AN XFEM BASED KINEMATIC LIMIT ANALYSIS FORMULATION FOR PLANE STRAIN CRACKED STRUCTURES USING SOCP

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ABSTRACT

This paper extends a numerical procedure for limit analysis based on extended finite element method (XFEM) and second-order cone programming (SOCP) to plane strain cracked structures. The cracked structures are easily modelled and simulated using XFEM because it allows discontinuities across elements, and these discontinuities are recognized by means of level set method. The resulting discretization formulation is then cast in a form which involves second-order cone constraints, ensuring that the underlying optimization problem can be solved by highly efficient primal-dual interior point algorithm. The efficiency of the present approach is illustrated by examing several numerical examples.

Keywords: Limit analysis (LA), Extend finite element method (XFEM), Second-order cone programming (SOCP).

Introduction

Limit analysis provides a direct tool for structural design and safety assessment of ductile structures containing of cracks, e.g. pressure vessels and reactors. Analytical upper-bound solutions of 2D cracked structures were originally reported by Hill [1], Ewing et al. [2, 3], in which slip-line method was employed to assume possible collapse mechanisms. However, the analytical method is not applicable for complicated problems in engineering practice, for which an appropriate failure mechanism may not be presupposed in advance. Consequently, various numerical procedures based on finite elements and bound theorems have been developed by Yan [4] and Vu [5]. These methods, known as finite element limit analysis, do not require assumptions to be made about the mode of failure, and require only simple strength parameters. However, in these numerical procedures the finite element meshes need to match the geometry of the cracks or discontinuities and mesh refinement near a crack tip must be made in order to achieve accurate solutions.

Recently, the extended finite element method (XFEM), which was originally proposed by Belytschko and Black [6], has been developed to overcome the above-mentioned shortcomings. In this method, discontinuities are permitted to cross elements, and are often realised by the level-set method. Therefore, it is relevant to investigate the performance of XFEM in the context of limit analysis of structures involving cracks or discontinuities.

In this paper, we extends a numerical procedure for limit analysis based on extended finite element method (XFEM) and second-order cone programming (SOCP) to plane strain cracked structures. The displacement fields with discontinuities and strong singularity at crack tip are approximated by XFEM shape functions which includes Heaviside functions (for introducing discontinuities) and asymptotic functions near tip field (dealing with singularity). The second-order cone programming (SOCP) is also combined with the XFEM based limit analysis problem so that engineering problem with a large number of variables can be solved using highly efficient primal-dual interior point algorithms. Numerical examples are presented to demonstrate the effectiveness of the proposed method.

Brief of the XFEM

The key idea of XFEM is to add an enrichment function to the standard finite element basis approximation using the partition of unity concept [8]. The enriched basis shape functions are associated to additional degrees of freedom, and the displacement field can be represented as [7].

$$u^h(x)=\sum_{i\in N}N_i(x)u_i+\sum_{j\in N^c}N_j(x)H(x)a_j+\sum_{k\in N^f}N_k(x)\sum_{\alpha=1}^{4}B_\alpha(x)b_K^\alpha \tag{1}$$

where

N is the set of the standard finite element nodes,

N^c is the set of nodes whose support is entirely split by the crack (circled nodes in Figure 1),

N^f is the set of nodes which contain the crack tip (square nodes in Figure 1),

$N_i(x)$ is the shape function associated with node i,

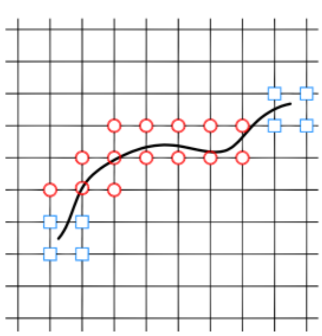
u_i, a_j, b_K^α are the displacement and enrichment nodal variables, respectively,

$H(x)$ is the modified Heaviside function which equal is 1 if x is above the crack and -1 if x is under the crack,

$B_\alpha(x)$ is a basis that spans the near tip asymptotic field:

$$[B_\alpha]=\left[\sqrt{r}\sin\frac{\theta}{2},\quad \sqrt{r}\cos\frac{\theta}{2},\quad \sqrt{r}\sin\frac{\theta}{2}\sin\theta,\quad \sqrt{r}\cos\frac{\theta}{2}\sin\theta\right] \tag{2}$$

