LIMIT ANALYSIS FOR 3-D STRUCTURES USING SECOND-ORDER CONE PROGRAMMING

***Tran Trung Dung*[1], *Le Van Canh*[2]**

[1]*Faculty of Construction & Electricity, HCMC Open University;*
e-mail: dung.ttrung@ou.edu.vn

[2]*Department of Civil Engineering, International University;*
e-mail: lvcanh@hcmiu.edu.vn

(Received: 23/02/2014; Revised: 02/04/2014; Accepted: 16/06/2014)

ABSTRACT

This paper extends a numerical procedure for limit ananlysis of 3-D structures using node-based smoothed finite element method (NS-FEM) in combination with second-order cone programming (SOCP). The obtained discretization formulation is then cast in a form which involves second-order cone constraints, ensuring that the underlying optimization problem can be solved by highly efficient primal-dual interior point algorithm. Furthermore, in the NS-FEM, the system stiffness matrix is computed using the smoothed strains over the smoothing domains associated with nodes. This ensures that the size of the resulting optimization problem is kept to a minimum. Moreover, it can alleviate volumetric locking for 3-D problem effectively. The efficiency of the present approach is illustrated by examing a benchmark example.

Keywords: *Limit analysis (LA), the node-based smoothed fem (NS-FEM), Second-order cone programming (SOCP).*

1. Introduction

Limit state criteria have been used to design and asses the safety of many engineering components and structures, from simple metal forming problems to large-scale engineering structures and nuclear power plants. A complete elasto-plastic analysis is generally quite complicated due to the need to specify initial stress conditions and to then carry out an analysis in an iterative manner. Difficulties in elasto-plastic analysis and its applications have motivated the development of a simplified direct method, limit analysis, which can be used to identify the collapse load (also known as the limit load, or load carrying capacity, or maximum load intensity) of a structural problem in a simple and more direct manner.

Current research in the field of limit analysis is focussing on the development of numerical tools which are sufficiently efficient and robust to be of use to engineers working in practice. Various numerical procedures for limit and shakedown analysis problems have been developed for decades. One of the most robust and popular discretisation methods is the finite element method (FEM). However, there are still many aspects which are in need of improvement, for instant locking problems, mesh distortion and highly sensitive to the geometry of the original mesh…

Moreover, in limit analysis procedures one must solve optimisation problems involving either linear or non-linear programming. When a non-linear yield condition is used, the resulting optimisation problem is non-linear, which presents major diffculties in the solution process. A traditional way of addressing the drawback is to linearise non-linear convex yield criteria, so that the resulting optimisation problem reduces to a linear program. Although this classical linear program can be solved effciently using Simplex **Error! Reference source not found.**, 2] or interior-point **Error! Reference source not found.**] algorithms, a large number of constraints generated in the linearisation process would be needed in order to provide accurate solutions (especially for three-dimensional problems), thereby increasing the computational cost. Attempts have also been made to solve problems involving exact convex yield function using non-linear programming packages. However, non-linear programming problems are often computationally expensive to solve, with the consequence that often only relatively small problems can be tackled.

To overcome these above shortcomings, the strain smoothing technique (the node-based SFEM) is used to to remove locking problems for 3D structures. Moreover the resulting optimization problem is cast in the form of a second-order cone programming problem so that a large-scale problem can be solved efficiently [Error! Reference source not found.**,** Error! Reference source not found.].

2. Kinematic formulation of limit analysis

Consider a rigid-perfectly plastic solid subject to body forces **F** in its volume **V** and surface tractions **f** on the free portion Γ_f of its boundary. The constrained boundary Γ_u is fixed. The basic values of external loads **F** and **f** are affected by a common multiplier α, and the value λ^+ of α for which collapse is attained (collapse multiplier) is sought. Let the vectors

$$\boldsymbol{\sigma} = \{\sigma_x \quad \sigma_y \quad \tau_{xy} \quad \sigma_z \quad \tau_{xz} \quad \tau_{yz}\}^t \tag{1a}$$

