ON NONLOCAL BVPs FOR DIFFERENTIAL INCLUSIONS OF FRACTIONAL ORDER

Phan Dinh Phung

Ho Chi Minh City University of Food Industry

Email: *pdphungvn@gmail.com*

Received: 23 March 2019; Accepted for publication: 5 June 2019

ABSTRACT

In this paper, we consider a class of boundary value problems (BVPs) in a separable Banach space *E*, which is a fractional differential inclusion associated with multipoint bounday conditions, of the form

$$\begin{cases} D^{\alpha}u(t) \in F(t,u(t),D^{\alpha-1}u(t)), \text{ a.e. } t \in [0,1], \\ I^{\beta}u(t)\big|_{t=0} = 0, \; u(1) = \sum_{i=1}^{m-2} \xi_i u(\eta_i), \end{cases}$$

where D^{α} is the Riemann-Liouville fractional derivative operator of order $\alpha \in (1,2]$, $\beta \in [0, 2-\alpha]$, *F* is a closed valued multifuction. With some certain suitable conditions we prove that the set of the solutions to the problem is nonempty and is a retract in space $W_E^{\alpha,1}(I)$.

Keywords: fractional differential inclusion, boundary value problem, Green's function, contractive set valued-map, retract.

1. INTRODUCTION

Differential equations of fractional or arbitrary order which is so-called fractional differential equations have recently demonstrated to be strongly tools in the modelling of many physical phenomena (see [1-4]). Consequently there has an increasing interest in studying the initial value problems and especially BVPs for fractional differential equations (see [5-17] and references therein).

El-Sayed and Ibrahim have initiated the study of fractional differential inclusions in [11]. In recent years, several qualitative results involving fractional differential inclusions are established, for instance, in [9, 18, 19]. However, most of that on fractional differential equations or inclusions are devoted to the solvability in the case that the nonlinear terms is independent of derivatives of unknown function. Moreover, there are very few studies considering such a problem in the general context, like Banach spaces. In this note, with *E* is a separable Banach space, we consider the following problem

$$D^{\alpha}u(t) \in F(t,u(t),D^{\alpha-1}u(t)), \text{a.e.}, t \in [0,1], \tag{1.1}$$

$$I^{\beta}u(t)\big|_{t=0} := \lim_{t\to 0}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}u(s)ds = 0, \quad u(1) = \sum_{i=1}^{m-2} \xi_i u(\eta_i), \tag{1.2}$$

where $\alpha \in (1,2]$, $\beta \in [0, 2-\alpha]$; $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$ and $\xi_i > 0$, $i = \overline{1, m-2}$, $m \geq 3$ are constants given satisfying $\sum_{i=1}^{m-1} \xi_i \eta_i^{\alpha-1} < 1$; Γ is Gamma function, D^{α} is fractional derivative operator of Riemann-Liouville kind; and $F : [0,1] \times E \times E \to 2^E$ is a closed valued multifunction. Problem (1.1)-(1.2) is also motivated from some our previous works [8, 12] extended to the multi-point condition which has increasing interest in the theory of BVPs. In the case that $\alpha = 2$, the equation (1.1) is a second-order differential inclusion which has been studied by many authors. We refer to [7, 20, 21] and references therein dealing with boundary value problem for regular order differential inclusion.

This paper is organized as follows. In Section **2** we introduce some notions and recall some definitions and needed results, in particular on the fractional calculus. Section **3** is to provide the results for existence of $W^{\alpha,1}(I)$ -solutions and properties of solutions set of the problem (1.1)-(1.2) via some classical tools such as fixed points theorem or retract property for the fixed points set of a contractive multivalued mapping.

2. PRELIMINARIES

Let I be the interval $[0,1]$ and let E be a separable Banach space; E' is its topological dual. For the convenience of the reader, we state here several notations that will be used in the sequel (see [22]).

- $\overline{B}_E$: the closed unit ball of *E*,
- $\mathcal{L}(I)$: the σ algebra of Lebesgue measurable sets on *I*,
- $\mathcal{B}(E)$: the σ algebra of Borel subsets of *E*,
- $L_E^1(I)$: the Banach space of all Lebesgue-Bochner integrable *E*-valued functions defined on I,
- $C_E(I)$: the Banach space of all continuous functions *f* from [0, 1] into *E* endowed with the norm

$$\| f \|_{\infty} = \sup_{t \in I} \| f(t) \| .$$

- $c(E)$: the set of all nonempty and closed subsets of *E*,
- $cc(E)$: the set of all nonempty and closed and convex subsets of *E*,
- $ck(E)$: the set of all nonempty and compact and convex subsets of *E*,
- $cwk(E)$: the set of all nonempty and weakly compact and convex subsets of *E*,
- $bc(E)$: the set of all nonempty bounded closed subsets of *E*,
- $d(x, A)$: the distance of a point *x* of *E* to a subset *A* of *E*, that is

$$d(x, A) = \inf \left\{ \|x - y\| : y \in A \right\}.$$

- $d_H(A, B)$: the Hausdorff distance between two subsets A and B of E, defined by

$$d_H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}.$$

Definition 2.1. ([2, pp. 45; 3, pp. 65]) Let $f: I \to E.$ The fractional Bochner-integral of order $\alpha > 0$ of the function f is defined by

$$I^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t > 0.$$

In the above definition, the sign "$\int$" stands for the Bochner integral. For more details on Bochner integral, we refer to [23, pp. 132].

