# A STABILIZED EQUILIBRIUM-BASED EFG MODEL FOR COMPUTATION OF COLLAPSE LOAD

*by Le Van Canh*[1]

**ABSTRACT**

*An equilibrium Element-Free Galerkin (EFG) based formulation for limit analysis of rigid-perfectly plastic plane strain problems is presented. In the formulation pure stress fields are approximated using a moving least squares technique, and a stabilized conforming nodal integration scheme is used in combination with the collocation method, ensuring that the equilibrium equations only need to be fulfilled at the nodes and instability problems can be eliminated. The von Mises yield criterion is enforced by introducing second-order cone constraints, ensuring that the resulting optimization problem can be solved using efficient interior-point solvers. Finally, the efficacy of the procedure is demonstrated by applying it to a benchmark Prandtl problem.*

**Keywords:** Limit analysis, meshless methods, EFG, equilibrium model, second-order cone programming.

## 1. Introduction

The load required to cause collapse of a body or structure can be estimated using lower bound theorem. In FEM-based numerical lower-bound limit analysis problem, a statically admissible stress for an individual element is chosen so that equilibrium equations and stress continuity requirements within the element and along its boundaries are met. This results in difficulties in building stress-based elements, and consequently stress-based elements are not popular compared with displacement - based elements. However, in this paper we will show that when a moving least squares approximation is used to construct stress fields, the field obtained is smooth over the entire problem domain. There is therefore no need to enforce continuity conditions at interfaces within the problem domain, which would be a key part of a comparable finite element formulation.

The equilibrium equations are frequently treated in one of two ways in numerical procedures: (i) equilibrium is enforced at nodes in the problem domain and also at boundaries (using the 'collocation' method), or (ii) the equilibrium equations are transformed into the equivalent weak-form (involving integrals), using the so-called 'weighted residual method' [1, 2]. The former method is simple and fast, but it has been reported to suffer from numerical stability problems [2, 3]. In contrast, formulations which use the weak-form can usually produce a stable set of discretized system equations, in turn leading to accurate solutions. Finite element based formulations have been developed by several authors [4, 5]. Considering meshfree methods, an equilibrium model for elastostatic problems was first introduced in [6], where stress fields were expressed by means of an Airy stress function, approximated using the moving least squares method. However, in this paper an alternative EFG equilibrium formulation in which the collocation method is used in combination with a smoothing technique is proposed.

## 2. Static Limit Analysis Formulation

A lower-bound solution to the problem involving a rigid-perfectly plastic body can

[1]*Department of Civil Engineering, International University, VNU HCMC, Vietnam. Email: lvcanh@hcmiu.edu.vn*

be obtained by using the static theorem of plasticity, which states that a stress field is statically and plastically admissible if (i) equilibrium and boundary conditions are fully satisfied, and (ii) the yield condition is not violated anywhere. The exact plastic collapse load multiplier, $\lambda_p$, is the largest value among a set of lower bound multipliers, $\lambda^-$, corresponding to any statically and plastically admissible stress distribution. The stress field is denoted as $\sigma=[\sigma_{xx}\ \sigma_{yy}\ \sigma_{xy}]^T$ and is constrained to belong to the domain

$$\mathrm{B} = \{\sigma | \psi(\sigma) \leq 0\}, \quad (1)$$

in which the so-called yield function $\psi(\sigma)$ is convex.

For plane strain problems, the von Mises failure criterion is expressed as

$$\psi(\sigma) = \sqrt{\frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2} - \sigma_0 \quad (2)$$

where $\sigma_0$ is the yield stress.

Finally, the lower-bound limit analysis of plane problems can be expressed in the form of a mathematical programming problem, as

$$\lambda^- = \max \lambda \quad (3)$$

$$s.t \quad \nabla\sigma = 0 \quad (4)$$

$$\sigma \in \mathrm{B} \quad (5)$$

where $\lambda^-$ is the numerically computed load multiplier, $\nabla = \left\{ \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \right\}$, and the stress field σ must also satisfy appropriate boundary conditions.

