MATHEMATICAL MODELS FOR DETERMINING THE RADIUS OF R-WAVE PROPAGATION DUE TO PILE DRIVING USING DIMENSIONAL ANALYSIS

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ABSTRACT

This paper represents a method of establishing a mathematical model for determining the radius of R-wave propagation during pile driving. It is necessary to point out the relation between the pile driving energy, duration of pile driving, soil medium, damping and other factors. Dimensiol Matrix Method has been used specifically to create two groups of dimensionless π-terms, i.e mathematical expressions regarding these variables or factors. Results indicates the radius of R-wave propagation: a) is proportioning to the square root of Energy; b) inversely varies to density of soil medium; and c) fastly decrease with distance to the power (-3/2) of distance. Power form of dimensionless products π-terms may be suitably studied to find out which variable affects most to predictor using sensitivity analysis.

Keywords: Dimensional analysis, R-wave, pile driving energy.

Introduction

Problems in engineering are necessary to be tackled for obtaining better understandings about what happens to a quantity (predictor or dependent variable) as other factors and parameters change or what the relationship between a quantity (normally called predictor) and other different factors or variables. A mathematical model is necessarily to create to govern the problems, i.e an investigation on behavior of the quantity under consideration in relationship with other variables and parameters is postulated. Based on prior knowledge and research, data or variables should be studied analytically or experimentally via other mathematical tools or by tests appropriately. In such case, for quantifying values of variables, parameters, coefficients and other related factors, it is *necessary to establish some information about the relationship between these variables and parameters*, so that some equations or experiments can be done to verify the theorem.

Dimension analysis is a useful method among many different approaches to check such the relations among parameters using their dimensions.

Besides, it is to determine which variable(s) or parameter(s) affects most on predictor and cause the greatest change for output results. Sensitivity Analysis then should be conducted to find out which variable(s) affect most to predictor or dependent variable.

The main idea for this article is to tentatively study how to connect the method of dimensional analysis and method of sensitivity analysis making use the fast divergence of power function. The procedure is to start gaining information for mathematical model by dimensional analysis first, then by analyzing sensitivity in order to point out the better information about how predictor varies as input data of variables change. Among popular specific problems in construction, determining the radius of Rayleigh wave propagation during pile driving is highly concentrated.

Problem statement and Procedure

During pile driving, energy is delivered to pile and makes the pile penetrates into soil. Most of the energy distributed to the soil around the shaft of pile and right under the pile tip makes the soil around pile vibrate.

This paper deals with determining the radius of wave propagation in terms of as many as possible parameters or factors related, including soil medium (density, physic - mechanical characteristics, damping…), energy of pile driving, and displacement or distance.

This is the advanced model as compared to previous paper published by author. In this study, variable relating to displacement is added to take the damping of medium into account. For a general procedure, energy of pile driving is simply studied to be energy, of which dimension is taken as that of work done, i.e force multiplied by distance.

Method

Dimensional analysis, developed by Buckingham in 1915, was chosen to be a simple tool for establishing mathematical relation between parameters of the problem under consideration.

The relationship between factors and physical parameters can be written as

R = Function of (para1, para2…, para *n*, factor1, factor2,…factor *m*)

Numerical values of para1, para2… are possible measurable; meanwhile factor1, factor 2… will be dimensionless. We say R is dependent variable or what we want to know its relation with the rest para1…para *n* and/or factors which are generally called independent variables or control variables. In dimensional analysis, para1 and others can be dependable mutually so that we can write:

[Constant] = $[\text{para1}]^a$ $[\text{para2}]^b$… $[\text{para } n]^i$

Whereas

[para1] stands for dimension of para1; a, b, …i are real number;

If expressed as another form of power function in terms of dimensionless parameters and variables which called π-terms as followed:

$$\pi = \frac{R}{para1^{m_1}\, para2^{m_2} ... parai^{m_i}}$$

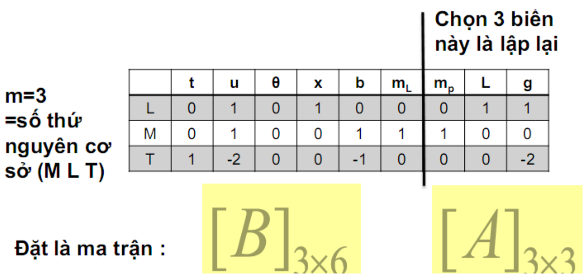
This is a dimensionless product through which we can come to another form of expression as follows:

$$\pi = \phi(\pi_1, \pi_2, ... \pi_{n-k})$$

Where n is the number of parameters and factors, and k is the number of dimension independent variables of both the n parameters and factors. We can understand hereinafter that k is the number of dimensions used.