Lemma 2.1 ([12]). *Let* $f \in L_E^1(I).$ *We have*

(i) If $\alpha \in (0,1)$ *then* $I^{\alpha} f(t)$ *exists for almost every* $t \in I$ *and* $I^{\alpha} f \in L_E^1(I).$

(ii) If $\alpha \geq 1$ *then* $I^{\alpha} f(t)$ *exists for all* $t \in I$ *and* $I^{\alpha} f \in C_E(I).$

Definition 2.2. ([2, pp. 82; 3, pp. 68]) Let $f \in L_E^1(I).$ The Riemann-Liouville fractional derivative of order $\alpha > 0$ of *f* is defined by

$$D^{\alpha} f(t) := \frac{d^n}{dt^n} I^{n-\alpha} f(t) = \frac{d^n}{dt^n} \int_0^1 \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f(s) ds,$$

where $n = [\alpha] + 1.$

In the case $E \equiv R$ (space of real numbers), we have the following well-known results.

Lemma 2.2 ([5]). *Let* $\alpha > 0.$ *The general solution of the fractional differential equation* $D^{\alpha} x(t) = 0$ *is given by*

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{2.3}$$

where $c_i \in R,\ i = 1, 2, ..., n\ (n = [\alpha] + 1).$

In view of Lemma **2.4**, it follows that

$$x(t) = I^{\alpha} D^{\alpha} x(t) + c_1 t^{\alpha-1} + \cdots + c_n t^{\alpha-n}, \tag{2.4}$$

for some $c_i \in R, i = 1, 2, ..., n.$

In the rest of the article we denote by $W_E^{\alpha,1}(I)$ the space of all continuous functions in $C_E(I)$ such that their Riemann-Liouville fractional derivative of order $\alpha - 1$ are in $C_E(I)$ and that of order α are in $L_E^1(I).$

3. MAIN RESULTS

Lemma 3.1. *Let E be a Banach space and let* $G(\cdot,\cdot): I \times I \to R$ *be a function defined by*

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1 \\ 0, & 0 \leq t \leq s \leq 1 \end{cases} + \frac{t^{\alpha-1}}{\Gamma(\alpha)\left(1 - \sum_{i=1}^{m-1} \xi_i \eta_i^{\alpha-1}\right)} \Psi(s), \tag{3.1}$$

where

$$\Psi(s)=\begin{cases}\sum_{i=1}^{m-1}\xi_i(\eta_i-s)^{\alpha-1}-(1-s)^{\alpha-1}, & 0\le s\le\eta_1,\\ \sum_{i=2}^{m-1}\xi_i(\eta_i-s)^{\alpha-1}-(1-s)^{\alpha-1}, & \eta_1\le s\le\eta_2,\\ \cdots & \\ \sum_{i=k}^{m-1}\xi_i(\eta_i-s)^{\alpha-1}-(1-s)^{\alpha-1}, & \eta_{k-1}\le s\le\eta_k,\\ \cdots & \\ -(1-s)^{\alpha-1}, & \eta_{m-2}\le s\le 1.\end{cases}\tag{3.2}$$

Then the following assertions hold.

(i) Function G satisfies the following estimate,

$$|G(t,s)|\le\frac{2}{\Gamma(\alpha)\left(1-\sum_{i=1}^{m-1}\xi_i\eta_i^{\alpha-1}\right)}.$$

(ii) If $u\in W_E^{\alpha,1}(I)$ *with* $I^\beta u(t)\big|_{t=0}=0$ *and* $u(1)=\sum_{i=1}^{m-1}\xi_i u(\eta_i)$, *then*

$$u(t)=\int_0^1 G(t,s)D^\alpha u(s)ds,\forall t\in I.$$

(iii) Let $f\in L_E^1(I)$ *and let* $u_f:I\to E$ *be the function defined by*

$$u_f(t)=\int_0^1 G(t,s)f(s)ds,\forall t\in I.$$

Then $I^\beta u_f(t)\big|_{t=0}=0$ *and* $u_f(1)=\sum_{i=1}^{m-1}\xi_i u_f(\eta_i)$. *Furthermore* $u_f\in W_E^{\alpha,1}(I)$ *and we get*

$$D^{\alpha-1}u_f(t)=\int_0^t f(s)ds+C_f,\ \forall t\in I,\tag{3.3}$$

$$D^\alpha u_f(t)=f(t),\ \text{a.e. } t\in I,\tag{3.4}$$

where

$$C_f=\frac{1}{1-\sum_{i=1}^{m-1}\xi_i\eta_i^{\alpha-1}}\left[\sum_{i=1}^{m-1}\xi_i\int_0^{\eta_i}(\eta_i-s)^{\alpha-1}f(s)ds-\int_0^1(1-s)^{\alpha-1}f(s)ds\right],$$

which depends only on f.