Figure 1. An arbitrary crack placed on a mesh-enrichment strategy



In order to derive the strain-displacement matrix, it is convenient to rewrite the enriched approximation as follows:

$$u^h(x)=\sum_{i\in N} N_i(x)u_i+\sum_{j\in N^c} N_j(x)\psi(x)a_j \qquad (3)$$

Note that in Eq.(3), the interpolation property, i.e. $u^h(x_i)=u_i$, is not valid. In order to retain this, the enriched part of Eq.(3) must be vanished at nodes. The following shifting is often made:

$$u^h(x)=\sum_{i\in N} N_i(x)u_i+\sum_{j\in N^c} N_j(x)\big(\psi(x)-\psi(x_j)\big)a_j \qquad (4)$$

As a result, the finite element matrix **B** can be expressed as

$$\mathrm{B}=\left[\mathrm{B}^{\mathrm{fem}} \quad \mathrm{B}^{\mathrm{enr}}\right] \qquad (5)$$

where $\mathbf{B}^{fem}$ is the standard finite element matrix **B**, and in two dimensions **B** reads:

$$\mathrm{B}_i^{fem}=\begin{bmatrix} N_{i,x} & 0 \\ 0 & N_{i,y} \\ N_{i,y} & N_{i,x} \end{bmatrix} \qquad (6)$$

and $\mathbf{B}^{\mathrm{enr}}$ is the enriched part of the finite element matrix **B**:

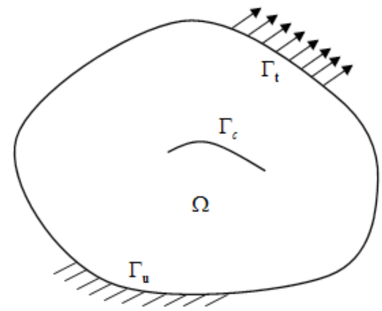
$$\mathrm{B}_i^{enr}=\begin{bmatrix} (N_i)_{,x}\psi_i+N_i(\psi_i)_{,x} & 0 \\ 0 & (N_i)_{,y}\psi_i+N_i(\psi_i)_{,y} \\ (N_i)_{,y}\psi_i+N_i(\psi_i)_{,y} & (N_i)_{,x}\psi_i+N_i(\psi_i)_{,x} \end{bmatrix} \qquad (7)$$

Limit analysis based on XFEM and SOCP

Kinematic formulation

Let Ω denote the domain of a rigid-perfectly plastic body defined in a domain with a boundary Γ of continuous and discontinuous parts such that $\Gamma=\Gamma_{\mathbf{u}}\cup\Gamma_{\mathbf{t}}\cup\Gamma_c$, $\Gamma_{\mathbf{u}}\cap\Gamma_{\mathbf{t}}\cap\Gamma_c=\varnothing$ as shown in Figure 2. The body is subjected to body forces **f** and to surface tractions g on the free portion Γ_t of Γ.

Figure 2. Structural model



Introducing the kinematically admissible velocity fields $\dot{\mathrm{u}}$ and strain rates $\dot{\boldsymbol{\varepsilon}}$, the external work rate can be expressed in the linear form as

$$F(\dot{\mathrm{u}})=\int_\Omega \mathrm{f}^T\dot{\mathrm{u}}\,\mathrm{d}\Omega+\int_{\Gamma_t}\mathrm{g}^T\dot{\mathrm{u}}\,\mathrm{d}\Gamma \qquad (8)$$

The kinematic principle states that the minimum value of load multiplier λ can be determined by the following mathematical programming (Martin [])

$$\lambda^+=\min\int_\Omega D(\dot{\boldsymbol{\varepsilon}})\mathrm{d}\Omega \qquad \text{(a)}$$

$$\text{s.t.}\quad\begin{cases} \dot{\boldsymbol{\varepsilon}}=\mathbf{L}\dot{\mathbf{u}} & \text{in } \Omega \qquad \text{(b)} \\ \dot{\mathbf{u}}=0 & \text{on } \Gamma_u \qquad \text{(c)} \\ F(\dot{\mathbf{u}})=1 & \qquad\qquad \text{(d)} \end{cases} \qquad (9)$$

where $D(\dot{\boldsymbol{\varepsilon}})$ is the plastic dissipation per unit domain and **L** is the differential operator

$$L=\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \qquad (10)$$

Without loss in generality, the kinematically admissible fields $\dot{\mathbf{u}}$ and $\dot{\boldsymbol{\varepsilon}}$ can be normalized such that

