lot A (or B) and gets the outcome x_i (or y_i). We represent such an event by (A, x_i) or (B, y_i) that denotes a joint occurrence of an act, and with a realised random outcome. Subjects assign utilities to the outcomes of the lotteries, $u(x_i)$ and $y(x_i)$ of (A, x_i) and (B, y_i), respectively. Here, $u(x)$ is a utility function of outcome that only depends on the total wealth of the agent as a result of lottery selection, x.

By using a utility function the agent evaluates various comparisons for forming the preference, $A \succeq B$, or $B \succeq A$. Expected utility theory, devised the following optimisation rule: an agent calculates the expectation values $E_A = \sum u(x_i)p_i$ and $E_B = \sum u(y_i)q_i$, to use their difference as a criterion for establishing her preference, [61].

3.2 QP Based Representation of Lotteries by Orthonormal Bases in a Belief-State Space

Consider the space of belief states of an agent in respect to different decision making tasks. Belief-states are represented by normalized vectors in a complex Hilbert space H. These are the so-called pure states, which depict the indeterminacy of the agent in respect to the realization of lottery outcomes. The lotteries A and B are mathematically realized as two orthonormal bases in H : $(|i_a\rangle)$ and $(|j_b\rangle)$. Any vector $|i_a\rangle$ represents the event (A, x_i) - "selecting the A-lottery, which will realise an outcome x_i." The same applies to the vectors of the B-basis. We should not that the lottery realization events are not real, but hypothetical. The agent conceives, which potential outcomes of the lotteries can realise, by the means of a state transition into lottery eigenbases, and compares the eigenvalues via the attached utility mappings. Here we also need to emphasise how the agent relates the lottery outcomes to the corresponding utilities. the utility (derived from some monetary amount) has not only a numerical value, but also a "color" determined by the circumstances surrounding the corresponding lottery selection. Mathematically, one can also represent lotteries by Hermitian operators:

$$A = \sum_i x_i |i_a\rangle \ , B = \sum_j y_j |j_b\rangle. \qquad (1)$$

As in the classical EUT, each outcome x_i has some utility $u_i = u(x_i)$ (say an amount of money). Starting with two lotteries A and B, with outcomes (x_i) and (y_j) these have corresponding utilities $u_i = u(x_i)$ and $v_j = u(y_j)$.

In the process of selection, an agent attaches these utilities to two orthonormal bases in the belief-state space H :

$$u_i \sim |i_a\rangle, \ v_j \sim |j_b\rangle. \qquad (2)$$

Since the bases are fixed in respect to the particular lottery observables, the final utilities are related to the specific lottery composition and the subjective beliefs of the decision maker, [8].

3.3 Belief State Representation And Subjective Probability Of Lottery Realisation

The state of a person's beliefs about the lottery A can be represented as a superposition:

$$|\psi_A\rangle = \sum_i \sqrt{p_i} e^{i\theta_{ai}} |i_a\rangle.$$

The probability of the realization of the event (A, x_i) is given by the Born rule and equals to $p_i = |\langle i_a|\psi_A\rangle|^2$. In the same way, the state of beliefs about the lottery B can be represented as superposition

$$|\psi_B\rangle = \sum_i \sqrt{q_i} e^{i\theta_{bi}} |i_b\rangle.$$

The agent superposes her belief-states about the lotteries and their respective outcomes. Her composite belief-state is given as a superposition of her beliefs about the A-lottery and the B-lottery. The overall state space of lottery selection is given by the composite state vector, Ψ that is the superposition of the ψ's s for two individual lotteries, i.e. $\Psi = \psi_A + \psi_B$.

