AN IMPLEMENTATION OF MESHLESS METHODS FOR MECHANICAL PROBLEMS
MỘT BỔ SUNG CỦA PHƯƠNG PHÁP KHÔNG LƯỚI TRONG TÍNH TOÁN CÁC BÀI TOÁN CƠ HỌC

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ABSTRACT

A meshless approach to the analysis of two-dimensional elasticity problems by the Element-Free Galerkin (EFG) method is presented. This method is based on moving least squares approximant (MLS). The unknown function of displacement is $u(x)$ approximated by moving least square approximants $u^h(x)$. These approximants are constructed by using a weight function, a monomial basis function and a set of non-constant coefficients. A subdivision similar to finite element method is used to provide a background mesh for numerical integration. The essential boundary conditions are enforced by Lagrange multipliers method. The results are obtained for a two-dimensional problem using different EFG weight functions and compared with the results of finite element method and exact methods.
Keywords: weight function, Lagrange multipliers method, moving least squares.

TÓM TẮT

Một phương pháp phân tích xấp xỉ không lưới cho bài toán đàn hồi hai chiều bằng phương pháp phần tử tự do Galerkin được giới thiệu. Phương pháp này dựa trên xấp xỉ bình phương tối thiểu động. Hàm chuyển vị $u(x)$ được xấp xỉ theo MLS thành $u^h(x)$. Xấp xỉ này được xây dựng bằng việc sử dụng các hàm trọng số dưới dạng hàm đa thức cơ sở. Việc chia nhỏ giống như phương pháp phần tử hữu hạn đóng vai trò lưới nền để thực hiện các tích phân số. Tuy nhiên các điều kiện biên chính phải được chỉnh lý bằng phương pháp nhân tử Lagrange. Kết quả thu được trong phân tích EFG cho bài toán hai chiều với hàm trọng số khác nhau được so sánh với các kết quả của phương pháp phần tử hữu hạn và phương pháp chính xác.

I. INTRODUCTION

The element free Galerkin (EFG) method is a meshless method for solving partial differential equations which uses only a set of nodal points and a CAD_like description of the body to formulate the discrete model. It has been used extensively for fracture problems and has yielded good results when adequate refinement is used near the crack tip.

In this paper, a meshless approach to the analysis of two-dimensional elasticity problems by the Element-Free Galerkin (EFG) method is presented. This method is based on moving least squares approximant (MLS) to construct the approximate function for the Galerkin weak-form. These approximations are constructed by using a weight function, a monomial basis function and a set of non-constant coefficients. A subdivision similar to finite element method is used to provide a background mesh for numerical integration. The essential boundary conditions are enforced by Lagrange multipliers method. In this study, EFG is applied to elastostatics analysis. Path test, plate with a central circular hole will be computed in this paper. The results are obtained for a two-dimensional problem using different EFG weight functions and compared with the results of exact methods. In addition, EFG method Matlab code also is offered in this paper. This Matlab code can be developed to meshfree application software or other meshfree method in the further.

II. MLS APPROXIMATIONS FUNCTIONS

MLS functions were developed by Lancaster and Salkauskas to approximate curves and surfaces. We approximate the displacement field by a discrete sum,

$$\mathrm{u}(\mathbf{x}) \cong \mathrm{u}^{\mathrm{h}}(\mathbf{x}) = \mathbf{p}^{\mathrm{T}}(\mathbf{x})\mathbf{a}(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_{\mathbf{x}} \qquad (1)$$

where $\mathrm{p}(\mathbf{x})$ is a linearly independent basis of m functions,

$$p^T(\mathbf{x}) = \begin{bmatrix} p_0(\mathbf{x}) & p_1(\mathbf{x}) & \dots & p_m(\mathbf{x}) \end{bmatrix} \qquad (2)$$

and $\mathbf{a}(\mathbf{x})$ collects the undetermined parameters of the approximation,

$$a^T(\mathbf{x}) = \begin{bmatrix} a_0(\mathbf{x}) & a_1(\mathbf{x}) & \dots & a_m(\mathbf{x}) \end{bmatrix} \qquad (3)$$

where each term is a function of the position $\mathbf{x} \in \Omega$

The parameters $\mathbf{a}(\mathbf{x})$ are found at any $\mathbf{x}$ point by minimizing the following weighted least squares discrete L_2 error norm (*Nayroles-1992*),