## 3. The EFG Equilibrium Model

### *3.1. Moving Least Squares Approximation*

Whereas in the kinematic formulation the displacement field is approximated, here the stress field needs to be approximated. In [6] the stress fields were expressed by means of an Airy stress function approximated using the moving least squares method. However, here these stress fields can be approximated directly by

$$\sigma^h(\mathbf{x}) = \begin{bmatrix} \sigma_{xx}^h \\ \sigma_{yy}^h \\ \sigma_{xy}^h \end{bmatrix} = \sum_{I=1}^{n} \Phi_I(\mathbf{x}) \begin{bmatrix} \sigma_{xxI} \\ \sigma_{yyI} \\ \sigma_{xyI} \end{bmatrix} \quad (6)$$

in which

$$\Phi_I(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}_I(\mathbf{x}) \quad (7)$$

$$\mathbf{A}(\mathbf{x}) = \sum_{I=1}^{n} w_I(\mathbf{x})\mathbf{p}^T(\mathbf{x}_I)\mathbf{p}(\mathbf{x}_I) \quad (8)$$

$$\mathbf{B}_I(\mathbf{x}) = w_I(\mathbf{x})\mathbf{p}(\mathbf{x}_I) \quad (9)$$

where $n$ is the number of nodes; **p(x)** is a set of basis functions; $w_I(\mathrm{x})$ is a weight function associated with node $I$. In this work, an isotropic quartic spline function is used, which is given by

$$w_I(\mathbf{x}) = \begin{cases} 1 - 6s^2 + 8s^3 - 3s^4 & if\ s \leq 1 \\ 0 & if\ s > 1 \end{cases} \quad (10)$$

with $s = \frac{\|\mathbf{x} - \mathbf{x}_I\|}{R_I}$, where $R_I$ is the support radius of node $I$ and determined by

$$R_I = \beta . h_I \quad (11)$$

where $\beta$ is the dimensionless size of influence domain and $h_I$ is the nodal spacing when nodes were distributed regularly, or the maximum distance to neighbouring nodes when nodes were distributed irregularly, see [7] for details.

### *3.2. Stabilized Equilibrium Equation*

A stabilized conforming nodal integration (SCNI) scheme [8] will be adapted in order to stabilize problems involving stress derivatives as follows

$$\tilde{\sigma}_{\alpha\beta,\alpha}^h(\mathbf{x}_j) = \frac{1}{a_j}\int_{\Omega_j} \sigma_{\alpha\beta,\alpha}^h(\mathbf{x}) \mathrm{d}\Omega$$

$$= \frac{1}{a_j}\int_{\Gamma_j}\left(\sigma^h_{\alpha\beta}(\mathbf{x})n_\alpha(\mathbf{x})\right)\mathrm{d}\Gamma \tag{12}$$

where $\tilde{\sigma}^h_{\alpha\beta,\alpha}$ is the smoothed value of the first-derivative of stress $\sigma^h_{\alpha\beta,\alpha}$ at node $j$; $\Gamma_j$ and $a_j$ are the boundary and area of the representative domain of node $j$.

Now introducing a moving least squares approximation of the stress fields, the smooth version of the stress first-derivative can be expressed as

$$\tilde{\sigma}^h_{\alpha\beta,\alpha}(\mathbf{x}_j) = \sum_{I=1}^{n}\tilde{\Phi}_{I,\alpha}(\mathbf{x}_j)\sigma_{\alpha\beta I} \tag{13}$$

with

$$\tilde{\Phi}_{I,\alpha}(\mathbf{x}_j) = \frac{1}{a_j}\int_{\Gamma_j}\Phi_I(\mathbf{x}_j)n_\alpha(\mathbf{x})\mathrm{d}\Gamma \tag{14}$$

where $\tilde{\Phi}$ is the smoothed version of $\Phi$, which can be determined using the technique presented in [9]

With the use of the smoothed value $\tilde{\sigma}^h_{\alpha\beta,\alpha}$ the equilibrium equation can be enforced at $n$ nodes, and Equation  can be rewritten as

$$\mathbf{A}_1\sigma_1 + \mathbf{A}_2\sigma_3 = \mathbf{0} \tag{15}$$

$$\mathbf{A}_1\sigma_3 + \mathbf{A}_2\sigma_2 = \mathbf{0} \tag{16}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \tilde{\Phi}_{1,x}(\mathbf{x}_j) & \tilde{\Phi}_{2,x}(\mathbf{x}_j) & \cdots & \tilde{\Phi}_{n,x}(\mathbf{x}_j) \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n\times n} \tag{17}$$

$$\mathbf{A}_2 = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \tilde{\Phi}_{1,y}(\mathbf{x}_j) & \tilde{\Phi}_{2,y}(\mathbf{x}_j) & \cdots & \tilde{\Phi}_{n,y}(\mathbf{x}_j) \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n\times n} \tag{18}$$

$$\sigma_1 = \begin{bmatrix} \sigma_{xx1} & \cdots & \sigma_{xxn} \end{bmatrix}^T \tag{19}$$

$$\sigma_2 = \begin{bmatrix} \sigma_{yy1} & \cdots & \sigma_{yyn} \end{bmatrix}^T \tag{20}$$

$$\sigma_3 = \begin{bmatrix} \sigma_{xy1} & \cdots & \sigma_{xyn} \end{bmatrix}^T \tag{21}$$