3.4 Comparison of Complementary Lotteries

As noted, the lottery selection process of the decision-maker is contextual. In many decision making problems the agent is not forming a joint probabilistic representation of her actions and the lottery outcomes and this is why, the lottery observables are evaluated *sequentially* by her comparison state. To put it differently, the agent is not thinking of the lotteries in terms of joint probability distribution of the outcomes (x_i, y_j). Hence, the lottery operators can be *non-commuting*, i.e., $[A, B] \neq 0$ corresponding to an impossibility of a joint measurement on the lottery observables. Instead of weighting probabilistically the pairs of outcomes, the DMr analyses the possibility of the realization of an outcome say x_i of the A-lottery, by accounting its utility $u(x_i)$. Then, under the assumption of such a realization, she thinks through each of the scenarios of the possible realizations (y_j) of the B-lottery, and compares corresponding utilities $u(y_j)$ and $u(x_i)$. Utility values are given through classical utility function, and are technically realised as mappings from the eigenstates of the lottery-operators to the actual numerical utilities. We can describe the comparison sequences as following: *"Suppose, I have selected the A-lottery and its outcome x_i was realized. What would be my gain (loss), if (instead) the B-lottery were to be selected, and an outcome y_j was realized?"* These reflections precede the formation of a firm preference in respect to a lottery choice and are described in the Hilbert state space via a specific *comparison operator*, D. This operator mathematically models a belief state transition from the A-basis to the B-basis and back, to obtain the final relative utility of the lots.

Operator D comprises of two transition operators that describe the process of transition from preferring the state $|i_a\rangle$ to preferring the state $|j_b\rangle$. We stress that state transitions take place between the different belief states of an agent before the selection of the lottery takes place. As the the agent transits from B to A, she evaluates a relative utility of selecting the lottery A in respect to selecting the lottery B. We can interpret relative utility as the difference between, $u(x_n)$ that the agent earns by choosing A and realizing a potential outcome x_n and the utility $u(y_m)$ of the possible outcome y_m of the lottery B. Hence, we can formulate a summary of decision criterion in a state dependent EUT:

Decision rule: *If the average of the comparison operator D is non-negative, then $A \succeq B$.*

Essentially, the agent evaluates average relative utility, from preferring A to B, respective, B to A, and if the relative utility of preferring the lottery A is positive she selects this lot.[g]

This operator captures contextuality and indeterminacy in decision making process, that goes beyond EUT approach based on calculation and comparison of weighted averages of the lotteries' utilities.

3.5 A Note on the Relationship of QP with the Agent's Subjective Beliefs

As noted, QP based subjective probabilities are closely reproducing a specific type of probability weighting function that captures ambiguity attraction to low probabilities, and ambiguity aversion to high probabilities that are close to one. These features of human judgements are captured with the aid of probability weighting functional, estimated from empirical data, [60], [46], [19].

This probability weighting function is of the form:

$$w_{\lambda,\delta}(x) = \frac{\delta x^{\lambda}}{\delta x^{\lambda} + (1-x)^{\lambda}}, \qquad (3)$$

The parameters λ and δ control the curvature and elevation of the function in eq.(3), see for instance, [19]. The smaller the value of the above concavity/convexity parameter the more 'curved' is the probability weighting function. The derivation of such a curvature of the probability weighting function from the QP amplitudes corresponds to one specific type of parameter function with $\lambda = 1/2$. In other words the interference angle of the magnitude 1/2 provides a good representation of agents' belief distortion in respect to the lottery outcomes. Hence, the interference term can provide a testable prediction for the estimation of agents' subjective probabilities from her initial superposition state in respect to each lottery observable.

4 ENDOWMENT AND DISPOSITION EFFECTS FROM AGENTS' STATE DEPENDENCE

Endowment effect characterises an asymmetric valuation of the item already in possession and the item to be acquired. Endowment effect was experimentally detected in [28] where the authors examined bid and offer prices for various items (mugs, pens etc.) and found a significant difference between the the selling price (WTA) and the preferred purchase price (WTP), more

[g] For a detailed account of the decision making dynamics via usage of the comparison operator and its mathematical form we refer to [8].

precisely $WTA > WTP$. The cause of such a discrepancy was attributed to a shift in agents' reference point, whereby they exhibit *loss aversion* in respect to the already possessed goods, and the dis-utility of selling them is higher than the utility from receiving the same amount of cash.

This effect is also present in finance setting, where investors continue to hold risky assets, which previously realized negative returns in respect to the purchase price. This effect is coined 'disposition effect' and is widely explored in terms of individual agents' behaviour and the implication for the capital market outcomes, see [53], [42], [63]. One natural consequence of disposition effect is that the trading volume drops, since the sellers do not want to part with the object that they posses, even if they are aware that is is essentially not worth the cash that they demand, if they were on the other side of the deal, see detailed elaboration in [28].