$$J = \sum_{I=1}^{n} \omega(\mathbf{x}-\mathbf{x}_I)\left[u^h(\mathbf{x}_I,\mathbf{x}) - u_I\right]^2 \qquad (4)$$

$$= \sum_{I=1}^{n} \omega(\mathbf{x}-\mathbf{x}_I)\left[p^T(\mathbf{x}_I)a(\mathbf{x}) - u_I\right]^2 \qquad (5)$$

where $\omega(\mathbf{x}-\mathbf{x}_I)$ is a weighting function which is nonzero on the influence domain of the node $\mathbf{x}_I$,thus generating a local approximation and sparse matrices. Only the $\mathbf{x}_I$ nodes whose influence domains contain the x point will appear in the sum (4). The dimension of the influence domain of each node and the choice of the weighting function are decisive parameters for the approximation by MLS [1].

Minimizing J in order to the unknown parameters $\mathbf{a}(\mathbf{x})$ results in

$$A(\mathbf{x})a(\mathbf{x}) = B(\mathbf{x})u \qquad (6a)$$

Or:

$$a(\mathbf{x}) = A^{-1}(\mathbf{x})B(\mathbf{x})u \qquad (6b)$$

With:

$$A(\mathbf{x}) = \sum_{I=1}^{n} \omega(\mathbf{x}-\mathbf{x}_I)p(\mathbf{x}_I)\,p^T(\mathbf{x}_I) \qquad (7)$$

$$B(\mathbf{x}) = \begin{matrix} \omega(\mathbf{x}-\mathbf{x}_1)\,p(\mathbf{x}_1)\,\omega(\mathbf{x}-\mathbf{x}_2)\,p(\mathbf{x}_2) \\ \dots\ \omega(\mathbf{x}-\mathbf{x}_n)\,p(\mathbf{x}_n) \end{matrix} \qquad (8)$$

Substituting the result (6) for $\mathbf{a}(\mathbf{x})$ in the initial approximation (1), this expression can be written in the usual form,

$$u^h(\mathbf{x}) = \sum_{I=1}^{n} \Phi_I(\mathbf{x})u_I = \Phi(\mathbf{x})U \qquad (9)$$

where the shape function is defined by

$$\Phi_I(\mathbf{x}) = \sum_{j=0}^{m} p_j(\mathbf{x})\left(A^{-1}(\mathbf{x})B(\mathbf{x})\right)_{jI} = p^T A^{-1} B_I \qquad (10)$$

where m is the order of the polynomial $\mathrm{p}(\mathbf{x})$. To determine the derivatives from the displacement (9), it is necessary to obtain the shape function derivatives. The spatial derivatives of the shape functions are obtained by

$$\Phi_{I,x} = \left(p^T A^{-1} B_I\right)_{,x}$$
$$= p^T_{,x} A^{-1} B_I + p^T_{,x}\left(A^{-1}\right)_{,x} B_I + p^T A^{-1} B_{I,x} \qquad (11)$$

where

$$B_{I,\mathbf{x}}(\mathbf{x}) = \frac{d\omega}{d\mathbf{x}}(\mathbf{x}-\mathbf{x}_I)\,p(\mathbf{x}_I);$$
$$A^{-1}_{,\mathbf{x}}(\mathbf{x}) = -A^{-1}A_{,\mathbf{x}}A^{-1}; \qquad (12)$$
$$A_{,\mathbf{x}} = \sum_{I=1}^{n} \omega(\mathbf{x}-\mathbf{x}_I)p(\mathbf{x}_I)\,p^T(\mathbf{x}_I)$$

It should be noted that EFG shape functions do not satisfy the Kronecker delta criterion: $\Phi_I(\mathbf{x}) \neq \delta_{ij}$. Therefore they are not interpolants and the name approximation is used. So $u^h(x_I) \neq u_I$, the nodal parameters u_I are not the nodal values of $u^h(x_I)$. The approximation to the displacement at the I^{th} node depends on the nodal parameter u_I as

well as the nodal parameters u_I through u_n corresponding to all other nodes within the domain of influence of node I. This property makes the imposition of essential boundary conditions more complicated than with finite elements [2]. We will use Lagrange multiplier method to enforce the essential boundary conditions.