## 4. Solution of The Discrete Problem

### *4.1. Second-Order Cone Constraints*

In this section, the von Mises criterion, Equation , will be formulated in the form of a standard second-order cone

$$\mathsf{L} = \left\{ x \in \mathrm{R}^m \mid x_1 \geq \sqrt{\sum_{j=2}^{m} x_j^2} = \left\| x_{2\to m} \right\|_{L^2} \right\} \tag{22}$$

If introducing a vector of additional variables $\rho$ as

$$\rho = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} = \begin{bmatrix} \sigma_0 \\ \frac{1}{2}(\sigma_{xx} - \sigma_{yy}) \\ \sigma_{xy} \end{bmatrix} \tag{23}$$

Equation  can be rewritten as

$$\mathsf{B} \equiv \mathsf{L} = \left\{ \rho \in \mathrm{R}^3 \mid \rho_1 \geq \left\| \rho_{2\to 3} \right\|_{L^2} = \sqrt{\rho_2^2 + \rho_3^2} \right\} \tag{24}$$

where $\mathsf{L}$ is the three-dimensional quadratic cone.

### *4.2. Limit Analysis Formulation*

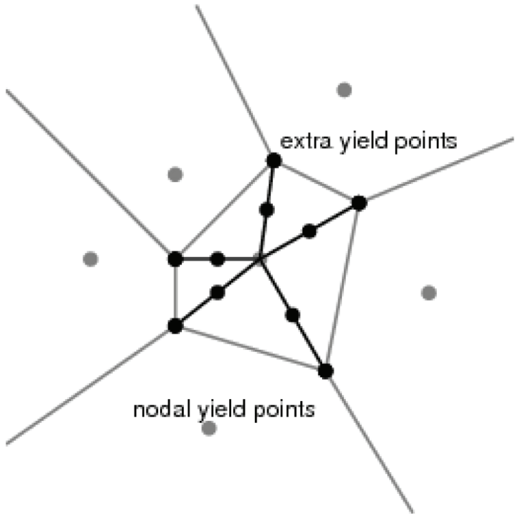
The limit analysis formulation can now be expressed in the form of a standard second-order cone programming problem as

$$\lambda^- = \max \lambda$$
$$s.t \begin{cases} \mathbf{A}_1\sigma_1 + \mathbf{A}_2\sigma_3 = \mathbf{0} \\ \mathbf{A}_1\sigma_3 + \mathbf{A}_2\sigma_2 = \mathbf{0} \\ \rho^k \in \mathsf{L}^k,\ k = 1, 2, \ldots, np \end{cases} \tag{25}$$

where $np$ is the number of yield points.

Using the existing Voronoi cell geometry, the yield condition can conveniently be enforced at vertex points within Voronoi cells, as well as at nodes, as indicated in Figure 1.