Endowment effect translates into disposition effect in the following: When making an investment the agents pay a certainty equivalent (CE) of cash to buy a share that can be considered as a risky lottery (L_s). When the investor buys a stock he is treating the purchase price as a reference point, i.e. this is the cash she would like to get back at t_1 (we ignore the time dimension and cost of money in this illustration). Assuming that the stock realized a negative return (P_-) at t_1, the investor prone to disposition effect, keeps the stock and essentially accepts another risky lot for the period t_2. Let us assume that CE $\sim L_s$, i.e. the agent is indifferent between the risky stock holding, L_s, and the cash (in fact she prefers the stock in this setting). Hence, assuming that in the next period the investment has the same degree of riskiness of outcomes, the preference of the agent becomes, $CE \sim (P_+ L_s)$. This means the CE decreases by the amount (P_-), and the agent becomes more risk taking by holding the stock, since she accepts a lower return on the stock over the composite investment period. This type of behaviour also implies that $WTA > WTP$ for the particular stock. The phenomenon is attributed to loss aversion in respect to the existing stock holding, coupled with the desire to break even in respect to the initial purchase price that was paid for the financial asset, cf. experimental evidence and detailed analysis in [27], [57], [60] and [52]

Following, [3], QP calculus can account for this type of asymmetry in valuation of risky assets via the special parameter λ that serves as a measure of "DER" (degree of risk evaluation) for a specific lottery. Consider a choice problem in which the CE in cash is x and a lot, whose outcome is $y(> 0)$ with probability p, or zero, with $q = 1 - p$. Then, the choice state is given as a superposition state:

$$\psi = \frac{1}{\sqrt{2}}\psi_{\text{lot}} + \frac{1}{\sqrt{2}}\psi_{\text{cash}}, \qquad (4)$$

The authors in cite [3] derive an indifference relation between the utilities of x and y for the comparison state, by introducing the interference parameter λ. A higher value of λ denotes a higher level of risk aversion in respect to the lottery outcomes. The coefficeint can be estimated for different outcome probabilities p, associated with the risky outcome u_y. The utility for u_y decreases as the value of λ goes up. We note that as the $\lambda \to 1$ the dislike for risk becomes very high.

One can derive the parameter from the indifference relation between CE (i.e the monetary amount x) and the lottery through following relationship: As noted, x in this case corresponds to the CE of the lot:

$$u_{CE} = u_x = \frac{\sqrt{p}(1 - \lambda\sqrt{q})u_y}{\sqrt{p} + \sqrt{q}}. \qquad (5)$$

This relationship can be expressed graphically, where the payoff amount of certainty equivalent per unit of the risky outcome will be a specific function of the probability of the risky outcome, with higher λ values denoting a more convex function and the more negative λ values denoting a 'curved' relationship between these two variables.[h]

In the formalism of QP representation, an agent exhibits endowment effect, when she possess state dependent DER, given by different λ parameters. This parameter treats the utility of the cash equivalent and the risky lot as being dependent on the subjective probabilities of the risky outcomes.

An agent has different valuation of the lot that she would like to acquire (such as purchasing a share) and the lot that she already possesses (the share that is already purchased). As noted in [3], (p.10): "*seller's λ will be smaller than the buyer has*," implying an increased risk seeking/loss aversion, on the part of the seller that translates into a higher WTA and a lower CE of the seller than of the buyer. Some concrete experimental findings for risky portfolio investments, were analysed in [24], and the results are consistent with a willingness of the subjects to hold the risky portfolio, after a negative return was realized. At the same time, a lower percentage of subjects is willing to invest into such a portfolio in a baseline setting. The authors provide a QP account of the state dependence in subjects' preferences.

We can take stock that the parameter λ is state dependent and shifts in a similar mode as a decision makers' reference point in Prospect Theory. More precisely, the comparison state, ψ, is characterised by different interference amplitudes, depending on the choice problem, e.g., to choose between a CE and the risky asset as a *buyer*, respective as a *seller*.