$$\dot{\boldsymbol{\varepsilon}} = \{\dot{\varepsilon}_x \quad \dot{\varepsilon}_y \quad \dot{\gamma}_{xy} \quad \dot{\varepsilon}_z \quad \dot{\gamma}_{xz} \quad \dot{\gamma}_{yz}\} \tag{1b}$$

$$\dot{\mathbf{u}} = \{\dot{u}_x \quad \dot{u}_y \quad \dot{u}_z\} \tag{1c}$$

collect the stress, strain rate and velocity components, respectively (for shear strains the engineering definition is adopted). The fields (1b) and (1c) are related by compatibility, which is symbolically written as

$$\dot{\boldsymbol{\varepsilon}} = \nabla_s \dot{\mathbf{u}} \quad in\ \mathbf{V}, \quad \dot{\mathbf{u}} = 0 \ on\ \Gamma_u \tag{2a,b}$$

The rigid-perfectly plastic assumption for the material implies that stresses are confined within the convex domain $f(\boldsymbol{\sigma}) \le 0$. If von Mises' criterion is considered, one has

$$f(s) = f(s) - s_0 \quad (3a)$$

$$f(\boldsymbol{\sigma}) = \frac{1}{\sqrt{2}}\left(\left(\sigma_x - \sigma_x\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + \left(\sigma_z - \sigma_x\right)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)\right)^{1/2} \text{ for 3D} \quad (3b)$$

wherer s_0 is the tensile yield stress. Deformations in the solid can only consist of plastic flow governed by the normality rule.

$$\dot{\boldsymbol{\varepsilon}} = \mu \frac{\partial \phi}{\partial \sigma}; \quad \mu \geq 0 \quad (4a,b)$$

The plastic multiplier μ can be positive only if the current stress point lies on the limit surface (q=0). For von Mises' materials, this establishes that the deformation process is *isochoric*, i.e. strain rates obey the condition

$$\dot{\varepsilon}_x + \dot{\varepsilon}_y + \dot{\varepsilon}_z = 0 \quad (5)$$

which can be written as

$$\boldsymbol{\chi}^t \dot{\boldsymbol{\varepsilon}} = 0; \qquad \boldsymbol{\chi} = \{1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0\}^t \quad (6)$$

Then the power of dissipation $\hat{D}(\dot{\varepsilon})$ can be formulated as a function of strain rates as [6]

$$\hat{D} = \frac{\sigma_0}{\sqrt{3}}\left[2\left(\dot{\varepsilon}_x^2 + \dot{\varepsilon}_y^2 + \dot{\varepsilon}_z^2\right) + \dot{\gamma}_{xy}^2 + \dot{\gamma}_{yz}^2 + \dot{\gamma}_{zx}^2\right]^{\frac{1}{2}} \qquad \text{for 3D} \quad (7a)$$

The kinematic theorem of limit analysis can now be cast in the well known format

$$\lambda^+ = \min \int_\Omega \hat{D}(\dot{\varepsilon}) d\Omega \quad (8a)$$

subject to

$$\dot{\boldsymbol{\varepsilon}} = \nabla_s \dot{\mathbf{u}} \quad \text{in } \mathbf{V}, \quad \dot{\mathbf{u}} = 0 \text{ on } \Gamma_u \quad (8b)$$

$$\boldsymbol{\chi}^t \dot{\boldsymbol{\varepsilon}} = 0 \quad \text{in } \mathbf{V} \quad (8c)$$

$$\int_V \mathbf{F}^t \dot{\mathbf{u}} dV + \int_{\Gamma_f} \mathbf{f}^t \dot{\mathbf{u}} d\Gamma = 1 \quad (8d)$$

The problem (8) is a convenient basis for finite element computations. Its objective function (8a) is convex; the compatibility conditions are accounted for automatically by a displacement model; the incompress-ibility constraint (8c) is linear and can be eliminated, thus reducing the number of free variables; Eq (8d) is easily dealt with by introducing a single Lagrangean multiplier, so that the problem is brought to the search of the minimum of an unconstrained convex function.