Proof. (i) From the definition of G it is easy to see that, for all $s,t\in[0,1]$,

$$|G(t,s)|\le\frac{2}{\Gamma(\alpha)\left(1-\sum_{i=1}^{m-1}\xi_i\eta_i^{\alpha-1}\right)}.$$

(ii) Let $y \in E'$. For all $t \in I$, we have

$$\left\langle y, \int_0^1 G(t,s) D^{\alpha} u(s) ds \right\rangle = \int_0^1 G(t,s) D^{\alpha} \langle y, u(s) \rangle ds$$

$$= I^{\alpha}\left(D^{\alpha}\langle y,u(t)\rangle\right) + \frac{t^{\alpha-1}}{1-\sum_{i=1}^{m-1}\xi_i \eta_i^{\alpha-1}}\left(\sum_{i=1}^{m-1}\xi_i I^{\alpha} D^{\alpha}\langle y,u(\eta_i)\rangle - I^{\alpha}D^{\alpha}\langle y,u(1)\rangle\right). \quad (3.5)$$

Using the assumption $\lim_{t\to 0^+} I^{\beta}u(t) = 0$ it follows from (2.4) that

$$\langle y,u(t)\rangle = I^{\alpha}D^{\alpha}\langle y,u(t)\rangle + c_1 t^{\alpha-1}, \quad (3.6)$$

for some $c_1 \in R$. So we have

$$\langle y,u(1)\rangle = I^{\alpha}D^{\alpha}\langle y,u(1)\rangle + c_1, \quad (3.7)$$

and

$$\left\langle y, \sum_{i=1}^{m-1}\xi_i u(\eta_i)\right\rangle = \sum_{i=1}^{m-1}\xi_i \langle y,u(\eta_i)\rangle = \sum_{i=1}^{m-1}\xi_i I^{\alpha}D^{\alpha}\langle y,u(\eta_i)\rangle + c_1\sum_{i=1}^{m-1}\xi_i\eta_i^{\alpha-1}. \quad (3.8)$$

As $u(1) = \sum_{i=1}^{m-1}\xi_i u(\eta_i)$ it follows from (3.7) and (3.8) that

$$c_1 = \frac{1}{1-\sum_{i=1}^{m-1}\xi_i\eta_i^{\alpha-1}}\left(\sum_{i=1}^{m-1}\xi_i I^{\alpha}D^{\alpha}\langle y,u(\eta_i)\rangle - I^{\alpha}D^{\alpha}\langle y,u(1)\rangle\right). \quad (3.9)$$

Combining (3.5), (3.6) and (3.9) we get

$$\left\langle y, \int_0^1 G(t,s)D^{\alpha}u(s)ds\right\rangle = \langle y,u(t)\rangle.$$

Since this equality holds for every $y \in E'$ so we have $u(t) = \int_0^1 G(t,s)D^{\alpha}u(s)ds, \forall t \in I.$

(iii) Let $f \in L_E^1(I)$ and $u_f(t) = \int_0^1 G(t,s) f(s) ds, \forall t \in I$. By the definition of *G* we have

$$u_f(t) = I^{\alpha}f(t) + \frac{t^{\alpha-1}}{1-\sum_{i=1}^{m-1}\xi_i\eta_i^{\alpha-1}}\left(\sum_{i=1}^{m-1}\xi_i I^{\alpha}f(\eta_i) - I^{\alpha}f(1)\right). \quad (3.10)$$

It's clear that $I^{\alpha}f \in C_E(I)$ by using Lemma **2.2**. So u_f is continuous on *I*. On the other hand, from (3.10), it follows that

$$u_f(1) = I^{\alpha}f(1) + \frac{1}{1-\sum_{i=1}^{m-1}\xi_i\eta_i^{\alpha-1}}\left(\sum_{i=1}^{m-1}\xi_i I^{\alpha}f(\eta_i) - I^{\alpha}f(1)\right) = \frac{\sum_{i=1}^{m-1}\xi_i\left(I^{\alpha}f(\eta_i) - \eta_i^{\alpha-1}I^{\alpha}f(1)\right)}{1-\sum_{i=1}^{m-1}\xi_i\eta_i^{\alpha-1}},$$

and

$$\sum_{i=1}^{m-1}\xi_i u_f(\eta_i)=\sum_{i=1}^{m-1}\xi_i I^\alpha f(\eta_i)+\frac{\sum_{i=1}^{m-1}\xi_i\eta_i^{\alpha-1}}{1-\sum_{i=1}^{m-1}\xi_i\eta_i^{\alpha-1}}\left(\sum_{i=1}^{m-1}\xi_i I^\alpha f(\eta_i)-I^\alpha f(1)\right)$$

$$=\frac{\sum_{i=1}^{m-1}\xi_i\left(I^\alpha f(\eta_i)-\eta_i^{\alpha-1}I^\alpha f(1)\right)}{1-\sum_{i=1}^{m-1}\xi_i\eta_i^{\alpha-1}}.$$