$$F(\dot{\mathrm{u}})=1 \qquad (11)$$

Finally, the limit load multiplier can be obtained by solving the following optimisation problem (in normalised form)

$$
\begin{aligned}
&\lambda^{+} = \min \int_{\Omega} D(\dot{\boldsymbol{\varepsilon}})\mathrm{d}\Omega && && \text{(a)} \\
&\text{s.t.} \begin{cases} \dot{\boldsymbol{\varepsilon}} = \mathbf{L}\dot{\mathbf{u}} & \text{in } \Omega \\ \dot{\mathbf{u}} = 0 & \text{on } \Gamma_u \\ F(\dot{\mathbf{u}}) = 1 & \end{cases} && && \begin{matrix}\text{(b)} \\ \text{(c)} \\ \text{(d)}\end{matrix} \quad (12)
\end{aligned}
$$

Although, there are many optimization techniques can be applied to solve the above problem. Unfortunately, the objective function in the associated optimization problem is not differentiable everywhere while powerful optimization algorithms require their gradients to be available everywhere. Various techniques have been proposed in the literature to overcome this singularity problem. These include linearization of the yield condition, regularization of the plastic dissipation function, and a direct iterative algorithm. Perhaps the SOCP is one of the most robust and efficient algorithms to overcome this difficulty because of efficient algorithms and implementations exist, and hence it is relevant to extend its use to our problem. Moreover, most commonly used yield criteria can be cast in the form of conic constraints, and therefore the plastic dissipation (objective function) can be formulated as conic constraints. In next section, a numerical procedure using SOCP in combination with XFEM is introduced to solve the limit analysis of cracked structures.

XFEM discretization and solution procedure

The nature of limit state is characterized by localized plastic deformations, meaning that plastic deformation exists at some elements. However, in numerical solution procedure these localized region is not known a priori, and hence we assume plastic strain rates to be existing at all elements in the formulation. When the structure reaches to limit state, the optimal solution of the discretization optimization problem, the collapse mechanism is formed. Consequently the associated velocity field includes rigid and plastic regions can be determined. Therefore, if the problem variable fields are approximated using the XFEM, and the von Mises failure criterion is employed, the plastic dissipation (12) can be expressed as

$$
D^{XFEM} = \sum_{e=1}^{ne} \int_{\Omega_e} \sigma_0 \sqrt{\dot{\varepsilon}^T D \dot{\varepsilon}^T} d\Omega = \sum_{i=1}^{NG} \sigma_0 w_i \sqrt{\dot{\varepsilon}_i^T D \dot{\varepsilon}_i^T} \tag{13}
$$

$$
\text{where } D = \begin{cases} \frac{1}{3}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{for plane stress} \\[2ex] \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{for plane strain} \end{cases}
$$

and s_0 is the yield stress, *ne* and *NG* are the number of elements and total Gauss points, respectively.

The problem (12) is a non-linear optimization problem with equality constraints. In fact, the objective function of this problem, i.e. the plastic dissipation, can be formulated in the form of a sum of norm as

$$
D^{XFEM} = \sum_{i=1}^{NG} \sigma_0 w_i \left\| \rho_i \right\| \tag{14}
$$

where $\|\rho_i\|$ are additional variables defined by

$$\rho_i = \begin{cases} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} = \dfrac{1}{\sqrt{3}} \begin{bmatrix} 2 & 0 & 0 \\ 1 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{\varepsilon}_i & \text{for plane stress} \\ \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} \dot{\varepsilon}_{xxi} - \dot{\varepsilon}_{yyi} \\ \dot{\varepsilon}_{xyi} \end{bmatrix} & \text{for plane strain} \end{cases} \tag{15}$$

Introducing auxiliary variables t_1, t_2, . . .t_{NG}, optimization problem (12) can be cast as a SOCP problem:

$$\lambda^+ = \min \sum_{i=1}^{NG} \sigma_0 w_i t_i \tag{16}$$

$$\begin{cases} \dot{u} = 0 & \text{on } \Gamma_u \\ \dot{\varepsilon}_i = B_i \dot{u} & \\ F(\dot{u}) = 1 & \\ \|\rho_i\| \le t_i & i = 1, 2, ..., NG \end{cases}$$

Note that for plane strain problems, incompressibility conditions must be introduced in order to ensure that the plastic dissipation D is finite. Furthermore, to ensure that strict upper bound can be obtained, this condition has to be satisfied everywhere. However, when low-order finite elements are used, the condition leads to a reduction in the available number of degrees of freedom, and therefore the true velocity field cannot be exactly described. To overcome such a problem, in this paper reduced integration technique [10] will be used.