3. A brief on the formulation of NS-FEM

In NS-FEM, using the mesh of

elements we further discretize the problem domain into smoothing domains based on nodes of the elements such that $\Omega \approx \sum_{k=1}^{N_n} \Omega^{(k)}$ and $\Omega^{(i)} \cap \Omega^{(j)} = \varnothing$, $i \neq j$, in which N_n is the total number of nodes of all elements in the entire problem domain. Moreover, NS-FEM shape functions are identical to those in the FEM. However, instead of using compatible strains, the NS-FEM uses strains smoothed over local smoothing domains. These local smoothing domains are constructed based on nodes of elements as shown in **Error! Reference source not found.**. A strain smoothing formulation is now defined by the following operation:

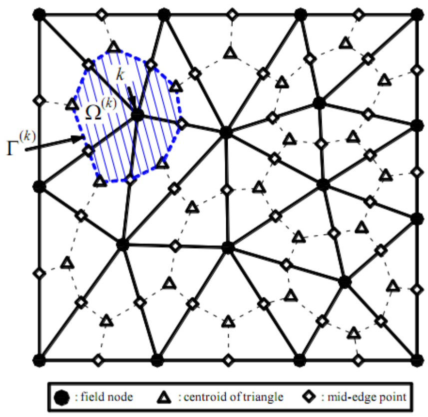

Figure 1. Three-node triangular mesh and smoothing domains

$$\bar{\boldsymbol{\varepsilon}}_k = \int_{\Omega^{(k)}} \boldsymbol{\varepsilon}^h(\mathbf{x})\Phi_k(\mathbf{x})d\Omega = \int_{\Omega^{(k)}} \nabla_s \mathbf{u}^h(\mathbf{x})\Phi_k(\mathbf{x})d\Omega \quad (9)$$

where $\Phi(\mathbf{x})$ is a given smoothing function that satisfies at least unity property

$$\int_{\Omega^{(k)}} \Phi(\mathbf{x})d\Omega = 1 \quad (10)$$

and in this work $\Phi(\mathbf{x})$ is assumed to be a step function given by

$$\Phi(\mathbf{x}) = \begin{cases} 1/V^{(k)} & \mathbf{x} \in \Omega^{(k)} \\ 0 & \mathbf{x} \notin \Omega^{(k)} \end{cases} \quad (11)$$

where $V^{(k)}$ is the 'volume' of the smoothing domain $\Omega^{(k)}$ and is calculated by $V^{(k)} = \int_{\Omega^{(k)}} d\Omega = \frac{1}{4}\sum_{j=1}^{N_e^{(k)}} V_e^{(j)}$ in which $N_e^{(k)}$ is the number of elements connected to the node k *and* $V_e^{(j)}$ is the volume of the j^{th} element around the node k.

In term of nodal displacement vectors d_I, the smoothing strains $\bar{\boldsymbol{\varepsilon}}_k$ can be written as

$$\bar{\boldsymbol{\varepsilon}}_k = \sum_{I \in N_n^{(k)}} \bar{\mathbf{B}}_I(x_k)\mathbf{d}_I \quad (12)$$

where $N_n^{(k)}$ is the number of nodes that are directly connected to node *k*, and $\bar{\mathbf{B}}_I\left(x_k\right)$ is the smoothed strain-displacement matrix on $\Omega^{(k)}$ the domain which is calculated numerically by an assembly process similarly as in the standard FEM

$$\bar{\mathbf{B}}_I\left(x_k\right)=\frac{1}{V^{(k)}}\sum_{j=1}^{N_e^{(k)}}\frac{1}{4}V_e^{(j)}\mathbf{B}_j^e \tag{13}$$

in which matrix $\mathbf{B}_j^e=\sum_{I\in S_e^j}\mathrm{B}_I$ is the compatible strain-displacement matrix for the j^{th} element around the node *k*. It is assembled from the compatible strain-displacement matrices $\mathbf{B}_I(\mathbf{x})$ of nodes in the set S_e^j which contains *nnel* nodes of the j^{th} linear element. Since linear shape functions are used, the entries of $\mathbf{B}_j^e$ are constants and therefore of $\bar{\mathbf{B}}_I\left(x_k\right)$ are also constants.