5 NARROW FRAMING AND MYOPIA: COMPLEMENTARY BELIEFS AND COMPLEMETARY ASSETS

On the level of the composite finance market, agents are often influenced by order effects, when forming beliefs about future dividends and price outcomes.

These effects are often coined 'overreaction' or 'underreaction' in behavioural finance literature [54]. The deviations from rational information processing can be considered as a manifestation of state dependence in agents' belief formation that affects their trading. Shiller, [54] notes that 'overre-

[h] We remind that λ is the interference angle and is mathematically bound by $-1 \leq \lambda\ 1$.

action' does not necessarily mean that new information about fundamentals is released, and can manifest itself in over-optimism and deviation from Bayesian update, due to observed price sequences.[i]

Recent market experiments show a persistent influence of previous gains and losses upon agents' investment behaviour, see for instance, [39], [56] and [24]. The experienced losses can trigger more pessimistic posterior beliefs that deviate from Bayesian information processing. It is also documented that agents exhibit narrow framing in respect to the evaluation of the risky assets' returns, by treating the future investment periods as complementary to each other in the process of belief formation, [18], [40] and [63].

Based on the assumptions in [36] on the non-classical correlations that assets' returns can exhibit, we depict a simple QP model of an agent's asset evaluation process. The model considers two risky assets, k and n and their price realisations. The agent is uncertain about the price dynamics of these assets and does not possess a *joint probability evaluation* of their price outcomes. Hence, interference effects exist in respect to the beliefs about price realization of these assets. The agent evaluates the future returns of the assets sequentially, and order effects in respect to the final evaluation of the price realization exist, see QP based works on order effects in judgements due to [59], [62] and [34].

By making a decision $\alpha = \pm 1$ on the asset k, an agent's state ψ is projected onto the eigenvector $|\alpha_i\rangle$ that corresponds to an eigenstate for a particular price realization for that asset.[j] After the next trading period price realization belief about the asset k, the agent proceeds by forming a belief about the possible price behaviour of the other asset n. The agent is in a different (updated) belief state $|+_i\rangle$ and her state transition in respect to the price behaviour of asset n with eigenvalues $\beta = \pm 1$ is given by Born rule, with the transition probabilities:

$$p_{k\to n}(\alpha \to \beta) = |\langle \alpha_k | \beta_n \rangle|^2. \quad (6)$$

The eigenvalues correspond to the possible price realizations of the considered assets.

The above exposition of state transition allows to obtain the quantum transition probabilities that denote agents' beliefs in respect to the asset n price distribution, when she firstly observes the price realization of asset k. Transition probabilities have also an objective interpretation. Consider an ensemble of agents in the same state ψ, who made a decision α, with respect to the price behaviour of the kth asset. As a next step, the agents form preferences about the nth asset and we choose only those, whose firm decision is β. In this way it is possible to find the frequency-probability $p_{k\to n}(\alpha \to \beta)$. Following the classical tradition, we can

[i] Overoptimism triggered by trends is related to the fallacy of small numbers, as the agent infers the future return or dividend distribution from a limited sequence of past observations.

[j] In the simple setup with two types of discrete price movements, we fix only two eigenvectors $|\alpha_+\rangle$ and $|\alpha_-\rangle$, corresponding to eigenvalues $a = \pm 1$.

consider these quantum probabilities as analogues of the conditional probabilities, $p_{k\to n}(\alpha \to \beta) \equiv p_{n|k}(\beta|\alpha)$. We remark that the belief formation about asset prices in this setup takes place under informational ambiguity. Hence, in each of the subsequent belief states about the price behaviour the agent is in a superposition in respect price behaviour of the complementary asset, and interference effects exist for each agent's pure belief state (that can be approximated by a notion of a representative agent).