**Figure 1: Locations of yield points (at nodes and elsewhere within Voronoi cells)**

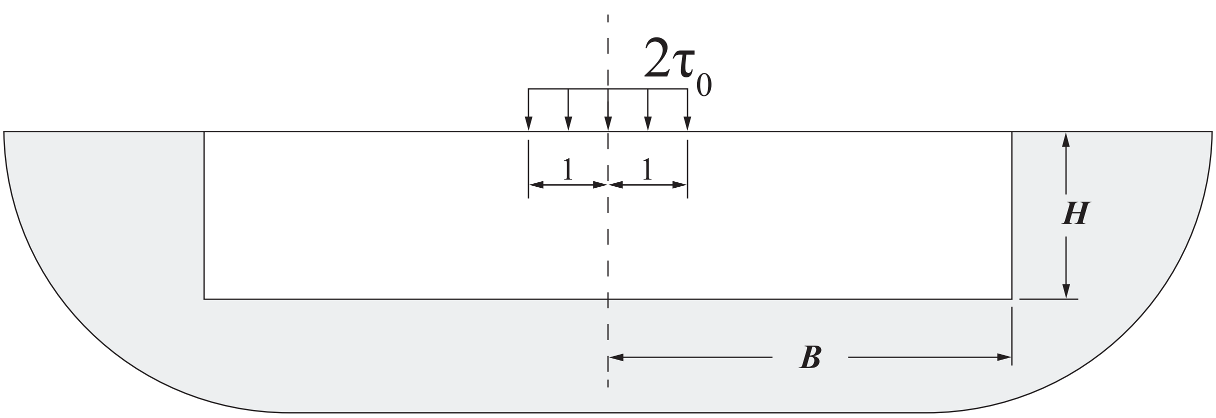



It should be emphasized that the collapse multiplier $\lambda^-$ determined using the described procedure is not guaranteed to represent a strict lower-bound on the exact value. This is because the smoothed moment derivative field may not fully satisfy equilibrium conditions everywhere in the domain, and because the yield condition is only enforced at a limited number of points. However, as the numerical discretization becomes increasingly fine one can expect to achieve an increasingly reliable approximation of the actual collapse load multiplier.

## 5. Numerical Examples

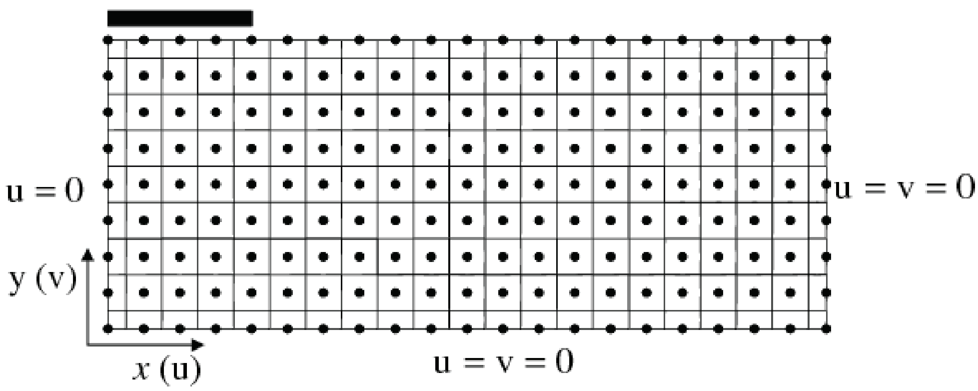
The performance of the limit analysis procedure described will now be tested by examining the classical plane strain problem, as shown in Figure 2. For a load of $2\tau_0$, the analytical collapse multiplier is $\lambda = 2 + \pi = 5.142$. The strong discontinuity at the edge of the indentor presents a severe challenge to many numerical analysis procedures. Problems were setup using MATLAB and the Mosek version 5.0 optimization solver was used to obtain all solutions.

**Figure 2: Prandtl problem: geometry and loading**



Due to symmetry, only half the domain needs to be considered. A rectangular region of dimensions *B=5* and *H=2* was used and the indentor (or 'punch') was represented by a uniform vertical load. Finally, appropriate boundary conditions were imposed, all as indicated in Figure 3.

**Figure 3: Prandtl problem: nodal layout, Voronoi cells and displacement boundary conditions**




Collapse multipliers and associated errors for various meshes are shown in Table 1. It can be observed that the solutions obtained using the present method show a very good accuracy when compared with the analytical solution. For all meshes, the relative errors to the exact solution are smaller than 1%. Furthermore, all the solutions obtained are below the exact value. This indicates that the presented procedure is capable of producing lower bound on the actual collapse load multiplier.

**Table 1: The punch problem: collapse multiplier**

| Models | Number of nodes | | | |
|---|---|---|---|---|
| | 189 | 697 | 2673 | 5929 |
| Proposed method | 5.1123 | 5.1192 | 5.1312 | 5.1372 |
| Error (%) | -0.58 | -0.44 | -0.21 | -0.09 |

Table 2 compares solutions obtained using the present method with upper and lower bound solutions that have previously been reported in the literature. The present results are proved to be competitive with those obtained by other methods, despite the fact that the number of nodes used in the present method is relatively small, compared with other methods.