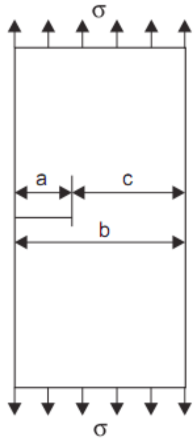
Numerical examples

In this section, the performance of the proposed solution procedure is illustrated via various benchmark problems in which analytical and other numerical solutions are available.

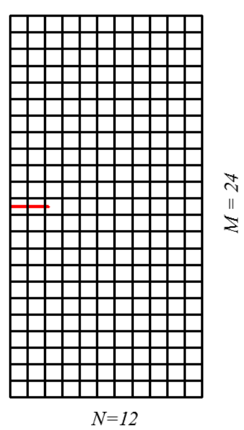
Simple-edge notched plate problem

The first example deals with a single-edge cracked plate under tension, which is often used for fatigue-crack propagation tests. The problem geometry, loadings and finite element mesh are shown in Figure 3a.

Figure 3. Single-edge cracked plate under tension



(a) Geometry and loading



(b) Element mesh

Analytical solution of this problem was proposed by Ewing and Richards [3], where a slip-line method was used. In plane stress condition, the limit load factor defined by $\lambda = \sigma_{limit}/\sigma_0$ can be computed as

$$\lambda = \left[\left(-\gamma x + \frac{\gamma - 1}{2}\right)^2 + \gamma(1-x)^2\right]^{1/2} - \left(\gamma x - \frac{\gamma-1}{2}\right) \quad \text{for deeply-cracked plate} \tag{17}$$

$$\lambda = 1 - x - x^2 \quad \text{for short-cracked plate } (x \le 0.146) \tag{18}$$

where $x = a/b$ and $\gamma = 2/3$.

In plane strain condition the limit load factor can be computed as

$$\lambda = 1.702\gamma\left[(0.206-x)^2 + 0.5876(1-x)^2\right]^{1/2} + (0.206-x) \quad \text{for } x > 0.545 \tag{19}$$

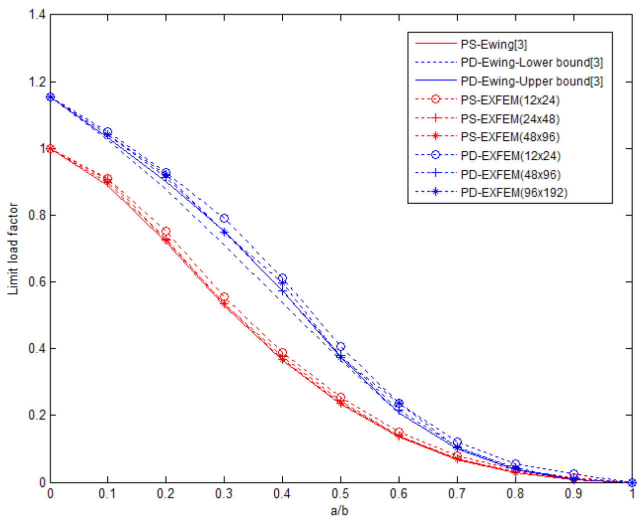
$$\lambda \ge \gamma(1 - x - 1.232x^2 + x^3)$$

$$\lambda \le \gamma\left[1 - x - 1.232x^2 + x^3 + 22x^3(0.545-x)^2\right] \tag{20}$$

for short-cracked plate ($x \le 0.146$)

Numerical solutions were also investigated by other authors e.g. Yan [4], Khoi [5] in which special elements is used to capture the singular strain field around crack tips. In our study, the full cracked plate is modelled with variation of *NxM* (number of nodes per short edge and length edge respectively) as shown in Figure 3b. Numerical results for different ratios a/b are shown in Figure 4 for both plane stress (PS) and plane strain (PD).