Hence $u_f(1)=\sum_{i=1}^{m-1}\xi_i u_f(\eta_i)$. Now, let $y\in E'$ be arbitrary. One has

$$\left\langle y, I^\beta u_f(t)\right\rangle = I^\beta\left\langle y,u_f(t)\right\rangle = I^\beta\left(\int_0^1 G(t,s)\left\langle y,f(s)\right\rangle ds\right)$$

$$=I^{\alpha+\beta}\left\langle y,f(t)\right\rangle+I^\beta\left(\frac{t^{\alpha-1}}{1-\sum_{i=1}^{m-1}\xi_i\eta_i^{\alpha-1}}\left\langle y,\sum_{i=1}^{m-1}\xi_i I^\alpha f(\eta_i)-I^\alpha f(1)\right\rangle\right)$$

$$=I^{\alpha+\beta}\left\langle y,f(t)\right\rangle+\frac{\Gamma(\alpha)\left\langle y,\sum_{i=1}^{m-1}\xi_i I^\alpha f(\eta_i)-I^\alpha f(1)\right\rangle}{\Gamma(\alpha+\beta)\left(1-\sum_{i=1}^{m-1}\xi_i\eta_i^{\alpha-1}\right)}t^{\alpha+\beta-1}. \tag{3.11}$$

Letting $t\to 0^+$ in (3.11) we get $\lim_{t\to 0^+}\left\langle y, I^\beta u_f(t)\right\rangle=0, \forall y\in E'$. This shows that $I^\beta u_f(t)\big|_{t=0}=0$.

It's enough to check the equalities (3.3)-(3.4). Indeed, since the function $I^\alpha f(\cdot)$ has Riemann-Liouville fractional derivatives of order γ, for all $\gamma\in(0,\alpha]$, so is the function $u_f(\cdot)$ by using (3.10). On the other hand, for each $y\in E'$, we have

$$\left\langle y, D^\gamma u_f(t)\right\rangle = D^\gamma\left\langle y,u_f(t)\right\rangle = D^\gamma\int_0^1 G(t,s)\left\langle y,f(s)\right\rangle ds$$

$$=D^\gamma I^\alpha\left\langle y,f(t)\right\rangle+\frac{1}{1-\sum_{i=1}^{m-1}\xi_i\eta_i^{\alpha-1}}\left(\sum_{i=1}^{m-1}\xi_i\left\langle y,I^\alpha f(\eta_i)\right\rangle-\left\langle y,I^\alpha f(1)\right\rangle\right)D^\gamma\left(t^{\alpha-1}\right) \tag{3.12}$$

Since $D^\gamma I^\alpha\left\langle y,f(t)\right\rangle=I^{\alpha-\gamma}\left\langle y,f(t)\right\rangle$ and

$$D^\gamma\left(t^{\alpha-1}\right)=\begin{cases}\dfrac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)}t^{\alpha-\gamma-1}, & 0<\gamma<\alpha,\\ 0, & \gamma=\alpha,\end{cases}$$

we deduce from (3.12) that

$$\langle y, D^{\alpha-1}u_f(t)\rangle = \int_0^t \langle y, f(s)\rangle ds + \frac{\Gamma(\alpha)}{1-\sum_{i=1}^{m-1}\xi_i\eta_i^{\alpha-1}}\left(\sum_{i=1}^{m-1}\xi_i\langle y, I^\alpha f(\eta_i)\rangle - \langle y, I^\alpha f(1)\rangle\right),$$

for all $t\in I,$ and

$$\langle y, D^\alpha u_f(t)\rangle = \langle y, f(t)\rangle, \text{ a.e. } t\in I.$$

These imply that (3.3) and (3.4) hold. The proof is completed.

Remark 3.1. *From Lemma **3.1**, it's easy to see that if* $u_f(t)=\int_0^1 G(t,s)f(s)ds,\ f\in L_E^1(I),$ *then*

$$\|u_f(t)\| \le M_G\|f\|_{L_E^1(I)} \quad \textit{and} \quad \|D^{\alpha-1}u_f(t)\| \le M_G\|f\|_{L_E^1(I)}, \tag{3.13}$$

for all $t\in I,$ *where*

$$M_G = \frac{2}{\Gamma(\alpha)}\left(1-\sum_{i=1}^{m-1}\xi_i\eta_i^{\alpha-1}\right)^{-1}.$$

Now we establish the main theorem of the existence of the solutions to problem (1.1)-(1.2) via applying the Covitz-Nadler fixed point theorem ([24]).

Theorem 3.1. *Let* $F:[0,1]\times E\times E\to c(E)$ *be a closed valued multifunction satisfying the following conditions*

(A1) F is $\mathcal{L}(I)\otimes\mathcal{B}(E)\otimes\mathcal{B}(E)$*-measurable,*