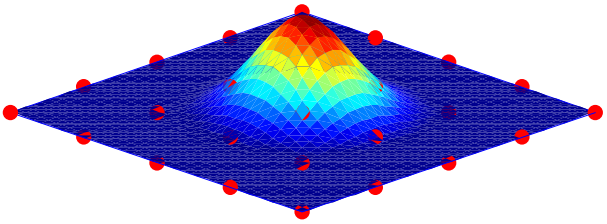

Fig. 1 Shape of the weight function

III. CHOICE OF SUPPORT DOMAIN AND WEIGHT FUNCTION

There is no difference if circular or rectangular support domain is used in the EFG method [3]. A weight function need to have following the properties:

- Compact support, i.e. zero outside the support domain.
- Adopt positive values for all points in the support domain
- Has its maximum value at the current point and decrease when moving outwards.

There are many kinds of function satisfying these properties, but the one used in this paper are the quartic spline function

$$\omega_j(r)=\begin{cases}\dfrac{2}{3}-4r_j^2+4r_j^3 & \text{Khi } r_j\le\dfrac{1}{2}\\ \dfrac{4}{3}-4r_j+4r_j^2-\dfrac{4}{3}r_j^3 & \text{Khi } \dfrac{1}{2}\le r_j\le 1\\ 0 & \text{Khi } r_j\ge 1\end{cases} \quad (13)$$

$$\text{With: } r_j=\frac{\|\mathbf{x}-\mathbf{x}_j\|}{d_{\max}c_j} \quad (14)$$

where $d_{\max}$ is a scaling parameter which is typically $2\div 4$ for a static analysis. The distance c_j is determined by searching for enough neighbor nodes for A to be regular.

IV. LAGRANGIAN MULTIPLIER METHOD

In EFG the shape functions do not fulfill the Kronecker delta property. So in order to invoke essential boundary, we have to use Lagrange multiplier. This will lead to a modified Lagrange function [4,5].

$$\tilde{L}=L+\int_{S_u}\lambda^T\left(u-\bar{u}\right)dS_u \quad (15)$$

The Lagrange multiplier (λ) can be interpreted as the reaction forces needed to fulfill the displacement conditions at the boundary.The approximation given by,

$$u(\mathbf{x})\approx u^h(\mathbf{x})=\sum_{I=1}^{n}\Phi_I(\mathbf{x})u_I=\Phi(\mathbf{x})U \quad (16)$$

Because both variations are independent, and not always equals to zero, the terms they are multiplied with have to be zero. This gives the following equation system,

$$\begin{bmatrix}K & G\\ G^T & 0\end{bmatrix}\begin{bmatrix}U\\ \Lambda\end{bmatrix}=\begin{bmatrix}f\\ q\end{bmatrix} \quad (17)$$

with:

$$K_{IJ}=\int_{\Omega}B_I^T D B_J\,d\Omega;\qquad G_{IK}=\int_{S_u}-N^T\Phi dS_u;$$

$$f_I=\int_{\Omega}\Phi^T b\,d\Omega+\int_{S_t}\Phi^T t\,dS_t \quad (18)$$

$$q_K=\int_{S_u}-N^T\bar{u}\,dS_u;\ B_I=\begin{bmatrix}\Phi_{I,x} & 0\\ 0 & \Phi_{I,y}\\ \Phi_{I,y} & \Phi_{I,x}\end{bmatrix};$$

$$N_K=\begin{bmatrix}N_K & 0\\ 0 & N_K\end{bmatrix} \quad (19)$$

V. NUMERICAL EXAMPLE

A global error indicator, the L_2-norm error in displacement, is defined by [6]

$$L_2 = \frac{\left\{\sum_{j=1}^{N}\left\{\left(u_j^{num}-u_j^{exact}\right)^2+\left(v_j^{num}-v_j^{exact}\right)^2\right\}\right\}^{1/2}}{\left\{\sum_{j=1}^{N}\left\{\left(u_j^{exact}\right)^2+\left(v_j^{exact}\right)^2\right\}\right\}^{1/2}} \quad (20)$$

Consider a plate with a central circular hole subjected to a unidirectional tensile load in the x direction as shown in Fig. 2. Due to symmetry, only the upper right quadrant of the plate is modelled (see, Fig. 3). Symmetry conditions are imposed on the left and bottom edges. The inner boundary is traction free. Plane strain conditions are assumed

- The material constants: $E = 2.10^{11} N/m^2$
- The unidirectional tensile load: $q = 1N/m$
- The demension of the plate: $L \times L = 10 \times 10\ m;\ d = 2m$