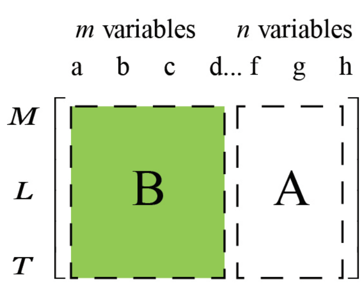
The correlation between dimensionless parameters and factors is valid regardless of the system of units because this is dimensionless correlation. The π-terms will give us information about the mathematical relationship between parameters and factors, so that we will be able to understand more about the dependent variable under consideration. Besides, using the method of dimensional analysis, many useful groups of information between parameters (variables) can be established in mathematical expressions (Abdelaziz, L. *et al.*, 2002). By the way, for the relationship between parameters is dimensionless, i.e independent of system of units, we can create physical model for testing, replacing and determining the values of this parameter in terms of other parameter(s) in a pilot plan for a similarity from dimensionless products of π numbers.

Dimension Matrix



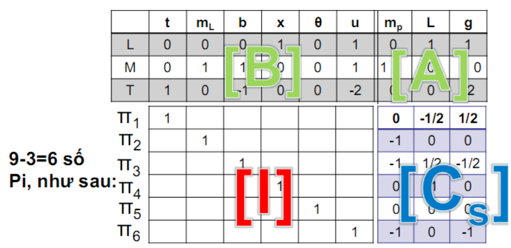
Method Rayleigh, which can be easily applied to create analytical expressions of power number gives result of less than four parameters, whilst Dimensional Matrix Method can be suitably used for models with more than 4 parameters for complicated models with more detailed information. The latter have proved to be very powerful in data processing using computer programming and unlimited number of variables. The method is as follows:

Variables including parameters with or without dimensions will be listed as many as possible. The more number of variables is, the better information about the relationship can be withdrawn. Dimensions of these parameters will be prescribed in matrice as follows:



M, L, T, are fundamental dimensions of Mass, Length, and Time respectively. B is matrix of independent variables and A is matrix of dependent variables. Independent variables can be measured to verify the variation of dependent variables. Meanwhile, dependent variables are commonly of geometrical attributes, material and external load or gravity (Bozida Stojadinovic, 2007).

For example, we consider what information to be provided to model related to displacement u, time *t*, mass *m* and span *L* …



Where independent variables are t, mass m_L, location x, size of structure b, temperature (for example) θ; dependent variables are m_p, L, gravity acceleration g. There are m=6 and n=3 out of 9 variables.

A notation is that even some constants or coefficient(s) may be inserted into dimensional matrix, in general it results in more complicate and troublesome matters and unwanted errors.

Matrix of unity I is $A^{-1}A$ and a diagonal one.

Matrix $C_S = (-A^{-1}B)^T$ will be the final results Matrix of Dimension has $m+n$ columns and 3 rows, hence there are m π-terms and C_S is a $m \times n$ matrix.

After finding out matrix C_S, we have m groups of informations as follows:

$$\pi_1 = a.f^{\alpha};\quad \pi_2 = \frac{b}{g^{\delta}};\ldots\quad \pi_m = d.h^{\gamma} \quad (1)$$

These m π-terms qualitatively stand for the model and because they are dimensionless, they can be powered, added, subtracted or multiplied altogether.

Some researchers preferred expressing them in power forms to other forms. The reason is that in power form they can be scaled. They are dimensionless products. They are so useful that many physical models in wide ranges of engineering fields and physical sciences were made and tested using similarity between real structures and model, originated from these dimensionless products.

Parameters taken into account for the problem

As described in all lectures of vibration, radius of wave propagation depends on time of propagation, the density of propagating medium and energy supplied for vibration exciting source. Damping characteristics is taken into account by decayed displacement of points in the surface of free domain vibration. So we have *at least 5* predictable parameters; those are

- Radius of wave propagation denoted as R (this is a dependent variable),
- Time of wave propagation, denoted as *t*
- Density of the wave propagation medium, denoted as ρ.
- Energy compensated for stirring the source, denoted as *E*.
- Displacement of a point on the surface of soil medium, denoted as *u*.