(A2) There exists positive functions $\ell_1,\ell_2\in L_R^1(I)$ *with* $M_G\|\ell_1+\ell_2\|_1<1$ *such that*

$$d_H\big(F(t,x_1,y_1),F(t,x_2,y_2)\big)\le \ell_1(t)\|x_1-x_2\|+\ell_2(t)\|y_1-y_2\|,$$

for all $(t,x_1,y_1),(t,x_2,y_2)\in I\times E\times E.$

(A3) The function $t\mapsto \sup\{\|z\|: z\in F(t,0,0)\}$ *is integrable.*

Then the problem (3.1)-(3.2) *has at least one solution in* $W_E^{\alpha,1}(I).$

Proof. We defined the set valued map $S: L_E^1(I)\to c(L_E^1(I))$ defined by

$$S(h)=\left\{f\in L_E^1(I): f(t)\in F\big(t,u_h(t),D^{\alpha-1}u_h(t)\big), \text{a.e.} t\in I\right\}, h\in L_E^1(I),$$

where $c(L_E^1(I))$ denotes the set of all nonempty closed subsets of $L_E^1(I)$ and $u_h\in W_E^{\alpha,1}(I),$

$$u_h(t)=\int_0^1 G(t,s)h(s)ds.$$

It is clear that *u* is a solution of (1.1)-(1.2) if and only if $D^\alpha u$ is a fixed point of *S*. We shall show that *S* is a contraction. The proof will be given in two steps.

Step 1. The subset $S(h)$ is nonempty and closed for every $h\in L_E^1(I).$ It's note that, by the assumptions, the multifunction $F(\cdot,u_h(\cdot),D^{\alpha-1}u_h(\cdot))$ is closed valued and measurable on *I*. Using the standard measurable selections theorem we infer that $F\left(\cdot,u_h\left(\cdot\right),D^{\alpha-1}u_h\left(\cdot\right)\right)$ admits a measurable selection z. One has

$$\begin{aligned}\left\|z\left(t\right)\right\| &\le \sup\left\{\left\|a\right\|: a\in F\left(t,0,0\right)\right\}+d_H\left(F\left(t,0,0\right),F\left(t,u_h\left(t\right),D^{\alpha-1}u_h\left(t\right)\right)\right)\\ &\le \sup\left\{\left\|a\right\|: a\in F\left(t,0,0\right)\right\}+\ell_1\left(t\right)\left\|u_h\left(t\right)\right\|+\ell_2\left(t\right)\left\|D^{\alpha-1}u_h\left(t\right)\right\|\\ &\le \sup\left\{\left\|a\right\|: a\in F\left(t,0,0\right)\right\}+M_G\left(\ell_1\left(t\right)+\ell_2\left(t\right)\right)\left\|h\right\|_{L^1_E(I)},\end{aligned}$$

for almost every $t\in I$, which shows that $z\in L^1_E\left(I\right)$ and $S\left(h\right)$ is nonempty. On the other hand, it is easy to see that, for each $h\in L^1_E(I)$, $S\left(h\right)$ is closed in $L^1_E\left(I\right)$.

Step 2. The multi-valued map *S* is a contraction.

We need to prove that there exists $k\in(0,1)$ satisfying

$$d_H\left(S(h),S(g)\right)\le k\left\|h-g\right\|_{L^1_E(I)},$$

for any $h,g\in L^1_E(I)$, where d_H denotes the Hausdorff distance on closed subsets in the Banach space $L^1_E(I)$. Let $f\in S\left(h\right)$ and $\varepsilon>0$. By a standard measurable selections theorem, there exists a Lebesgue-measurable $\phi: I\to E$ such that

$$\phi\left(t\right)\in F\left(t,u_g\left(t\right),D^{\alpha-1}u_g\left(t\right)\right),$$

and

$$\left\|\phi\left(t\right)-f\left(t\right)\right\|\le d\left(f(t),F\left(t,u_g(t),D^{\alpha-1}u_g(t)\right)\right)+\varepsilon,$$

for all $t\in I$. As $f\in S(h)$ we have

$$\begin{aligned}\left\|\phi\left(t\right)-f\left(t\right)\right\| &\le d_H\left(F\left(t,u_h(t),D^{\alpha-1}u_h(t)\right),F\left(t,u_g(t),D^{\alpha-1}u_g(t)\right)\right)+\varepsilon\\ &\le \ell_1\left(t\right)\left\|u_g\left(t\right)-u_h\left(t\right)\right\|+\ell_2\left(t\right)\left\|D^{\alpha-1}u_g\left(t\right)-D^{\alpha-1}u_h\left(t\right)\right\|+\varepsilon,\end{aligned}$$

for all $t\in I$. This follows that

$$\left\|\phi-f\right\|_{L^1_E(I)}\le M_G\left\|\ell_1+\ell_2\right\|_{L^1_R(I)}\left\|g-h\right\|_{L^1_E(I)}+\varepsilon,\forall f\in S\left(h\right).$$

Hence $\phi\in S(g)$ and

$$\sup_{f\in S(h)} d\left(f,S\left(g\right)\right)\le M_G\left\|\ell_1+\ell_2\right\|_1\left\|g-h\right\|_{L^1_E(I)}+\varepsilon.$$

Whence we get

$$\sup_{f\in S(h)} d\left(f,S\left(g\right)\right)\le M_G\left\|\ell_1+\ell_2\right\|_1\left\|g-h\right\|_{L^1_E(I)},$$

since ε can be arbitrarily small. By interchanging the variables g,h we obtain

$$d_H\left(S\left(g\right),S\left(h\right)\right)\le M_G\left\|\ell_1+\ell_2\right\|_1\left\|g-h\right\|_{L^1_E(I)},\ \forall g,h\in L^1_E\left(I\right).$$

Since $k:=M_G\left\|\ell_1+\ell_2\right\|_1<1$ by assumption, this shows that *S* is a contractive map. Applying the Covitz-Nadler fixed point theorem to *S* proves that *S* has a fixed point. The theorem is proved.