Given the probabilities, in (6) we can define a quantum joint probability distribution for forming beliefs about both of the two assets k and n.

$$p_{kn}(\alpha, \beta) = p_k(\alpha)p_{n|k}(\beta|\alpha). \quad (7)$$

This joint probability respects the order structure, as such:

$$p_{kn}(\alpha, \beta) \neq p_{nk}(\beta, \alpha), \quad (8)$$

Sequential information processing is a manifestation of order effects, or state dependence in belief formation that is not in accord with the classical Bayesian probability update, see e.g., [45], [62]. Order effect corresponds to a non-satisfaction of the joint probability distribution and brings a violation of the commutativity principle that is central to classical probability theory, [38].[k]

The obtained results with the QP formula can be also interpreted as subjective probabilities or an agent' degree of belief about the distribution of asset prices. As an example, the agent in the belief-state ψ considers two possibilities for the dynamics of the kth price. She speculates: suppose that kth asset would demonstrate the $\alpha(=\pm 1)$ behaviour. Under this assumption (which is a type of 'counter-factual' update of her state ψ), she forms her beliefs about a possible outcome for the nth asset price. Starting with the counter-factually updated state $|\alpha_k\rangle$, she generates subjective probabilities for the price outcomes of both of these assets. These probabilities give the conditional expectations of the asset n price value $\beta = \pm$, after observing price behaviour of asset k, with a price value $\alpha = \pm 1$.

To sum up, in the setting of narrow framing and sequential information processing, when ambiguity is present, QP frameworks aids to depict agents' non-definite opinions about the prices behaviour for the 'complementary assets' on her portfolio. Non-classical information processing can be boosted by a presence a vague probabilistic composition of the future price state realizations of the set of traded assets.

In the case of such assets, an agent forms her beliefs *sequentially*, and not jointly as is the case in the standard finance portfolio theory. She firstly resolves her uncertainty about the asset k, and only with this knowledge can she resolve the uncertainty about other assets (in our simple example the asset n.) The quantum probability belief formation scheme based on non-commuting asset price-observables can

[k] Agents can show order effects for: i) information processing related to the observation of some informational signals; ii) preference formation related to the sequence of asset price observation, or in respect to the actual asset trading behaviour. Non-commuting observables allow to depict agents' state dependence in preference formation. As noted, when state dependence is absent, the observable operators do commute.

be applied to describe subjective belief formation of a representative agent by exploring the 'bets', or price observations of an ensemble of agents and approximate the frequencies by probabilities, see similar studies on order effects, [12, 44, 22, 36, 24].

6 AMBIGUITY, HETEROGENEOUS BELIEFS AND SPECULATIVE BUBBLES

State dependence in beliefs and preferences that is modelled via non-commuting projectors can also explain the existence of periods, in which assets are overpriced. Instances of overpricing are known as speculative bubbles and are accompanied by excess trading volume and volatility of asset prices that is not predicted by the fundamentals. The main explanatory causes behind the deviation of the prices from their fundamental values that are explored in finance literature are due to agents' *shifts in risk aversion*, where staring with the works of [53], disposition effect coupled with myopia is attributed to cause an excessively high risk premium on stocks. These individual effects result in underpricing of assets in the present, followed by high price increase in the future, due to the increased risk premiums.[l] While disposition effect and myopia well explain the past returns, due to agents non-classical beliefs and shifts is risk attitude, the question remains open on why the prices are extremely high for some classes of assets for longer periods? These high prices that are evaluated as not being justified, in respect to the known fundamentals are alluded to as 'bubbles'. These bubbles are also speculative nature, since their roots are irrational from the point of view of traditional finance, and the only reasoning to trade in such as setting is due to 'betting' on an even higher future price, see an extensive treatment in [54], [56] and [58].

Another stream of financial literature focuses on the impact of *agents' divergence in beliefs upon price formation* coupled with informational ambiguity that surrounds the financial markets.[m] Investors can have a heterogeneous attitude towards ambiguity, and also, exhibit state dependent shifts in their attitude towards some types of uncertainties. For instance, 'ambiguity seeking' expectations that are manifest in an overweighting of uncertain probabilities can also take place under specific decision making states, see [47], and references herein.

[l] See foundational works focused on the impact of disposition effect by [9] and [42], followed by experimental studies in [40], [63] and a recent analysis in [5] and [25]. Myopia means that agents to not possess a joint probability distribution in the assessment of price outcomes. This bias acts as a catalyst for the effect of loss aversion, as discussed in the section, 5.