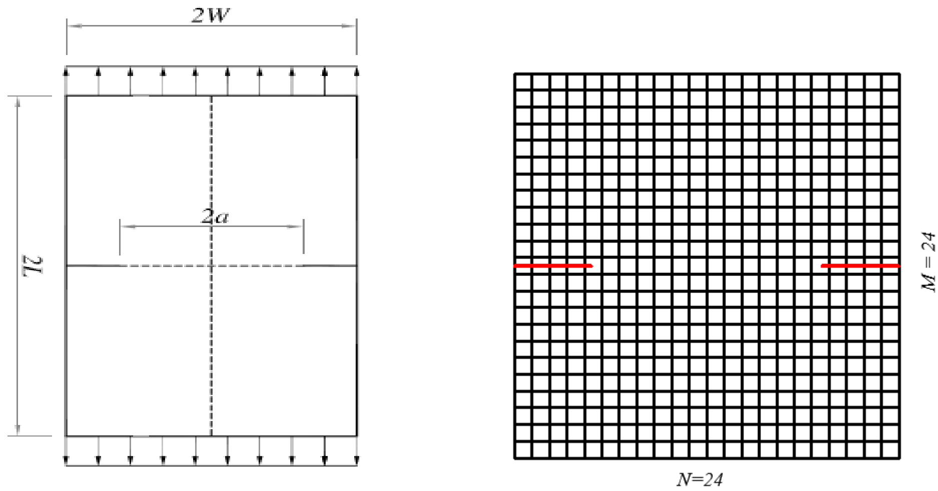
Figure 4. Limit load factor of single-edge cracked plate; PS – Plane stress; PD – Plane strain



From the results, it can be observed that the present solution converges to the exact solution with the error less than 2.5% for PS condition and 3.9% for PD condition. When compared with solution previously obtained in [5], very good agreements can be observed in both PS and PD conditions. For case $a/b = 0.5$ with plane strain condition, the upper bound result with dual algorithm [5] $\lambda^+ = 0.3729$ only error of 0.4% compared to the present solution. Although present solution is a little higher than the previous solutions, in the present procedure mesh generation is simple and cracks are realised by means of level set method. Moreover, the underlying optimization problem is cast in a suitable form so that it can be solved using highly efficient primal-dual interior point algorithms.

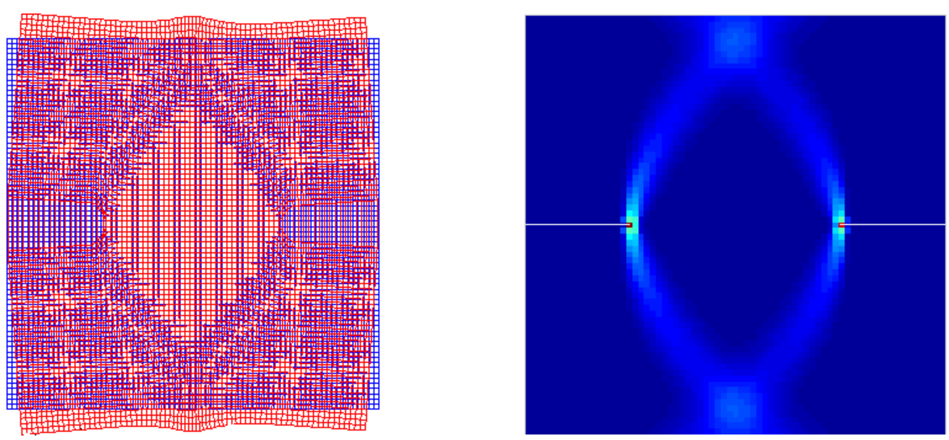
Double-edge notched plate problem

Figure 5. Single-edge cracked plate under tension



Finally, the double-edge notched tensile plate is consider, as shown Figure 5. The problem was first given by Nagtegaal et al. [1] in order to illustrate the locking phenomena in strain problem. It has become a popular benchmark for various numerical models in the field of rigid-plastic limit analysis.

Figure 6. Collapse mechanisms and plastic dissipation distribution for $a = \frac{1}{2}$



(a) Collapse mechanisms (b) Plastic dissipation distribution

Again taking advantages of the XFEM, the full cracked plate is modelled with variation of *NxM*. In order to shown the effectiveness of the present method, the obtained results will be compared with upper bound solutions [12, 13], lower bound solutions [12, 14, 15] and mixed solutions [16, 17]. These methods do not allow discontinuities across elements or domains of influence, cracks must be recognized in advance and care must be made to generate mesh around the cracks and its tips. From Table 1, it can be observed from that the present solutions are generally in reasonably good agreement with those obtained previously. The plastic dissipation distribution for the case $a = \frac{1}{2}$ are shown in Figure 6.