Corollary 3.1. *Let* $f: I\times E\times E\to E$ *be a mapping satisfying the following conditions*

(B1) for every $\left(x,y\right)\in E\times E$, *the function* $f\left(\cdot,x,y\right)$ *is measurable on I,*

(B2) for every $t \in I, f(t,\cdot,\cdot)$ *is continuous and there exists positive functions* $\ell_1, \ell_2 \in L^1_R(I)$ *for which* $M_G \|\ell_1 + \ell_2\|_1 < 1$ *such that*

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \le \ell_1(t)\|x_1 - x_2\| + \ell_2(t)\|y_1 - y_2\|,$$

for all $(t,x_1,y_1),(t,x_2,y_2) \in I \times E \times E,$

(B3) the function $t \mapsto f(t,0,0)$ *is Lebesgue-integrable on I.*

Then the fractional BVP

$$\begin{cases} D^{\alpha}u(t) = f\left(t, u(t), D^{\alpha-1}u(t)\right), \text{ a.e. } t \in I, \\ I^{\beta}u(t)\big|_{t=0} = 0, \ u(1) = \sum_{i=1}^{m-1} \xi_i u(\eta_i), \end{cases} \tag{3.14}$$

has a unique solution $u \in W_E^{\alpha,1}(I).$

Proof. The existence of solution u is guaranteed by Theorem **3.3**. Let u_1, u_2 be two $W_E^{\alpha,1}(I)$ -solutions to the problem (3.14). For each $t \in I,$ we have

$$\begin{aligned}\left\|D^{\alpha}u_1(t) - D^{\alpha}u_2(t)\right\| &= \left\| f\left(t, u_1(t), D^{\alpha-1}u_1(t)\right) - f\left(t, u_2(t), D^{\alpha-1}u_2(t)\right)\right\| \\ &\le \ell_1(t)\|u_1(t) - u_2(t)\| + \ell_2(t)\left\|D^{\alpha-1}u_1(t) - D^{\alpha-1}u_2(t)\right\|. \end{aligned} \tag{3.15}$$

On the other hand, it follows from Lemma **3.1** that

$$\|u_1(t) - u_2(t)\| \le M_G \left\|D^{\alpha}u_1 - D^{\alpha}u_2\right\|_{L^1_E(I)}, \tag{3.16}$$

And

$$\left\|D^{\alpha-1}u_1(t) - D^{\alpha-1}u_2(t)\right\| \le M_G \left\|D^{\alpha}u_1 - D^{\alpha}u_2\right\|_{L^1_E(I)}. \tag{3.17}$$

Combining (3.15), (3.16) and (3.17) we deduce that

$$\left\|D^{\alpha}u_1 - D^{\alpha}u_2\right\|_{L^1_E(I)} \le M_G \left\|\ell_1 + \ell_2\right\|_{L^1_{\mathbb{R}}(I)} \left\|D^{\alpha}u_1 - D^{\alpha}u_2\right\|_{L^1_E(I)},$$

which ensures $D^{\alpha}u_1 = D^{\alpha}u_2,$ and hence, by (3.16), we get $u_1 = u_2.$

Theorem 3.2. *Let* $F:[0,1]\times E \times E \to bc(E)$ *be a bounded closed valued multifunction satisfying the conditions* (A1)-(A3) *in Theorem 3.3. Then the* $W_E^{\alpha,1}(I)$ *-solutions set,* $\mathcal{S},$ *of the problem* (1.1)-(1.2) *is retract in* $W_E^{\alpha,1}(I),$ *here the space* $W_E^{\alpha,1}(I)$ *is endowed with the norm*

$$\|u\|_W = \|u\|_\infty + \left\|D^{\alpha-1}u\right\|_\infty + \left\|D^{\alpha}u\right\|_{L^1_E(I)}.$$

Proof. According to Theorem **3.3** and our assumptions, the multifunction

$$S : L^1_E(I) \to c\left(L^1_E(I)\right)$$

defined by

$$S(h) = \left\{ f \in L^1_E(I) : f(t) \in F\left(t, u_h(t), D^{\alpha-1}u_h(t)\right), \text{a.e.} t \in I\right\}, \ \ h \in L^1_E(I),$$

where $c\left(L^1_E(I)\right)$ denotes the set of all nonempty closed subsets of $L^1_E(I)$ and $u_h \in W_E^{\alpha,1}(I),$

$$u_h(t) = \int_0^1 G(t,s)h(s)ds,$$

is a contraction with the nonempty, bounded, closed and decomposable values in $L^1_E(I)$. So by a result of Bressan-Cellina-Fryszkowski ([25]), the set Fix(S) of all fixed points of S is a retract in $L^1_E(I)$. Hence there exists a continuous mapping $\psi : L^1_E(I) \to \text{Fix}(S)$ such that

$$\psi(h) = h, \ \forall h \in \text{Fix}(S).$$

For each $u \in W^{\alpha,1}_E(I)$, let us set

$$\Phi(u)(t) = \int_0^1 G(t,s)\psi\left(D^\alpha u\right)(s)ds, \ t \in I. \tag{3.18}$$