$\nu = 0,3$

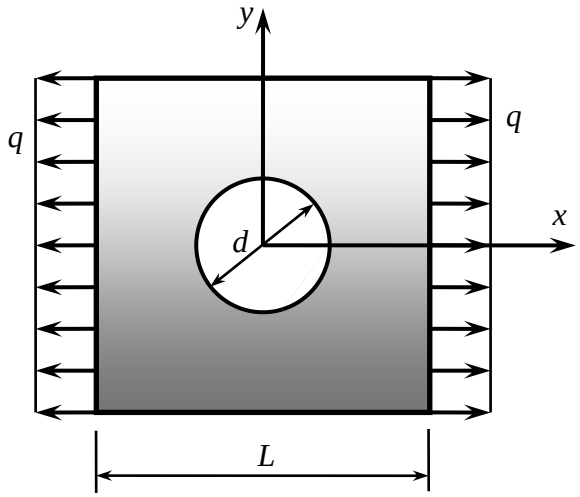

Fig. 2 A 2D solid with a central hole subjected to a unidirectional tensile load

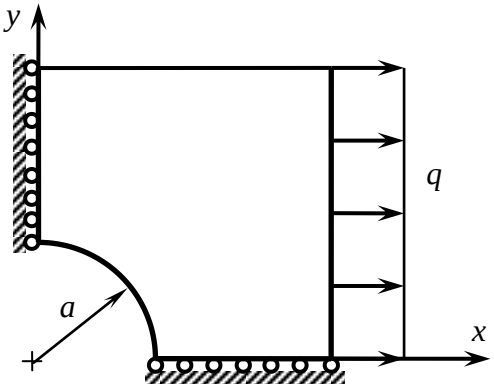

Fig. 3 One quarter model of plate

The analytical solution for the stresses of an infinite plate [7]

$$\sigma_x = 1 - \frac{a^2}{r^2}\left(\frac{3}{2}\cos 2\theta + \cos 4\theta\right) + \frac{3}{2}\frac{a^4}{r^4}\cos 4\theta$$

$$\sigma_y = -\frac{a^2}{r^2}\left(\frac{1}{2}\cos 2\theta - \cos 4\theta\right) - \frac{3}{2}\frac{a^4}{r^4}\cos 4\theta \quad (21)$$

$$\sigma_{xy} = -\frac{a^2}{r^2}\left(\frac{1}{2}\sin 2\theta + \sin 4\theta\right) + \frac{3}{2}\frac{a^4}{r^4}\sin 4\theta$$

$$u_r = \frac{1}{4G}\left\{r\left(\frac{k-1}{2}+\cos 2\theta\right) + \frac{a^2}{r}\left[1+(1+k)\cos 2\theta\right] - \frac{a^4}{r^3}\cos 2\theta\right\}$$

$$u_\theta = \frac{1}{4G}\left[(1-k)\frac{a^2}{r} - r - \frac{a^4}{r^3}\right]\sin 2\theta \quad (22)$$

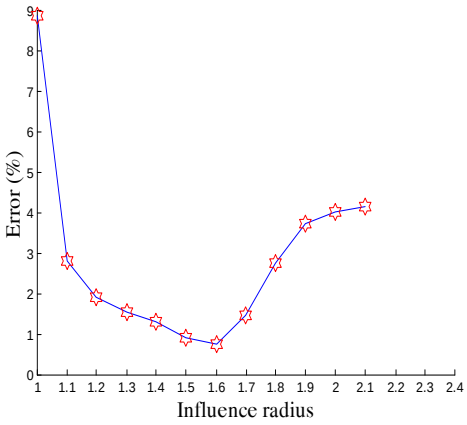

Fig. 4 Comparison between the exact and EFG solution for σ_x with influence radius $r_i = 1 \div 2.1$