The smoothed domain stiffness matrix is then calculated by

$$\bar{\mathbf{K}}^{(k)}=\int_{\Omega^{(k)}}\bar{\mathbf{B}}_I^T\mathbf{C}\bar{\mathbf{B}}_I d\Omega=V^{(k)}\bar{\mathbf{B}}_I^T\mathbf{C}\bar{\mathbf{B}}_I \tag{14}$$

where **C** is the matrix of material constants, note that due to the smoothed strains $\bar{\boldsymbol{\varepsilon}}_k$ in Eq. (**Error! Reference source not found.**) are constants, the stresses $\bar{\boldsymbol{\sigma}}_k=\mathbf{C}\bar{\boldsymbol{\varepsilon}}_k$ are also constants in the smoothing domain $\Omega^{(k)}$.

4. Solution procedure with second order cone programming

If the von Mises failure criterion is employed, the plastic dissipation, i.e. the objective function can now be written in the form:

$$\lambda^+=\sum_{k=1}^{N_n}V^{(k)}\sigma_0\left\|\boldsymbol{\rho}_i^k\right\| \tag{15}$$

where $\left\|\boldsymbol{\rho}_k\right\|$ are additional variables defined by

$$\boldsymbol{\rho}_k=\begin{bmatrix}\rho_1\\\rho_2\\\rho_3\\\rho_4\\\rho_5\\\rho_6\end{bmatrix}=\frac{1}{\sqrt{3}}\begin{bmatrix}2&0&0&0&0&0\\0&2&0&0&0&0\\0&0&2&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\dot{\boldsymbol{\varepsilon}}_k \tag{16}$$

Introducing auxiliary variables t_1, t_2, . . . , t_{N_n} , the optimization problem becomes:

$$\min \qquad \lambda^{+} = \sum_{k=1}^{N_n} V^{(k)} \sigma_0 t_k \tag{17a}$$

$$\text{s.t.} \qquad \begin{cases} \dot{\mathbf{u}} = 0 & \textbf{on } \boldsymbol{\Gamma}_{\mathbf{u}} & (17b) \\ \dot{\bar{\boldsymbol{\varepsilon}}}_k = \bar{\mathbf{B}}_k \dot{\mathbf{u}} & \forall k = \overline{1, N^n}\ , & (17c) \\ \mathbf{D}_v \dot{\bar{\boldsymbol{\varepsilon}}}_k = 0 & & (17d) \\ F(\dot{\mathbf{u}}) = 1 & & (17e) \\ \|\boldsymbol{\rho}_k\| \le t_k & k = 1, 2, \ldots, N^n & (17f) \end{cases}$$

The third constraint, Eq.(17d), ensures that the incompressibility condition must be satisfied on all smoothing domains $\Omega^{(k)}$ and $\mathbf{D}_v$ has the form

$$\mathbf{D}_v = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{\bar{\boldsymbol{\varepsilon}}}_k \tag{18}$$

5. Numerical examples

In this section, the performance of the proposed solution procedure is illustrated via a benchmark problem in which analytical and other numerical solutions are available.

The example deals with a square plate with a central circular hole with constant modulus of elasticity and thickness under independently varying pressure loads p_1 and p_2 as in **Error! Reference source not found.**(a). The limit load factor was obtained analytically by Gaydon and McCrum **Error! Reference source not found.**] using plane stress hypothesis and von Mises yield criterion. Numerical limit analyses were also investigated by some authors, e.g. Garcea et al. **Error! Reference source not found.**] for the case of *D/L*=0.2 and Heitzer **Error! Reference source not found.**], Vu **Error! Reference source not found.**], for different ratios of *D/L* to evaluate the elastic–plastic behaviour of the structure. Moreover, H. Nguyen-Xuan **Error! Reference source not found.**] has applied the NS-FEM to limit analysis problems of 3D structures using Koiter's theorem, in which fictitious elastic stresses are assumed. However, in this paper the NS-FEM is formulated associated with Markov's kinematic theorem, and the resulting optimization problem is cast in the form of a second-order cone programming problem so that a large-scale problem can be solved efficiently **Error! Reference source not found.**]. Owing to its symmetry, only the upper-right quarter of the plate is modeled, see **Error! Reference source not found.**(b). Symmetry conditions are enforced on the left and bottom edges.