Using Lemma **3.1** obtains that

$$I^\beta\left(\Phi(u)\right)(t)\Big|_{t=0} = 0, \quad \Phi(u)(1) = \sum_{i=1}^{m-1}\xi_i\Phi(u)(\eta_i),$$

$$D^{\alpha-1}\left(\Phi(u)\right)(t) = \int_0^t \psi\left(D^\alpha u\right)(s)ds + C_{\psi\left(D^\alpha u\right)}, \tag{3.19}$$

and

$$D^\alpha\left(\Phi(u)\right)(t) = \psi(D^\alpha u)(t), \text{ a.e. } t \in I. \tag{3.20}$$

This shows that $D^\alpha\left(\Phi(u)\right) \in \text{Fix}(S)$. So $\Phi(u)$ is a $W^{\alpha,1}_E(I)$-solution of problem (1.1)-(1.2), that is $\Phi(u) \in \mathcal{S}$. It remains to prove that Φ is continuous mapping from $W^{\alpha,1}_E(I)$ in to $\mathcal{S}$. Let $u \in W^{\alpha,1}_E(I)$ and $\varepsilon > 0$. As ψ is continuous on $L^1_E(I)$, there exists $\delta > 0$ such that

$$\left\|h - D^\alpha u\right\|_{L^1_E(I)} < \delta \ \Rightarrow \ \left\|\psi(h) - \psi\left(D^\alpha u\right)\right\|_{L^1_E(I)} < \varepsilon, \tag{3.21}$$

for all $h \in L^1_E(I)$. Let us consider the ball $B_{W^{\alpha,1}_E(I)}(u,\delta)$ of center u with radius δ in $\left(W^{\alpha,1}_E(I), \|\cdot\|_W\right)$. Then, for $v \in B_{W^{\alpha,1}_E(I)}(u,\delta)$, one has $\left\|D^\alpha v - D^\alpha u\right\|_{L^1_E(I)} < \delta$ using the definition of the norm $\|\cdot\|_W$. So it follows from (3.20) and (3.21) that

$$\left\|D^\alpha\left(\Phi(v)\right) - D^\alpha\left(\Phi(u)\right)\right\|_{L^1_E(I)} = \left\|\psi(D^\alpha v) - \psi\left(D^\alpha u\right)\right\|_{L^1_E(I)} < \varepsilon. \tag{3.22}$$

Using Lemma **3.1** again we deduce, from (3.18), (3.19) and (3.22), that

$$\left\|\Phi(v)(t) - \Phi(u)(t)\right\| \le M_G\left\|D^\alpha\left(\Phi(v)\right) - D^\alpha\left(\Phi(u)\right)\right\|_{L^1_E(I)} < M_G\varepsilon,$$

$$\left\|D^{\alpha-1}\left(\Phi(v)\right)(t) - D^{\alpha-1}\left(\Phi(u)\right)(t)\right\| \le M_G\Gamma(\alpha)\left\|D^\alpha\left(\Phi(v)\right) - D^\alpha\left(\Phi(u)\right)\right\|_{L^1_E(I)} < M_G\Gamma(\alpha)\varepsilon,$$

for all $t \in I$. Combining (3.22)-(3.24) we obtain the continuity of Φ. Finally, for $u \in \mathcal{S}$, we have $D^\alpha(u) \in \text{Fix}(S)$. So

$$\psi(D^\alpha(u)) = D^\alpha(u),$$

by the property of ψ. It follows that

$$\Phi(u)(t) = \int_0^1 G(t,s)\psi\left(D^\alpha u\right)(s)ds = \int_0^1 G(t,s)D^\alpha u(s)ds = u(t),$$

for all $t \in I$. The proof is thus completed.

4. CONCLUSION

Our study of the fractional inclusion

$$D^{\alpha}u(t)\in F(t,u(t),D^{\alpha-1}u(t)),\ \text{a.e. } t\in[0,1]$$

provides a new technique to deal with the problem associated to the nonlocal boundary condition of multi-point type. After finding the Green function for the linearization problem, the existence is obtained via the multi-value contraction mapping Covitz-Nadler and the the solution set is then a retract with the additional assumption of boundedness of F. This results, especially existence result, can also be applied to get some results for relaxation and control problem, as the way in [6, 8, 12].