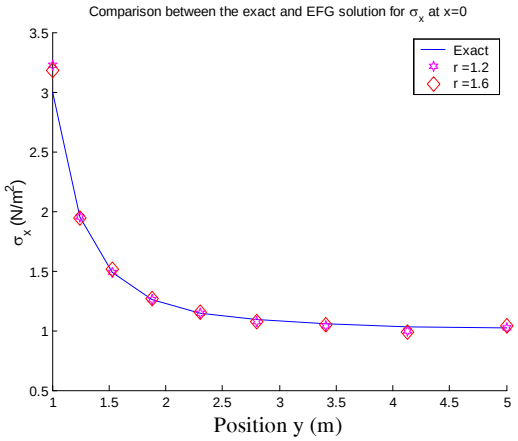

Fig. 5 Comparison between the exact and EFG solution for σ_x at $x = 0$

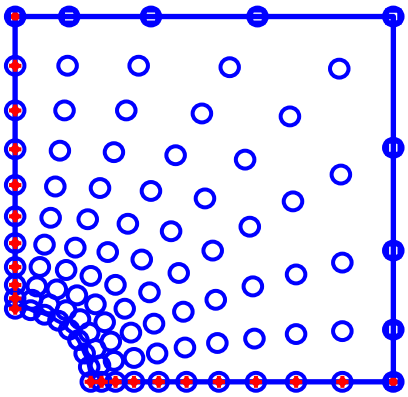

Fig. 6 Nodal arrangement

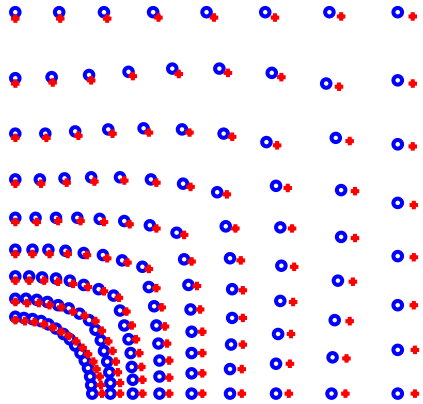

Fig. 7 Displacement of nodes with EFG method

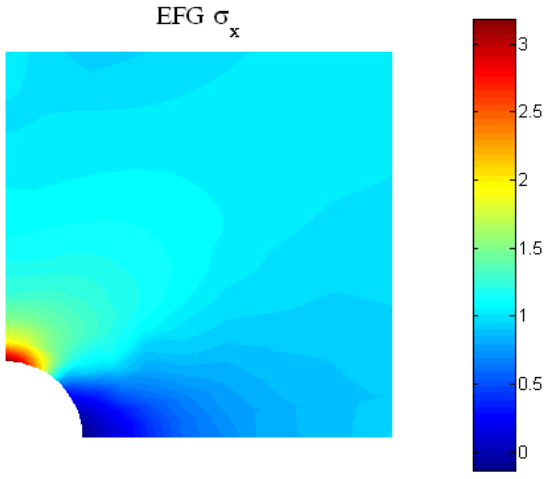

Fig. 8. EFG stress field σ_x

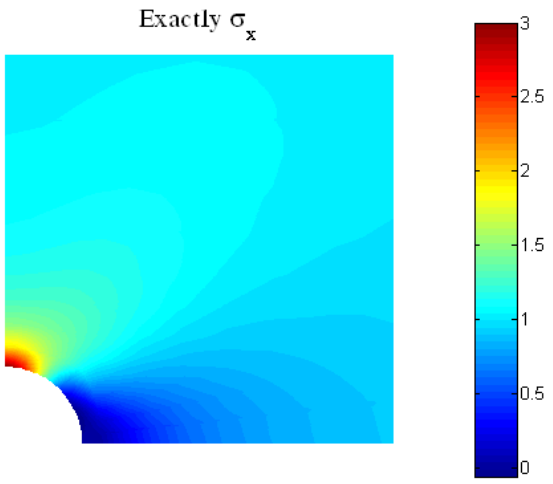

Fig. 9. Analytical stress field σ_x

VI. CONCLUSIONS AND DISCUSSION

- An Element Free Galerkin method was implemented in Matlab for linear statics. The method seems accurate enough and converge to the analytical solution when the number of nodes approach infinity.

- A little remark has to be mentioned about the numerical integration. To have

accurate result, the numerical integration has to be very fine, i.e. many integration points. Note that it converge even with a small amount of integration points, but it is not accurate enough. The only requirement to converge is that the number of nodes in the support domain is greater than the number of components in p. This to guarantee that the matrix A is invertible. The many integration points combined with the enlarged system because of the multiplier Lagrange function, results in a very time consuming program. Even for a so simple problem with linear statics. But no optimization where made on the code to improve the performance.

- Meshless methods still require considerable improvement before they equal the prominence of finite elements and finite differences in computer science and engineering. The greatest challenges appear to be in developing the speed and robustness in meshless methods which characterize low-order finite elements. The quality of the approximations are exemplary, only the cost is too high. Breakthroughs in these directions will have considerable impact.

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