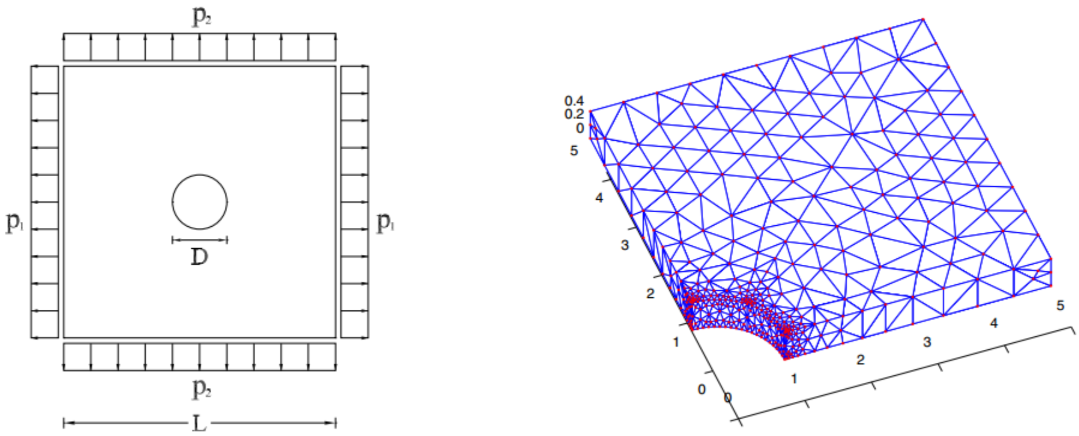

Figure 2. A square plate with a circular hole:
(a) geometry and loading, (b) finite element mesh

The procedure is applied to the case of *D/L*=0.2.**!Unexpected End of Formula** compares the best solutions obtained using the present method with solutions obtained previously by different limit analysis approaches (kinematic or static) using other FEM and meshfree models for case. It can be seen that the NS-FEM solutions agree well with published ones. Specially, NS-FEM can effectively alleviate volumetric locking for 3D problem.

Table 1. Collapse load multiplier with different loading cases and compared with previously obtained solutions *D/L*=0.2

Approach	Authors	Loading cases		
		$p_2 = p_1$	$p_2 = p_1/2$	$p_2 = 0$
Kinematic (upper bound)	da Silva and Antao **Error! Reference source not found.**]	0.899	0.915	0.807
	Le et al. **Error! Reference source not found.**]	0.895	0.911	0.801
	ES-FEM **Error! Reference source not found.**]	0.896	0.911	0.801
	NS-FEM-T3	0.894	0.911	0.802
	NS-FEM-T4	0.893	0.917	0.807
Mixed formulation	Zouain et al. **Error! Reference source not found.**]	0.894	0.911	0.803
Analytical solution	Gaydon and McCrum **Error! Reference source not found.**]	–	–	0.800
Static (lower bound)	Chen et al. **Error! Reference source not found.**]	0.874	0.899	0.798
	Gross-Weege **Error! Reference source not found.**]	0.882	0.891	0.782
	Belytschko **Error! Reference**	–	–	0.780

	source not found.]			
	Nguyen-Dang and Palgen **Error! Reference source not found.**]	0.704	–	0.564

6. Conclusion

In the paper, NS-FEM associated with second-order cone programming has been further investigated in efforts to provide more robust and effcient procedures and overcome drawbacks when applying to limit analysis. The extension of the NS-FEM formulation to the limit and of 3D problems is straightforward. It can alleviate volumetric locking for 3D problem effectively. In addition, limitation of large-scale problems in engineering practice can no longer difficult because of associating with the MOSEK software package.