Results

As mentioned above, the problem under consideration has started with the minimum numbers of variables, they are:

- Dependent variable R: dimension is length [R]= [length]
- Time of vibration, with [t] = [time];
- Density ρ, its dimension is mass divided by volume, [ρ]=[Mass]/ [Length]^3
- Energy E, of which, [E]= [M] [Length]^2 / [time]^2
- Displacement of a point on the surface of soil medium, its dimension [*u*]= [Length]

So, dimension matrix is:

$$\begin{array}{c} \\ \mathrm{M} \\ \mathrm{L} \\ \mathrm{T} \end{array} \begin{array}{c} \begin{array}{ccccc} R & t & \rho & E & u \end{array} \\ \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & -3 & 2 & 1 \\ 0 & 1 & 0 & -2 & 0 \end{array}\right] \end{array} \xrightarrow{\quad} [\mathrm{A}]$$

The number of variables is m+n=5, as fundamental dimension is 3. So we predict that there is 5-3=2 dimensionless products (π- terms).

Let matrices A and B be:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -3 & 2 & 1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

After several algebraic calculations, with

$C_S = (-A^{-1}.B)^T$

we come to obtain two π- terms as follow:

$$\begin{array}{c|ccccc} & R & t & \rho & E & u \\ \hline M & 0 & 0 & 1 & 1 & 0 \\ L & 1 & 0 & -3 & 2 & 1 \\ T & 0 & 1 & 0 & -2 & 0 \\ \hline \pi_1 & 1 & 0 & 0 & 0 & -1 \\ \pi_2 & 0 & 1 & -0.5 & 0.5 & 2.5 \end{array}$$

$[A]_\times$ $[I]_{2\times 2}$ $[C_S]_\times$ (ii)

We can write:

$$\pi_1 = \frac{R}{u}$$
$$\pi_2 = \frac{t.\sqrt{E}}{\sqrt{\rho}} u^{-5/2} \qquad (2)$$

Based on basic characteristics of dimensionless products, and after carrying out some algebraic tricks of calculation over these π-terms, for doing to the power of these π-terms, add and or multiply or combine many of them, we come to an interesting expression (3). This is an expression dealing with the interrelation between all required quantities in the problem under consideration: Energy of pile driving, soil density of medium, displacement of a point on the surface of free domain vibration, time of wave propagation and the radius of propagation.

Because π-terms can be compared this one to other, so we have:

$$\pi_1 = const \times \pi_2$$

$$\Leftrightarrow \quad \frac{R}{u} \propto \frac{t.\sqrt{E}}{\sqrt{\rho}} u^{-5/2}$$

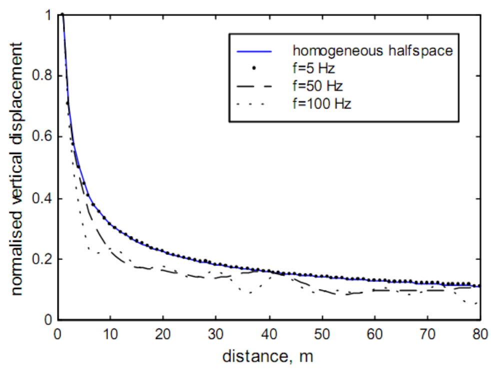
$$\Leftrightarrow \quad R \propto const \times \sqrt{\frac{E}{\rho}}.t.u^{-3/2} \qquad (3)$$

or we can empower pi-terms to come to more frequent formulae:

$$const \propto \frac{1}{\sqrt{R}} \sqrt{\frac{E}{\rho}}.t.u^{-2} \qquad (4)$$

We can compare (3) to result postulated by Richard *et al*, (1970):

Figure 1. Plot diagram indicates the inverse correlation between displacement and distance of wave spreading (Richard et al., 1970)



Note that (3) somehow looks like the same formula $v = 0.5\frac{E^{0.75}}{R}$ (5)

Experiments or site tests can be designed to validate the relationship (3), or to find out the values of some definite parameters from other measurable parameters or factors; results postulated by author (Tham DH, 1997) as following:

$$v = 6.351 + 1.462[E] - 2.14[R] \qquad (mm/s)$$

$$u = 147.873 + 52.906[E] - 58.196[R] \mu m$$

[R] =1,=5 when when distance equals to 10m, 40m respectively. For example pile driving energy E equals to 7500 kgm, ρ = 1500 kgf/m^3.