[m] We can emphasise that the complexity of risks and hence the ambiguity in respect to realization of economic states is increasing in the recent years. To give some concrete examples, we can allude to the Global Financial Crisis and its adverse impact upon market outcomes. The notion of an 'increasing uncertainty' associated with the impact of the Brexit Referendum in the UK and the forthcoming political and economic events are a frequent subject of the analysis in UK's news. Finally, the rise of the opaque asset classes, with limited or no history of past returns, such as crypto-currencies, contributes to the complexity of risks faced by today's investors.

A popular view is that some types of agents can be optimistic in some trading periods (i.e., overprice the assets in the next coming period), followed by shifts in their beliefs. This results in a high trading volume and overpricing of the risky assets, see works by [20], followed by [50] and [6] to mention few. Agents' beliefs can also switch, as a result of the observed asset prices, or realized returns, where an overreaction to negative outcomes can take place, [39]. In [25], the author arrives at similar conclusions, by showing that agents' beliefs are state dependent and follow price trends. This pattern in beliefs results in a trading behaviour that is in accord with a 'reverse disposition effect'. Traders are selling losing stocks and keeping the wining stocks for longer periods, thereby expecting an unjustified price growth from the previously well performing stocks. The findings by [5] also indicate that a reverse disposition effect among agents can take place, as they invest in previous winners and sell fast the losing stocks.

The works that build upon the heterogeneity of beliefs are taking as the benchmark classical probability based information processing, where the heterogeneity is due to the lack of information or some present 'noise' in agents' belief evolution about the fundamentals and their interconnection with asset values.

In the recent work by [36] the classical probability based Markov process of price dynamics was generalized, to include a more deep type of uncertainty about the asset prices. The price realizations in this QP model are resulting from agents' non-classical expectations, future fundamentals and the resulting prices. The agents are trading upon their beliefs, and the prices are affected by their actions. This process is modelled with the aid of a so called 'bath of agents' expectations, following a quantum Markovian dynamics. With a simulation of two asset price states the model shows that stationary equilibrium prices can be obtained in the long term, following the evolution of agents' non-classical beliefs. The distinguishing feature of the model is that the agents are not forming joint distribution beliefs about the future price outcomes with an implication to deviations of their valuation from the rational expectation equilibrium. A QP based framework with agents holding heterogeneous ambiguous beliefs is formalised in [37]. The agents can belong to two agents types, the optimists and the pessimists and their beliefs are state dependent. The agents trade in discrete time and their beliefs can switch as the observe signals on the realisation of fundamentals. The interference effects due to ambiguity about future prices and fundamentals correspond to agents degree of overoptimism or pessimism that results in overpricing of the risky asset in a trading periods, and price decrease in a subsequent trading period.

7 CONCLUDING REMARKS

We presented an introductory survey on the advances of QP based decision theory with some applications to the domain of behavioural finance, given the wide range of revealed behavioural anomalies. These anomalies are related with non-classical mode of information processing by investors and an existence of state dependence in their trading preferences.

The core premise QP based decision theoretic framework is that *non-commutativity of lottery observables* can give raise to agents' belief ambiguity in respect to the subjective probability evaluation, in a similar mode, as captured by the probability weighing function. Interference effects that are present in an agent's ambiguous comparison state, translate into over-, or under-weighting of objective probabilities associated with the riskiness of the lots. The interference term (more precisely the interference angle λ) and its size allows to quantify an agent's fear to obtain an undesirable outcome that is a part of her ambiguous comparison state. The agent compares the relative utilities of the lottery outcomes that are given by the eigenstates associated with the lottery specific orthonormal bases in the complex Hilbert space. This setup creates a lottery dependence of an agent's utility, where the lottery payoffs and probability composition play a role in her preference formation.

The main motivation for the application of QP mathematical framework as a mechanism of probability calculus under non-neutral ambiguity attitudes among agents coupled with a state dependence of their utility perception derived from its ability to generalise the rules of classical probability theory, and capture the indeterminacy state before a preference is formed through the notion a *superposition.*

QP has also a potential to serve as generalised framework of contextual belief formation and update in asset trading, due to its axiomatic completeness that is derived from the rules of quantum physical formalism.

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