REFERENCES

Anderheggen, E. & Knopfel, H 1972, ‘Finite element limit analysis using linear programming’, *International Journal of Solids and Structures,* 8 1413–1431.

Andersen, K. D. & Christiansen, E 1995, ‘Limit analysis with the dual affine scaling algorithm’, *Journal of Computational and Applied Mathematics,* 59 233–243.

C. V. Le, H. Nguyen-Xuan, H. Askes, S. Bordas, T. Rabczuk and H. Nguyen-Vinh. 2010, ‘A cell-based smoothed nite element method for kinematic limit analysis’, *International Journal for Numerical Methods in Engineering*, 83:1651-1674

Christiansen, E 1981, ‘Computation of limit load’, *International Journal for Numerical Methods in Engineering,* 17 1547–1570.

Garcea G, Armentano G, Petrolo S, Casciaro R 2005, ‘Finite element shakedown analysis of two-dimensional structures’, *International Journal for Numerical Methods in Engineering, 63*:1174–1202.

Gaydon FA, McCrum AW 2005, ‘A theoretical investigation of the yield point loading of a square plate with a central circular hole’. *Journal of the Mechanics and Physics of Solids 1954,* 2:156–169.

H. Nguyen-Dang and L. Palgen 1979, *Shakedown analysis by displacement method and Equilibrium finite elements,* page L3/3. Proceedings of SMIRT-5, Berlin.

H. Nguyen-Xuan, T. Rabczuk, T. Nguyen-Thoi ,T.N.Tran and N. Nguyen-Thanh 2012, ‘Computation of limit and shakedown loads using a node-based smoothed finite element method’, *International journal for numerical methods in engineering*, 90:287–310

Heitzer M. Traglast- und Einspielanalyse zur Bewertung der Sicherheit passiver Komponenten. 1999, *Berichte des Forschungszentrums Julich*, Jul-3704, Dissertation, RWTH Aachen, Germany

J. Gross-Weege 1997, ‘On the numerical assessment of the safety factor of elasto-plastic structures under variable loading’, *International Journal of Mechanical Sciences,*39:417-433.

Le CV, Askes H, and Gilbert M 2012, ‘A locking-free stabilized kinematic EFG model for plane strain limit analysis’, *Computers and Structures*, 106-107:1–8.

Le CV, Gilbert M, and Askes H 2009, ‘Limit analysis of plates using the EFG method and second-order cone programming’, *International Journal for Numerical Methods in Engineering*, 78:1532–1552.

M. Vicente da Silva and A. N. Antao 2007, ‘A non-linear programming method approach for upper bound limit analysis’, *International Journal for Numerical Methods in*

Engineering, 72:1192-1218.

N. Zouain, L. Borges, J. L. Silveira 2002, '*An algorithm for shakedown analysis with nonlinear yield functions', Comput Methods Appl. Mech. Engrg*, 191, 2463–2481.

Riccardo Aceti, Antonio Capsoni, Leone Corradi 2005, 'A 3D finite element with planar symmetry for limit analysis computations', *Comput. Methods Appl. Mech. Engrg*, 194: 1823–1838.

S. Chen, Y. Liu, and Z. Cen 2008, 'Lower-bound limit analysis by using the EFG method and non-linear programming', *International Journal for Numerical Methods in Engineering*, 74:391-415.

T. Belytschko 1972, Plane stress shakedown analysis by nite elements. *International Journal of Mechanical Sciences*,14:619-625.

T. N. Tran, G. R. Liu, H. Nguyen-Xuan and T. Nguyen-Thoi 2010, 'An edge-based smoothed finite element method for primal-dual shakedown analysis of structures', *International Journal for Numerical Methods in Engineering*, 82:917–938.

Vu DK 2001, *Dual limit and shakedown analysis of structures*. Dissertation, Universit´ edeLiège, Belgium.