Velocity of free domain point

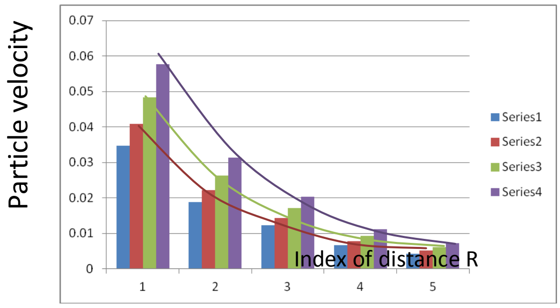
Pile driving Energy

R	1800	2500	3500	5000
10	13.817	17.678	22.752	29.73
15	9.2116	11.785	15.168	19.82
20	6.9087	8.8388	11.376	14.87
30	4.6058	5.8926	7.584	9.91
40	3.4543	4.4194	5.688	7.433

Results from this mathematical model:

R\E	1800	2500	3500	5000
10	0.0346	0.0408	0.0483	0.058
15	0.0189	0.0222	0.0263	0.031
20	0.0122	0.0144	0.0171	0.02
30	0.0067	0.0079	0.0093	0.011
40	0.0043	0.0051	0.006	0.007

Figure 2. Trend of decrease of the estimated velocity which was manipulated from (3) to increasing of distance at the definite time (series 1 is the lowest energy, series 3 is the highest)



Discussion

Some key points can be withdrawn from results of mathematical model built by dimensional analysis:

By replacing particle velocity of a surface point v = u/t, we have

$$v = const1.\sqrt{\frac{E}{\rho}.\frac{1}{R}}.u^{-0.5}$$

$$v = const2.\sqrt{\frac{E}{\rho}}.R^{-3/2}$$

This formula indicates a more general expression as compared to empirical formula described herefafter in code of practice "*Quytrinh dong coc trong vung xay chen (Bo Xay Dung, 1994)*":

$$v = 0.5\frac{E^{0.75}}{R}$$

- Through the mathematical expression of radius of R- wave propagation found in (3), it is easy to come to conclusion that: the more rigid the soil medium of propagation (high value of ρ), the smaller the radius of wave propagation, i.e R-wave will run farther (*not faster*) through soft soil (smaller value of density) than through harder soil. This result is properly relevant to the R-wave, which stores large part of total energy of pile driving (more than 67 percentage) and causes in more serious damages of construction works in its vicinity. Besides, the radius of wave propagation will be proportioned to squared root of energy supplied to vibration.

- The more complicated model in which damping characteristic of soil medium and frequency of source will be postulated by considering periods (its dimension is time) into dimension matrix.
- According to method, if we change the position of u and R in dimension matrix (i), the result will not change (ii), so we can predict that the amplitude of displacement will decrease to power (-3/2) of distance R. So we can conclude that at distance more than 10 meters from the pile driving site, the amplitude will be negligeable.
- It is possible to recognize from (3) that radius R depends on Energy, time (velocity of wave spreading), damping and density of soil medium. Site test using accelerometers to pick up displacement and time of wave propagation can be recorded to calculate the wavelength of R-wave.
- Power form of radius of wave propagation in terms of Energy imparted to piling E, density of soil medium ρ, displace u and time of pile driving *t* as in (3) is convenient for sensitivity analysis. At first glance, we can see that energy and density is sensitive to radius of wave spreading.
- Formulae (3) points out damping consequence on amplitude relevant as previously postulated by Richard et al (1970).
- Frequency and velocity of wave spreading was taken into account by time variable and distance variables (R and u).
- Results from this study may be developed referring previous researches to give a better understanding about the effect of pile driving to surrounding construction works.

Conclusion

Using dimensional analysis, many problems can be solved qualitatively and supply many useful information about the mathematical relationship between parameters involved. Results are dimensionless π-terms which are independent of system units.

Results taken by applying this method i.e dimensional analysis, indicate that at the same time of wave transferring, the more rigid the soil foundation, the smaller the radius of propagation, and it depends upon the energy supplied to the vibration; by the way, amplitude or vertical displacement at points on the surface of soil medium will decay exponentially, i.e very fast to power of (-3/2) of distance from the source of vibration. Power function of results is helpful to figure out an approach of sensitivity analysis.

Although we can upgrade this method with a more detailed relationship and many invisible factors can be taken into account, it is necessary to carry out further experimental and theoretical research to come closer to a more rigorous solution(s) for the problem under consideration.

This method proved to be an efficient tool for providing information about relationship between factors and parameters in many engineering problems, and served as an important step before bringing piling and construction activities in reality.

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