**Table 2: Collapse load multiplier compared with previously obtained solutions**

| Approach | Authors | Collapse multiplier | Error (%) |
|---|---|---|---|
| Kinematic | Vicente da Silva and Antao [10] | 5.264 | +2.37 |
| | Sloan & Kleeman [11] | 5.210 | +1.32 |
| | Makrodimopoulos & Martin [12] | 5.148 | +0.12 |
| | Le et al. [13] | 5.143 | +0.02 |

| | | | |
|---|---|---|---|
| Mixed formulation | Capsoni & Corradi [14] | 5.240 | +1.90 |
| Analytical solution | Prandtl [15] | 5.142 | -- |
| Static | Present method | 5.137 | -0.09 |
| | Makrodimopoulos & Martin [16] | 5.141 | -0.02 |
| | Tin-Loi and Ngo [17] | 5.173 | +0.60 |

## 6. Conclusions

An equilibrium EFG-based model for limit analysis of plane strain problems has been proposed. This uses a moving least squares approximation of the moment field, which means that the resulting field is smooth over the entire problem domain. The collocation method is used in combination with the stabilized conforming nodal integration (SCNI) scheme to ensure that equilibrium needs only to be enforced at nodes. The von Mises yield criterion is formulated as second-order cones so that the underlying limit analysis problem becomes a standard second-order cone programming problem, which can be solved efficiently using primal-dual interior point solvers.

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## REFERENCES

1. O. C. Zienkiewicz and R. L. Taylor. The Finite Element Method, Volume 1: The Basis. Butterworth-Heinemann; 5 edition, 2000.
2. G. R. Liu and Y. T. Gu. An Introduction to Meshfree Methods and Their Programming. Springer, 2005.
3. Y. Chen, J. Lee, and A. Eskandarian. Meshless Methods in Solid Mechanics. Springer, 2006.
4. B. Fraeijs de Veubeke. Displacement and equilibrium models in the finite element method. International Journal for Numerical Methods in Engineering, Reprinted, 52:287-342, 2001.
5. B. Fraeijs de Veubeke and O. C. Zienkiewicz. Strain-energy bounds in finite element analysis by slab analogy. Journal of Strain Analysis, 2:265-271, 1967.
6. M. Duot and H. Nguyen-Dang. Dual analysis by a meshless method. Communications in Numerical Methods in Engineering, 18:621-631, 2002.
7. C. V. Le, M. Gilbert, and H. Askes. Limit analysis of plates and slabs using a meshless equilibrium formulation. International Journal for Numerical Methods in Engineering, 83 1739-1758, 2010.
8. J. S. Chen, C. T. Wu, S. Yoon, and Y. You. A stabilized conforming nodal integration for Galerkin mesh-free methods. International Journal for Numerical Methods in Engineering, 50:435-466, 2001.
9. C. V. Le, H. Askes, and M. Gilbert. Adaptive Element-Free Galerkin method applied to the limit analysis of plates. Computer Methods in Applied Mechanics and Engineering, 199:2487-2496, 2010.
10. M. Vicente da Silva and A. N. Antao. A non-linear programming method approach for upper bound limit analysis. International Journal for Numerical Methods in Engineering, 72:1192-1218, 2007.

11. S. W. Sloan and P. W. Kleeman. Upper bound limit analysis using discontinuous velocity fields. Computer Methods in Applied Mechanics and Engineering, 127:293-314, 1995.
12. Makrodimopoulos and C. M. Martin. Upper bound limit analysis using simplex strain elements and second-order cone programming. International Journal for Numerical and Analytical Methods in Geomechanics, 31:835-865, 2006.
13. C.V. Le, H. Nguyen-Xuan, H. Askes, S. Bordas, T. Rabczuk, and H. Nguyen-Vinh. A cell-based smoothed finite element method for kinematic limit analysis. International Journal for Numerical Methods in Engineering, 83:1651-1674, 2010.
14. Capsoni and L. Corradi. A finite element formulation of the rigid-plastic limit analysis problem. International Journal for Numerical Methods in Engineering, 40:2063-2086, 1997.
15. L. Prandtl. Ueber die haerte plastischer koerper. Nachrichtex der Akademie der Wissenschaften in Gottingen. II. Mathematisch-Physikalische Klasse II, 12:74{85, 1920}.
16. Makrodimopoulos and C. M. Martin. Lower bound limit analysis of cohesive-frictional materials using second-order cone programming. International Journal for Numerical Methods in Engineering, 66:604-634, 2006.
17. F. Tin-Loi and N. S. Ngo. Performance of the p-version finite element method for limit analysis. International Journal of Mechanical Sciences, 45:1149-1166, 2003.