Table 1. The limit load factor of the present method in comparison with those of other methods for the double-edge notched plate problem

Approach	Author and methods	a=1/3	a=1/2	a=2/3
Kinematic	Ciria et al. [12]		1.1390	
	Le et al. [13]	0.9412	1.153	1.4097
	XFEM (24x24)	0.9846	1.2198	1.4722
	XFEM (48x48)	0.952	1.1747	1.4353
	XFEM (96x96)	0.9373	1.1527	1.4094
Static	Ciria et al. [12]		1.1315	
	Krabbenhoft and Damkilde [14]		1.1315	
	Tin-loi and Ngo [15]	0.947	1.166	1.434
Mixed solution	Andersen [16]	0.9271	1.1366	1.3894
	Christiansen and Andersen [17]	0.9276	1.1358	1.3884

Conclusion

A numerical limit analysis procedure that uses the extend finite element method (XFEM) and second-order cone programming (SOCP) has been proposed. Advantages of applying the XFEM to limit analysis of cracked structure problems can be drawn as follows:

(1) Numerical solutions of the XFEM are, in general, close to the exact solutions and show good agreement with numerical results available in the literature for plane strain condition.

(2) Numerical examples are given to demonstrate the efficiency of the present method. It is shown that the proposed procedure is able to solve large-scale problems in engineering practice.

Although only two-dimensional problems were considered, the presented method can be extended to tackle more complex structural configurations, subject to a variety of loading regimes.

REFERENCES

[1] Hill R. (1952). On discontinuous plastic states, with special reference to localized necking in thin sheets. *Journal of the Mechanics and Physics of Solids*, Vol. 1, pp. 19-30.

[2] Ewing D. J. F. (1967). A series-method for constructing plastic slip-line fields. *Journal of the Mechanics and Physics of Solids*, Vol. 15, pp. 105-114.

[3] Ewing D. J. F., Richards C. E. (1974). The yield-point loading of singly-notched pin loaded tensile strips. *Journal of the Mechanics and Physics of Solids*, Vol 22, pp. 27-36.

[4] Yan A. M. (1997). Limit analysis of cracked structures by mathematical programming and finite element technique. *Computational Mechanics* 24 (1999), 319-333.

[5] Vu Duc Khoi (2001). Dual limit and shakedown analysis of structures. *Doctoral thesis*, University of Liege.

[6] Black T., Belytschko T. (1999), Elastic crack growth in finite elements with minimal

remeshing, *International Journal for Numerical Methods in Engineering*, 45 601–620.

[7] Moes N., Dolbow J., Belytschko T. (1999), A finite element method for crack growth without remeshing, *International Journal for Numerical Methods in Engineering*, 46 (1) 133–150.

[8] Babuska I., Melenk J.M. (1997) , The partition of unity method, *International Journal for Numerical Methods in Engineering* 40, 727–758.

[9] Martin J. B. (1975). *Plasticity: Fundamentals and General Results*. MIT.

[10] S.W. Sloan and M.F. Randolph (1982). Numerical prediction of collapse loads using finite element methods. *International Journal for Numerical and Analytical Methods in Geomechanics*, 6:47–76.

[11] J. C. Nagtegaal, D. M. Parks, and J. C. Rice (1974). On numerically accurate finite element solutions in the fully plastic range. *Computer Methods in Applied Mechanics and Engineering*, 4:153–177.

[12] H. Ciria, J. Peraire, and J. Bonet (2008). Mesh adaptive computation of upper and lower bounds in limit analysis. *International Journal for Numerical Methods in Engineering*, 75:899–944.

[13] C.V. Le, H. Askes, and M. Gilber. A locking-free stabilized kinematic model for plane strain limit analysis. *Computers and Structures*, 2012.

[14] K. Krabbenhoft and L. Damkilde. A general nonlinear optimization algorithm for lower bound limit analysis. *International Journal for Numerical Methods in Engineering*, 56:165 -184, 2003.

[15] F. Tin-Loi and N.S. Ngo. Performance of the p-version finite element method for limit analysis. *International Journal of Mechanical Sciences*, 45:1149–1166, 2003.

[16] K. D. Andersen, E. Christiansen, and M. L. Overton. Computing limit loads by minimizing a sum of norms. *SIAM Journal on Scientific Computing*, 19:1046–1062, 1998.

[17] E. Christiansen and O. S. Pedersen. Automatic mesh refinement in limit analysis. *International Journal for Numerical Methods in Engineering*, 50:1331–1346, 2001.