REFERENCES

1. El-Sayed A.M.A. - Nonlinear functional differential equations of arbitrary orders, Nonlinear Analysis **33** (1998) 181-186.
2. Miller K.S., Ross B. - An Introduction to the fractional calculus and fractional differential equations, Willey, New York (1993), 366p.
3. Podlubny I. - Fractional differential equation, Academic Press, New York (1999).
4. Samko S. G., Kilbas A. A., Marichev O. I. - Fractional integrals and derivatives: Theory and applications, Gorden and Breach, New York (1993).
5. Bai Z., Lu H. - Positive solutions for boundary value problem of nonlinear fractional differential equation, Journal of Mathematical Analysis and Applications **311** (2005) 495-505.
6. Castaing C., Godet-Thobie C., Phung P.D., Truong L.X. - On fractional differential inclusions with nonlocal boundary conditions, Fractional Calculus and Applied Analysis **22** (2) (2019) 444-478.
7. Castaing C., Truong L.X. - Second order differential conclusions with m-point boundary conditions, Journal of Nonlinear and Convex Analysis **24** (2013) 451–482.
8. Castaing C., Truong L.X. and Phung P.D. - On a fractional differential inclusion with integral boundary condition in Banach spaces, Journal of Nonlinear and Convex Analysis **17** (3) (2016) 441-471.
9. Cernea A. - On a fractional differential inclusion with boundary condition, Studia Universitatis Babeș-Bolyai Mathematica **LV** (2010) 105-113.
10. El-Sayed A.M.A., El-Salam S.A.A. - Nonlocal boundary value problem of a fractional-order functional differential equation, International Journal of Nonlinear Science **7** (2009) 436-442.
11. El-Sayed A. M. A., Ibrahim A. G. - Set-valued integral equations of arbitrary (fractional) order, Applied Mathematics and Computation **118** (2001) 113-121.
12. Phung P.D., Truong L.X. - On a fractional differential inclusion with integral boundary conditions in Banach space, Fractional Calculus and Applied Analysis **16** (3) (2013) 538-558.
13. Phung P.D., Truong L.X. - Existence of solutions to three-point boundary value problems at resonance, Electronic Journal of Differential Equations **115** (2016) 1-13.
14. Phung P.D., Minh H.B. - Existence of solutions to fractional boundary value problems at resonance in Hilbert spaces, Boundary Value Problems (2017) 2017:105.
15. Phung P.D. - Positive solutions of a fourth-order differential equation with multipoint boundary conditions, Vietnam Journal of Mathematics **43** (1) (2015) 93-104.

16. Phung P.D., Truong L.X. - On the existence of a three point boundary value problem at resonance in $\mathbb{R}^n$, Journal of Mathematical Analysis and Applications **416** (2014) 522-533.
17. Truong L.X., Phung P.D. - Existence of positive solutions for a multi-point four-order boundary value problem, Electronic Journal of Differential Equations **129** (2011) 1-10.
18. Benchohra M., Henderson J., Ntouyas S. K., Ouahab A. - Existence results for fractional functional differential inclusions with infinite delay and applications to control theory, Fractional Calculus and Applied Analysis **11** (2008) 35-56.
19. Ouahab A. - Some results for fractional boundary value problem of differential inclusions, Nonlinear Analysis **69** (2008) 3877-3896.
20. Azzam D.L., Castaing C. and Thibault L. - Three boundary value problems for second order differential inclusions in Banach spaces, Control Cybernet **31** (2001) 659-693.
21. Gomaa A.M. - On the solution sets of the three-points boundary value problems for nonconvex differential inclusions, Journal of the Egyptian Mathematical Society **12** (2004) 97-107.
22. Castaing C., Valadier M. - Convex Analysis and Measurable Multifunctions. Lecture Notes in Mathematics, Vol. 580, Springer-Verlag, Berlin, Heidelberg, New York (1977).
23. Yosida K. - Functional Analysis, Springer, 6th edition, Germany (1995).
24. Covitz H., Nadler S. B. - Multivalued contraction mappings in generalized metric spaces, Israel Journal of Mathematics **8** (1970) 5-11.
25. Bressan A., Cellina A. and Fryszkowski A. - A class of absolute retracts in spaces of integrable functions, Proceedings of the American Mathematical Society **112** (1991) 413-418.

TÓM TẮT

VỀ BÀI TOÁN BIÊN PHI ĐỊA PHƯƠNG CHO BAO HÀM THỨC VI PHÂN BẬC KHÔNG NGUYÊN

Phan Đình Phùng
Trường Đại học Công nghiệp Thực phẩm TP.HCM
Email: *pdphungvn@gmail.com*

Trong bài báo này, tác giả xét một lớp bài toán biên trong không gian Banach khả ly *E*, gồm một bao hàm thức vi phân cấp không nguyên liên kết với điều kiện biên nhiều điểm, có dạng

$$\begin{cases} D^{\alpha}u(t) \in F(t, u(t), D^{\alpha-1}u(t)), \text{ a.e. } t \in [0,1], \\ I^{\beta}u(t)\big|_{t=0} = 0,\ u(1) = \sum_{i=1}^{m-2} \xi_i u(\eta_i), \end{cases}$$

trong đó, D^{α} là toán tử đạo hàm cấp $\alpha \in (1,2]$, $\beta \in [0, 2-\alpha]$, F là một ánh xạ đa trị nhận giá trị đóng. Với một số điều kiện thích hợp, tác giả chứng minh bao hàm thức trên có nghiệm, hơn nữa tập nghiệm là một tập co rút trong không gian $W_E^{\alpha,1}(I)$.

Từ khóa: Bao hàm thức cấp không nguyên, bài toán biên, hàm Green, ánh xạ đa trị co, tập co rút.